Journal of Al-Nahrain University Vol.13 (4), December, 2010, pp.194-198 Science

S-QUASI-INJECTIVE MODULES

Ahmed Hanoon Abud
Department of Mathematics, College of Science, Al-Mustansirya University.

Abstract

Let R be a ring with non-zero identity and let M be a right R-module. This paper study the concept of s-quasi-injectivity. Some properties of this concept are investigated and some conditions has been given for s-quasi-injective R-modules to be injective or quasi-injective. Some conditions stated for submodules with this property to be s-quasi-injective.

Keywords: injective, quasi-injective, stable.

Introduction and Definitions

The concept of s-quasi-injectivity was introduced by S.A. Al-Saadi in [1]. A submodule N of an R-modul M is called stable if for each R-homomorphism $f:N \to M$ implies $f(N) \subseteq N$. An R-module M is called fully stable in case every submodule of M is stable [2]. An R-module M is called s-quasi-injective if for each stable submodule N of M, each R-homomorphism from N into M can be extended to an R-endomorphism of M. He related this concept with the extended modules. In our work we gives some characterization for this concept and generalized some properties of quasi-injectivity to s-quasi-injectivity and give s ome conditions under which these concepts be coincides. We show that these concepts are equivalent in case that the R-module is fully stable. Also we proved that an s-quasi-injective R-module M with $J(S)M \subseteq M$ is injective if there is an s-ess-epimorphism from M onto its injective envelope. We define a kind of subset of the ring of endomorphism of the injective envelope of an R-module and we give a characterization of s-quasi-injectivity in term of it. If M is an R-module we write $ann_R(M) = \{r \in R \mid rm=0, \forall m \in M\}$ and we write $ann_M(R)=\{m \in M \mid rm=0, \forall r \in R\}$.[3]

Lastly through this work, all rings are with 1 and all modules are unitary R-modules.

(1) Definition:

An R-module M is said to be s-quasi-injective if each R-homomorphism of any stable submodule N of M into M can be extended to an R-endomorphism of M.[1]

(2) Examples and Remarks:

(a)Every quasi-injective R-module(and hence injective) is s-quasi-injective .

(b) Infinite cyclic groups are S-quasi-injective for,

Let G =<a> bean infinite cyclic group generated by a and N be any stable subgroup of G , then N =<b> for some $b \in N$. For each Z-homomorphism $f: N \to G$ and since N is stable, then $f(N) \subseteq N$ and hence f(b)=nb for some $n \in Z$. Now define $g: G \to G$ by g(x)=nx $\forall x \in G$. It is clear that g is an extension of f.

In particular Z as Z-module is s-quasi-injective while is not quasi-injective, since:

Assume that Z is quasi-injective Z-module and let $f:2Z \to Z$ be the Z-homomorphism defined by f(2n)=n for each $n \in Z$. Then there is an endomorphism $g: Z \to Z$ such that g extends f (i.e. $g|_{2Z}=f$) thus $n=f(2n)=g(2n)=2n\ g(1)$ and hence g(1)=1/2 which is a contradiction. Therefore Z is not a quasi-injective Z-module.

We start our result by the following proposition which give us the relationship between the s-quasi-injective R-modules and the s-quasi-injective R/I-modules, where I is an ideal of R.

(3)Proposition:

Let M be an R-module and I be an ideal of R, if M is s-quasi-injective R/I-module then M is s-quasi-injective R-module. Conversely, if M is s-quasi-injective R-module such that $I \subseteq ann_R(M)$, then M is s-quasi-injective R/I-module.

Proof:

The relation (r+I)m=rm for each $m \in M$ and $r \in R$ is used in each case to define M as a module over R or over R/I where M is s-quasi-injective R/I-module or R-module (respectively). Let N be any stable R-submodule of M (respectively R/I-submodule) and $f:N \rightarrow M$ be any R-homomorphism (respectively R/I-homomorphism), then N is an R/I-submodule (respectively R-module) and f is an R/I-homomorphism (respectively R-homomorphism). Since M is s-quasi-injective R/I-module (respectively R-module), then there exist an extension in both cases, thus M is s-quasi-injective R-module (respectively R/I-module).■

Next we will give a characterization of s-quasi-injectivity. First we need the following definition.

