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DAIF AND SCP – DERIVATIONS ON SEMIGROUP IDEAL *I* OF A NEAR–RING *N*.

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Abstract.

In this paper we study two kinds of derivations on a semigroup ideal *I* of a near–ring *N*. The first kind called Daif–derivation (Daif 1–derivation and Daif 2–derivation), the second kind is called strong commutativity–preserving derivations. Bell and Mason, showed that a prime near–ring *N* must be commutative if it admits any of these kinds of derivations, and we generalize this to a semigroup ideal *I*.

Keywords and phrases : Prime near-ring, semiprime near-ring, semigroup ideal, Daif 1– derivation, Daif 2–derivation, SCP– derivation.
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1.Introduction

A left near-ring is a set N together with two binary operations (+) and (.) such that $(N, +)$ is a group (not necessarily abelian).and $(N, .)$ is a semigroup, for all $a, b, c\in N$; we have $a.(b+c) =a.b+ a.c$. A near ring N is called zero symmetric if $0x = 0$, for all $x \in N$. An additive mapping $D: N\to N$ is called a derivation if $D(xy)= xD(y)+ D(x)y$, for all $x,y\in N$. Further an element $x \in N$ for which $D(x)= 0$ is called a constant. D is called Daif 1– derivation if D is a derivation with the property that $-xy+D(xy) = -yx+D(yx)$, for all $x,y\in N$, and Daif 2–derivation if D is a derivation with the property that $xy+D(xy) = yx+D(yx)$, for all $x,y\in N$, [6].A mapping D is called strong commutativity – preseriving derivation (scp_ derivation), if D is a derivation such that $[D(x),D(y)] = [x,y]$, for all $x,y\in N$, [5]. A non empty subset I of N will be called a semigroup ideal if $IN\subseteq I$ and $NI \subseteq I$. A near-ring N is said to be a 2-torsion free if for all $a \in N$, $2a = 0$ implies $a = 0$. According to Bell and Mason [3], and Bell and Kappe [2], a near – ring N is said to be prime if $xNy = 0$ for $x, y\in N$ implies $x = 0$ or $y = 0$, and semiprime if $xNx = 0$ for $x\in N$ implies $x = 0$. For $x,y\in N$, the symbol $[x,y]$ will denote the commutator $xy-yx$, while the symbol (x,y) will denote the additive – group commutator $x+y-x-y$. In [2] the derivation D was called commuting if $[x,D(x)] = 0$, for all $x \in N$. As for terminologies used here without mention, we refer to [7]. Throughout this paper N will denote a zero–symmetric left near – ring with multiplicative center $Z(N)$.

2. The Results

Lemma 1:

Let D be a derivation on a near–ring N and I semigroup ideal of N. Then $D(xy) = D(x)y + xD(y)$, for all $x,y \in I$.

Proof:

For all $x,y\in I$, we have $x(y+y)=xy+xy$. Applying D for both sides, we get

$$\begin{aligned} D(x(y+y)) &= xD(y+y)+D(x)(y+y) \\ &= xD(y)+xD(y)+D(x)y+D(x)y. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} D(xy+xy) &= D(xy)+D(xy) \\ &= xD(y)+D(x)y+xD(y)+D(x)y. \end{aligned}$$

Comparing these two expressions gives

$$\begin{aligned} xD(y)+D(x)y &= D(x)y+xD(y) \\ D(xy) &= D(x)y+xD(y) \end{aligned}$$

for all $x,y \in I$. □

Lemma 2:

Let D be a derivation on a near–ring N and I a semigroup ideal of N. Then $(aD(b) + D(a)b)c = aD(b)c + D(a)bc$, for all $a,b\in I$ and $c\in N$.

Proof:

For all $a,b\in I$, $c\in N$, we get

$$\begin{aligned} D((ab)c) &= abD(c)+D(ab)c \\ &= abD(c)+(aD(b)+D(a)b)c \end{aligned}$$

On the other hand,

$$\begin{aligned} D(a(bc)) &= aD(bc)+D(a)bc \\ &= abD(c)+aD(b)c+D(a)bc \end{aligned}$$

For these two expressions of D(abc), for all a,b∈ I, c∈ N we obtain that.
(aD(b) + D(a)b)c = aD(b)c + D(a)bc. □

Lemma 3

Let N be a near–ring and I a semigroup ideal of N that admits a Daif 1–derivation D. Then

(i) D(c) = c, for each commutator c in I.
(ii) D(z)[x,y] = [x,y]D(z) , for all x,z∈ I and y∈ N.