(4) Definition:

Let M be an R-module and N be any stable submodule of M, then N is said to be complement stable (com-stable) if N have a stable complement in M. M is called fully complement stable (fully com-stable) in case each stable submodule of M is com-stable.

(5) Examples and Remarks:

(a) Each fully stable R-module is fully com-stable.

(b) An R-module M is called cl-fully stable if each closed submodule of M is stable.[2]

(c) The converse of (a) is not true in general since any cl-fully stable is a fully com-stable but it is not fully stable [2].

(d) the complement of a submodule is not unique in general, but it is unique for the stable submodules.[2]

A submodule N of an R-module M is called essential in case N have non-zero intersection with each non-zero submodule of M.[3]

The following proposition give as a relationship between the essential submodules and complement submodules.

(6) Proposition:

Let N be a submodule of an R-module M, if K is the complement of N in M then $N \oplus K$ is essential in M.■ [4]

Now we are ready to state our result which is a characterization of s-quasi-injectivity.

(7) Theorem:

Let M be a fully com-stable R-module. Then M is s-quasi-injective if and only if for each essential stable submodule N of M, each R-homomorphism of N into M can be extended to an endomorphism of M.

Proof:

Let N be any stable submodule of M and $f:N \rightarrow M$ be any R-homomorphism of N into M. Assume that K is the complement of N in M, then K is stable (since M is fully com-stable) and $N \oplus K$ is stable submodule of M [2] and essential in M (by above proposition), moreover f can be extended to an R-endomorphism g of $N \oplus K$ by putting g(K)=(0). Therefore by hypothesis there is an R-endomorphism h of M which is an extension of g and hence of h. The converse is trivial.■

Johnson and Wong show that an R-module M is quasi-injective if and only if M is invariant under every endomorphism of its injective envelope. Next we give a characterization of s-quasi-injectivity in term of special kind of endomorphism of the injective envelope.

First we need to define the following

(8) Definition:

Let M be an R-module and E(M) be its injective envelope, an R-endomorphism $\alpha \in End_R(E(M))$ is said to be stable-essential endomorphism simply (s-ess-endomorphism) if there exist a stable essential submodule N of M such that N is invariant under α.

$FixedT=\{\alpha \in End_R(E(M)) \mid \alpha(N) \subseteq N\}$

where $End_R(E(M))$ is the endomorphism ring of the injective envelope of M.

Note that if $S= End_R(E(M))$, then the Jacobson radical of S is :

$J(S) = \{\alpha \in S \mid ker(\alpha)$ is essential submodule of E(M) } [4].

To show that $J(S) \subseteq T$, let $\alpha \in J(S)$ then $ker(\alpha) \cap M$ is an essential submodule of M. Furthermore, $\alpha(ker(\alpha) \cap M) = 0 \subseteq ker(\alpha) \cap M$, which implies that $\alpha \in T$.

Now, we can give our characterization of s-quasi injectivity.

(9) Theorem:

Let M be a fully com-stable R-module, E(M) be its injective envelope and $S= End_R(E(M))$. Then M is s-quasi injective and $J(S)M \subseteq M$ if and only if M is invariant under T.

Proof:

Suppose that M is invariant under T (i.e TM$\subseteq$ M), since J(S) $\subseteq$T as we show above, we have J(S)M$\subseteq$M. By theorem (7) it is sufficient to prove this direction on the essential submodules. Let N be any stable essential submodule of M and f any R-homomorphism, f:N$\rightarrow$M. Injectivity of E(M) implies that there exist an R-homomorphism h:E(M)$\rightarrow$E(M) such that h(N) = f(N)$\subseteq$N, hence h$\in$T, so h(M)$\subseteq$M, thus $h|_M$:M$\rightarrow$M is an extension of f and hence M is s-quasi-injective.

Conversely, let f$\in$T then there exist a stable essential submodule N of M such that f(N)$\subseteq$N. By s-quasi-injectivity of M, there exist an R-homomorphism g:M$\rightarrow$M such that g extends f. Now the injectivity of E(M) implies that there is h$\in$S such that $h|_M$ = h(M) = g(M)$\subseteq$M. Hence (f-h)(N) =(0) which implies N$\subseteq$ker(f-h) so ker(f-h) is essential submodule of E(M), hence f-h$\in$J(S). By hypothesis (f-h)M$\subseteq$M, therefore for each x$\in$M we have (f-h)(x)=m for some m$\in$M and hence f(x)=m-h(x)$\in$M, therefore f(M)$\subseteq$M.■

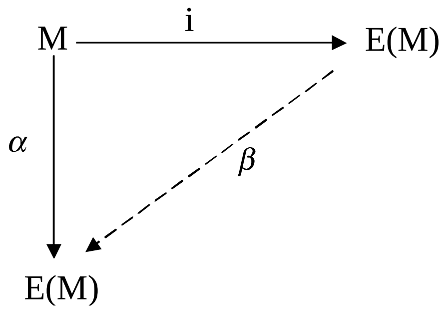
As we show before that each quasi-injective R-module (and hence injective) is s-quasi-injective but the converse is not true in general. It is natural to ask when s-quasi-injectivity consequence with injectivity or quasi-injectivity. The following theorem gives us a relationship between injectivity and s-quasi-injectivity.

(10) Theorem:

Let M be an s-quasi-injective R-module, E(M) be its injective envelope and S= End_R(E(M)) with J(M)$\subseteq$M. If there is an s-ess-epimorphism α from M onto E(M), then M is injective.

Proof:

Consider the following diagram



where i is the inclusion mapping of M into E(M). By injectivity of E(M), there is an R-homomorphism $\beta\in$S such that β extends α. But α is s-ess-epimorphism, then there is a stable essential submodule N of M. Therefore β(N) = $\beta\circ$i(N) = α(N)$\subseteq$N (since N is stable), hence $\beta\in$T and by theorem(6) we have β(M)$\subseteq$M and hence E(M)= α(M)= $\beta\circ$i(M)= β(M)$\subseteq$M which is implies that M=E(M) and therefore M is injective.■

The following corollary is immediate consequence from theorem (10)

(11) Corollary:

Let M be an R-module as in the theorem (10), then the following are equivalent:

(a) M is s-quasi-injective.
(b) M is injective.
(c) M is quasi-injective.■

The following proposition shows us conditions under which the class of quasi-injective R-modules and the class of s-quasi-injective R-modules will be equivalent.

(12) Proposition:

Let M be an s-quasi-injective R-module and J(S)M$\subseteq$M. If M contains an essential quasi-injective submodule, then M is quasi-injective.

Proof:

Let N be an essential quasi-injective submodule of M. Then N is essential submodule of E(M) and hence E(N) =E(M) [5],then End_R(E(N))= S. But N is quasi-injective R-module, then SN$\subseteq$N [6] which is means α(N)$\subseteq$N $\forall\alpha\in$S which implies that $\alpha\in$T, thus T= S. But M is s-quasi-injective and J(S)M$\subseteq$M, then by theorem(9) we have TM$\subseteq$M and hence SM$\subseteq$M, therefore M is quasi-injective [6].■

The following proposition gives us another condition under which the property of s-quasi-injectivity and the property of quasi-injectivity are equivalent.

(13) Proposition:

Let M be a fully stable R-module. Then M is quasi-injective if and only if M is s-quasi-injective.

Proof:

Suppose that M is s-quasi-injective and let N be a submodule of M and f:N$\to$ M be an R-homomorphism, then N is stable (since M is fully stable) and hence by s-quasi-injectivity of M, there is an R-endomorphism g of M such that g extends f. Therefore M is quasi-injective. The other direction is trivial.■

The following theorem was proved in [2] which gives some equivalent conditions for the fully stability.

(14) Theorem:

The following statements are equivalent for an R-module M:

(a) M is fully stable module.

(b)Every cyclic submodule of M is stable.

(c) For each x,y in M, y$\notin$(x) implies $ann_R(x)\not\subset ann_R(y)$.

(d) For each x in M, $ann_M(ann_R(x))$ = (x).■

So by using proposition (13) and the above theorem we get the following corollaries.

(15) Corollary:

Let M be an s-quasi-injective R-module such that every cyclic submodule of M is stable, then M is quasi-injective.■

(16) Corollary:

Let M be an s-quasi-injective R-module such that for each x,y in M, y$\notin$(x) implies $ann_R(x)\not\subset ann_R(y)$, then M is quasi-injective.■

(17) Corollary:

Let M be an s-quasi-injective R-module such that for each x in M, $ann_M(ann_R(x))$ = (x) then M is quasi-injective.■

The following theorem is useful to show that s-quasi-injective R-modules inherit the property of s-quasi-injectivity to some kinds of its submodules.