Proof:

(i) Let c = [x,y], where x∈ I, y∈ N.So that c∈ I.By the definition of Daif 1– derivation, we have D([x,y])=[x,y], for all x∈ I, y∈ N. Thus, D(c) = c for each commutator c in I.
(ii) Since D is a Daif 1– derivation on I, we have
–[x,y]z + D([x,y]z) = –z[x,y] + D(z[x,y]), for all x,z∈ I, y∈ N.
By Lemma(1), we get –[x,y]z+D([x,y])z + [x,y]D(z) = –z[x,y] + zD([x,y]) + D(z)[x,y].
By application (i), we get
[x,y]D(z) = D(z)[x,y], for all x,z∈ I, y∈ N. □

Lemma 4:

Let N be a prime near–ring and I a semigroup ideal of N, that admits a Daif 1– derivation D. Then

(i) If c is a commutator in I and uc = vc, where u,v ∈ I, then cD(u–v)=0.
(ii) If c1 and c2 are commutators in I with c1c2 = 0, then c1 = 0 or c2= 0.

Proof:

(i)Let c=[x,y] for all x∈I,y∈ N. Then,by the hypothesis,we have u[x,y]=v[x,y], for all x,u,v∈ I, y∈ N.Applying D for both sides, we get uD([x,y]) +D(u)[x,y]= vD([x,y]) +D(v)[x,y], for all x,u,v∈ I, y∈ N.By Lemma (3) (i, ii), we get [x,y]D(u)= [x,y]D(v), for all x,u,v∈ I,y∈N. Hence, [x,y]D(u–v) =0. Thus, cD(u–v) = 0, for all commutator c in I and u,v∈ I.

(ii) If c1c2 = 0 = 0c2, since c2 is a commutator in I, (i) yields

$c_2D(c_1) = 0$ .... (1)

By Lemma (3) (i), since c_1 is commutator in *I* we get

$c_2c_1 = 0$ .... (2)

Replace c_1 by yc_1, where $y\in I$, in equation (1), we get

$c_2D(yc_1) = 0 = c_2\, yD(c_1) + c_2D(y)c_1$ .... (3)

Using Lemma (3) (ii), and equation (2) in equation (3), we get $c_2yD(c_1) = 0$, for all commutators c_1, c_2 in *I* and $y\in I$. Hence, $c_2\, I\, D(c_1) = 0$, by Lemma (3) (i), since c_1 is commutator, we get $c_2\, I\, c_1 = 0$. Since *I* is a nonzero semigroup ideal of *N* and *N* is a prime near–ring,we get $c_1 = 0$ or $c_2 = 0$. □

Lemma 5:

Let *N* be a prime near–ring and *I* be a nonzero semigroup ideal of *N*, then $Z(I)\subseteq Z(N)$.

Proof:

Let $a \neq 0\in Z(I)$.That means, $[a,x] = 0$, for all $x\in I$. Taking *xy* instead of *x*, where $y\in N$, we get $[a, xy] = 0 = x[a,y]$, for all $a,x\in I$, $y\in N$, since $a\in Z(I)$. Hence, $I[a,y] = 0$, since *I* is a nonzero semigroup ideal of *N* and *N* is a prime near–ring, we obtain $[a,y] = 0$, for all $a\in I$, $y\in N$. Hence, $a\in Z(N)$. □

Lemma 6:

Let *N* be a prime near–ring and *I* be a semigroup ideal of *N*.

(i) If *z* is a nonzero element in *Z*(*N*), then *z* is not a zero divisor.
(ii) If there exists a nonzero element *z* of *Z*(*N*) such that $z+z \in Z(N)$, then $(I,+)$ is abelian.

Proof:

(i) If $z\in Z(N)\backslash\{0\}$, and $zx = 0$ for some $x\in I$. Left multiplying this equation by *b*, where $b\in N$, we get $bzx = 0$. Since $z\in Z(N)$, we get $zbx = 0$, for $b\in N$, $x\in I$. Hence, $zNx=0$, since *N* is a prime near–ring and *z* is a nonzero element, we get $x=0$.

(ii) Let $z\in Z(N)\backslash\{0\}$ be an element, such that $z+z\in Z(N)$,and let $x,y\in I$, then $(x+y)(z+z)= (z+z)(x+y)$.Hence,$xz+ xz+ yz + yz = zx+zy+zx+zy$. Since $z\in Z(N)$,we get $zx + zy = zy + zx$.Thus, $z(x+y–x–y) = 0$. The Left multiplicative this equation by *b*, where $b\in N$, we get $bz(x,y)= 0$, for all $x,y \in I$ and $b\in N$.Since *N* is multiplicative with center *Z*(*N*),we get $zb(x,y)= 0$,for all $x,y\in I$, $b\in N$. Hence, $zN(x,y) = 0$.Since *N* is a prime near-ring and *z* is a nonzero element, we get $(x,y)=0$, for all $x,y\in I$.Thus,$(I, +)$ is abelian.□

Lemma 7:

Let D be a nonzero derivation on a prime near–ring N and I be a nonzero semigroup ideal of N. Then $xD(I) = 0$ implies $x = 0$ and $D(I)x = 0$ implies $x = 0$, where $x\in N$.

Proof:

Let $xD(I) = 0$,and let $r\in N$, $s\in I$. Then, $0 = xD(sr) = xsD(r) + xD(s)r$, thus $xsD(r) = 0$, for $x,r\in N$ and $s\in I$. Hence, $xID(r) = 0$. Since I is a semigroup ideal of N, we get $xIND(r) =0$. Since N is a prime near–ring, I is a nonzero semigroup ideal of N and D is a nonzero derivation on N , we get $x = 0$. By similar way, we can show that if $D(I)x = 0$, for all $x\in N$, then $x=0$. □

Lemma 8:

Let N be a prime near–ring and I a nonzero semigroup ideal of N. If N is 2 – torsion free and D is a derivation on N such that $D^2(I)=0$, then $D(I) = 0$.

Proof:

Suppose D is a nonzero derivation, For arbitrary $x,y\in I$, we have

$0 = D^2(xy)= D(D(xy))=D(xD(y) +D(x)y)= xD^2(y)+D(x)D(y)+D(x)D(y)+ D^2(x)y$.

By the hypothesis, we get $2D(x)D(y) = 0$, for all $x,y\in I$. Since N is a 2 – torsion free, we get $D(x)D(y) = 0$. Thus, $D(x)D(I) = 0$, for all $x\in I$. By Lemma (7), we get $D(I) = 0$.□

Lemma 9:

Let N be a prime near–ring and I a nonzero semigroup ideal of N.If I is commutative, then N is a commutative ring.

Proof:

For all $a,b\in I,[a,b] = 0$. Taking ax instead of a and, by instead of b, where $x,y\in N$, we get $[ax,by] = 0$. Since I is commutative semigroup ideal of N, we have $0 = axby – byax = baxy – byax = abxy – abyx = ab[x,y]$, for all $a,b\in I$, $x,y\in N$. Thus, $ab[x,y] = 0 = I^2[x,y]$. Since I is a semigroup ideal of N, we get $I^2N[x,y] = 0$, for all $x,y\in N$. Since N is a prime near–ring and I is nonzero, we get $[x,y] = 0$, for all $x,y\in N$. Hence, N is a commutative ring. □

Lemma 10:

Let D be a derivation on a near–ring N and I a semigroup ideal of N, suppose $u\in I$ is not a left zero divisor. If $[u,D(u)] = 0$, then (x,u) is a constant for every $x\in I$.

Proof:

From $u(u+x) = u^2+ux$, apply D for both sided we have

$uD(u+x)+D(u)(u+x)=uD(u)+D(u)u+uD(x)+ D(u)x$. Which reduces to $uD(x)+D(u)u = D(u)u+ uD(x)$, for all $u,x\in I$. Using the hypothesis $[u,D(u)]=0$,this equation is expressible as $u(D(x)+ D(u) –D(x)– D(u))= 0 = uD((x ,u))$. Since u is not a left zero divisor, we get $D((x,u)) = 0$.Thus, (x,u) is a constant for every $x\in I$. □

Theorem 1:

Let N be a near–ring and I a semigroup ideal of N with no nonzero divisors. If N admits a nonzero derivation D which is commuting on I, then $(N, +)$ is abelian.

Proof:

Let c be any additive commutator in I. Then, by Lemma (10) yields that c is a constant. For any $x\in I$, xc is also an additive commutator in I. Hence, also a constant. Thus, $0 = D(xc) = xD(c) + D(x)c$. First summand in this equation equals zero, hence $D(x)c = 0$, for all $x\in I$ and an additive commutator c in I. Since $D(x) \neq 0$, for some $x\in I$ and I has no nonzero divisors of zero, we get $c = 0$, for all additive commutator c in I. Hence, $(I, +)$ is abelian. By [1], we get $(N, +)$ is abelian. □

Lemma 11:

Let N be a prime near–ring which admits a nonzero derivation D and let I be a semigroup ideal of N such that $D(I)\subseteq Z(N)$, then $(I, +)$ is abelian. If N is a 2 – torsion free and $D(I)\subseteq I$, then I is a central ideal.

Proof:

Since $D(I)\subseteq Z(N)$ and D is a nonzero derivation, there exists a nonzero element x in I, such that $z=D(x)\in Z(N)\backslash\{0\}$.And, $z+z =D(x)+ D(x)=D(x+x)\in Z(N)$.Hence, $(I,+)$ is abelian by Lemma (6) (ii). Using hypothesis, for any $a,b\in I$ and $c\in N$, $cD(ab) = D(ab)c$. Using Lemma(2), we have $caD(b)+cD(a)b=aD(b)c+D(a)bc$. But $D(I)\subseteq Z(N)$ and $(I, +)$ is abelian, so we get $caD(b)+ D(a)cb = acD(b) + D(a)bc$. So, we have $[c,a]D(b) = D(a)[b,c]$, for all $a,b\in I$, $c\in N$. Suppose that I is not a central ideal. Choosing

$b\in I$ and $c\in N$ such that $[b,c]\neq 0$. And since $D(I)\subseteq I$, let $a = D(x)\in Z(N)$, where $x\in I$, we get $[c,D(x)]D(b) = D^2(x)[b, c]$, for all $x,b\in I$, $c\in N$. Then, $D^2(x)[b,c] = 0$, for all $x\in I$. By Lemma (6) (i), the central element $D^2(x)$ can not be a nonzero divisor of zero, then we conclude that $D^2(x) = 0$, for all $x\in I$. By Lemma (8), we get $D(x) = 0$, which is a contradiction since D is a nonzero derivation on N.So, we get $[b,c] = 0$, contradiction with assumption. Hence, I is a central ideal. □

Theorem 2:

Let N be a prime near–ring that admits a nonzero derivation D and let I be a semigroup ideal of N such that $D(I)\subseteq Z(N)$. Then $(N, +)$ is abelian. If N is 2 – torsion free and $D(I)\subseteq I$, then N is a commutative ring.

Proof:

By Lemma (11), we have $(I,+)$ is abelian,and by [1], $(N,+)$ is abelian. Now, assume N is 2 – torsion free. By applying Lemma (11), we get I is a central ideal. Thus, I is a commutative. By Lemma (9), we get N is a commutative ring. □

Theorem 3:

Let N be a prime near–ring and let I be a nonzero semigroup ideal of N admits a nonzero Daif 1– derivation D.Then $(N, +)$ is abelian. If N is 2 –torsion free and $D(I)\subseteq I$, then N is a commutative ring.

Proof:

Since $[x,xy] = x[x,y]$, for all $x\in I$, $y\in N$. By Lemma (3) (ii), we have

$D(z)x[x,y] = x[x,y]D(z) = xD(z)[x,y]$, for all $x,z\in I$ and $y\in N$.

By Lemma (4) (i), we get $[x,y]D(D(z)x - xD(z)) = 0$, for all $x,z\in I$ and $y\in N$. Hence, $[x,y]D([D(z), x]) =0$, for all $x,z\in I$, $y\in N$. By Lemma (3) (i), we get $[x,y][D(z), x] = 0$. By Lemma (4)(ii), we obtain either $[x,y]=0$ or $[D(z),x]=0$, for all $x,z\in I$, $y\in N$. If $[D(z), x] = 0$, for all $x,z\in I$, then, $D(I)\subseteq Z(I)$.By Lemma(5),we get $D(I)\subseteq Z(N)$. Thus, by Theorem (2), we complete the proof of the theorem. Now, if $[x,y]=0$, for all $x\in I$, $y\in N$. Taking $xD(z)$ instead of x, where $z\in I$, we get $[xD(z), y] = 0 = x[D(z), y]$, for all $x,z\in I$, $y\in N$. Hence, $I[D(z),y] =0$,for all $z\in I$, $y\in N$. Since I is a nonzero semigroup ideal of N and N is a prime near–ring, we get $[D(z), y] = 0$, for all $z\in I$, $y\in N$. So, $D(I)\subseteq Z(N)$. Again by Theorem (2), we complete the proof in this case. □

Theorem 4:

Let N be a prime near–ring and let I be a nonzero semigroup ideal of N admits a Daif 2–derivation D.Then N is a commutative ring.

Proof:

Since D is a Daif 2 – derivation on I, we have $D([x,y]) = -xy + yx$, for all $x\in I, y\in N$. Replacing y by xy in this equation, we get

$D([x,xy])= -xxy+xyx =x(-xy+yx)$,for all $x,y\in I$.

On the other hand,

$$\begin{aligned} D([x,xy])=D(x[x,y]) &= xD([x,y])+D(x)[x,y] \\ &= x(-xy+yx)+D(x)[x,y] \end{aligned}$$

It follows from the two expressions for $D([x,xy])$ that

$D(x)xy =D(x)yx$, *for all* $x\in I, y\in N$ .... (4)

Replacing y by yz in equation (4), where $z\in I$, we get

$D(x)xyz = D(x)yzx$, *for all* $x,z\in I$, $y\in N$ ...... (5)

Right – multiplbing equation (4) by z, where $z\in I$, we obtain

$D(x)xyz = D(x)yxz$, *for all* $x,z\in I$, $y\in N$ ..... (6)

Combining equation (5) and equation (6), we get $D(x)y[x,z]=0$, for all $x,z\in I$, $y\in N$. Hence, $D(x)N[x,z] = 0$. Since N is a prime near–ring, we get $D(x)= 0$ or $[x,z]= 0$. If $D(x)=0=D(w)$, for all $x,w\in I$. Then by the defining equation

$D([x,w])=D(xw-wx)=D(x)w+x D(w)-D(w)x-wD(x)= -xw+wx$, for all $x,w\in I$.

We get $[w,x] = 0$, for all $x,w\in I$. Thus, I is a commutative and by Lemma (9), we get N is a commutative ring. If $[x,z] = 0$, for all $x,z\in I$. Then I is commutative. And by Lemma (9), we get N is a commutative ring. □

Lemma 12:

Let N be a prime near–ring and let I a nonzero semigroup ideal of N. If D is an scp–derivation on I, then the constants in I are in $Z(N)$.

Proof:

Let x be a constant in I. i.e. $D(x) = 0$. Since D is a scp–derivation on I, we have $[x,y] = [D(x), D(y)] =[0, D(y)] = 0$, for all $y\in I$. Then, $x\in Z(I)$. By Lemma (5), we get $x\in Z(N)$. □

<u>Theorem 5</u>:

Let N be a prime near–ring and let I be a nonzero semigroup ideal of N, with no nonzero divisors. If I has right cancellation, D is a nonzero scp–derivation on I and $D(I)\subseteq I$, then D is commuting and $(N, +)$ is abelian.

<u>Proof:</u>

$[x, xD(x)]=[D(x),D(xD(x))]$, *for all* $x\in I$ ....(7)

By Lemma (1) and (2), the right – hand side of equation (7) equals

$$D(x)xD^2(x)+D(x)^3 - D(x)^3 - xD^2(x)D(x) = D(x)xD^2(x) - xD^2(x)D(x).$$

The left – hand side of equation (7) equals

$$x[x,D(x)] = x[D(x),D^2(x)] = xD(x)D^2(x) - xD^2(x)D(x), \text{ for all } x\in I.$$

It follows from equation (7) that

$xD(x)D^2(x) = D(x)xD^2(x)$, *for all* $x\in I$ .... (8)

If $D^2(x) = 0$, for all $x\in I$. Then, $D(x)$ is constant in I, by Lemma (12), we get D is central. Thus, D is commuting in I. By Theorem (1), we get $(N, +)$ is abelian. Otherwise, $D^2(x)$ can be cancelled on the right in equation (8). In either event, $[x,D(x)] = 0$, for all $x\in I$. Thus, by Theorem (1), we get $(N, +)$ is abelian. □

Theorem 6:

Let N be a near–ring and let I be a nonzero semigroup ideal of N, that has no nonzero divisors of zero. If D is a nonzero scp–derivation on I which is commuting on I, then N is a commutative ring.

<u>Proof</u>:

For all $x,y\in I$,we have $[x,xy]=[D(x), D(xy)] = [D(x),xD(y)+D(x)y]$. By Lemma (2), we get $[x,xy]=D(x)xD(y)+D^2(x)y-xD(y)D(x)-D(x)yD(x)$. Since D is commuting, by Theorem (1), $(N, +)$ is abelian, then we get

$x[x,y]= x[D(x),D(y)]= x[D(x),D(y)]+ D(x)[D(x),y]$, for all $x,y\in I$

Hence, $D(x)[D(x),y] = 0$, for all $x,y\in I$. Since I has no nonzero divisors and D is nonzero, we conclude that $[D(x),y] = 0$, for all $x,y\in I$. In particular, for all $x,y,z\in I$, we have $[D(x),zD(y)] = 0 = z[D(x),D(y)]$. Hence, I $[D(x),D(y)]=0$, for all $x,y\in I$. Since I is a nonzero semigroup ideal of N and N is a prime near–ring, we get $[D(x),D(y)] = 0$, for all $x,y\in I$. Thus, we conclude that $[x,y] = 0$, for all $x,y\in I$. Hence, I is commutative. By Lemma (9), we get N is a commutative ring. □

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References

[1] N. Argac, “On near-rings with two sided α –derivations”, Turk. J. Math., Vol. 28, 2004, 195 – 204.

[2] H. E. Bell and L. C. Kappe, "Rings in which derivations satisfy certain algebraic conditions", Act Math. Hungar, Vol. 53. No. 3–4, 1989,pp. 339 – 346.

[3] H. E .Bell and G. Mason," On derivation in near–rings, Near–rings and Near–fields", (G. Betsch, ed), North–Holland, Amsterdam, 1987, pp. 31 – 35.

[4] H. E.Bell and G. Mason, "On derivations in near–rings and rings”, Math. J. Okayama Univ, Vol. 34, 1992, pp. 135 – 144.

[5] M. Bresar," Commuting traces of biadditive mappings, commutativity – preserving mappings, and Lie mappings", Trans. Amer. Math, Soc. 335, 1993, pp. 525 – 546.

[6] N. N. Daif and H. E.Bell," Remarkes on derivations on semiprime rings", Internat. J. Math & Math.,Sci. 15 ,1992, pp. 205 – 206.

[7] G. Pliz," Near-rings, 2nd Ed", North – Holland, Amsterdam.1983.

الخلاصة

في هذا البحث سوف ندرس نوعين من الاشتقاق علـــى المثالي شبة زمرة I في الحلقة المقتربة N ، النـــوع الاول من النمط ديف -1 واشتقاق من النمط ديـــف -2 والنـــوع الثاني هو اشتقاق المحافظ على الابدالية القوية .بل وماسون نرى ان الحلقة المقتربه الاولية N يجب ان تكون ابدالية اذا وجد اي نوع من الاشتقاق وسوف نعمم تلك علـــى المثـــالي شبه زمرة I.