(18) Theorem:

Let M be an s-quasi-injective R-module and let N be a closed submodule of M. Then any mapping α of a stable submodule K of M into N can be extended to a mapping β of M into N.

Proof:

By Zorn's lemma we can assume that K is such that α cannot be extended to a mapping of T into N for any submodule T of M which properly contains K. since M is s-quasi-injective, then α induced by a map. β: M$\to$M. suppose β(M)$\not\subset$N and let L be the complement of N in M, and since N is closed, then N is the complement of L in. Since β(M)+N$\supseteq$N, we see that (β(M)+N)) $\cap$ L$\neq$(0). Let $0\neq$x=a+b$\in$ (β(M)+N)) $\cap$ L where a$\in\beta$(M) and b$\in$N. If a$\in$N, then x$\in$N $\cap$ L=0, a contradiction. Therefore a$\notin$N, and a= x-b$\in$L$\oplus$N. Now T={y$\in$M | β(y)$\in$ L$\oplus$N } is a submodule of M containing K. If y$\in$M such that β(y)=a then y$\in$T, but y$\notin$K (since a$\notin$N). let π denote the projection of L$\oplus$N on N. Then $\pi\beta$ is a map. of T in N, and $\pi\beta$(y)= β(y)= α(y) $\forall$ y$\in$K. Thus $\pi\beta$ is proper extension of α, a contradiction. Therefore β(M)$\subseteq$N and β is the desired extension of α .■

(19) Corollary:

A stable closed submodule N of an s-quasi-injective R-module M is a direct summand of M.

Proof:

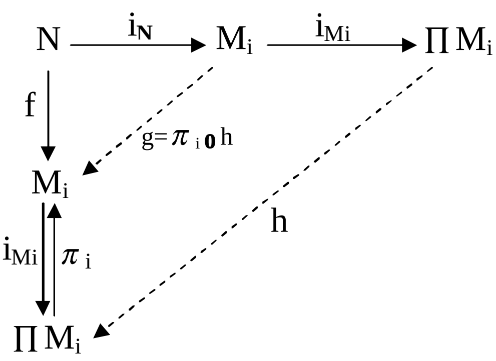
Consider the injection mapping i:N$\to$N, then by theorem(18) i can be extended to some R-homomorphism g:N$\to$M which implies that M=N$\oplus$ker(g) and hence N is a direct summand of M.■

(20) Proposition:

If the direct product $\prod M_i$ of R-modules {Mi | i$\in$I} is s-quasi-injective, then M_i is s-quasi-injective for each i$\in$I.

Proof:

Suppose that $\prod M_i$ is s-quasi-injective R-module, to prove M_i is s-quasi-injective, let N be a stable submodule of M_i and f:N$\to M_i$ be any R-homomorphism. Since $\prod M_i$ is s-quasi-injective, then f induced by an R-homomorphism h:$\prod M_i\to\prod M_i$. Put g=π_{i_0}h, where π is the natural projection of $\prod M_i$ on M_i .(see the following Figure)



Then g is the desired extension of f. Hence M_i is s-quasi-injective.■

The following corollary is immediate consequence from proposition (20).

<u>(21) Corollary:</u>

A direct summand of s-quasi-injective is s-quasi-injective.■

The following corollary follows from corollary (19) and corollary (21).

(22) Corollary:

A stable closed submodule of an s-quasi-injective R-module is s-quasi-injective.■

References

[1] S.A. Al-Saadi : *On s-extended modules and*, Ph.D. thesis, Al-Mustansiriya Univ., (2007).
[2] M.S. Abbas: *On fully stable* modules, Ph.D. thesis, Univ. of Baghdad,(1990).
[3] R. Wisbaur: *Foundations of module and ring theory,* Gordon and Breach Science Publishers, Reading (1990).
[4] C. Faith: *Lectures on injective modules and quotient rings,* Springer-Verlage, Berlin, Heidelberg, New York (1976).
[5] D.W. Sharp and P. Va'mos: *Injective modules,* Cambridge Univ. press, London (1972).
[6] R.E. Wong and E.T. Wong: *Quasi-injective modules and irreducible rings,* J. London Math. Soc. (1961) 39, 260-268.

الخلاصة

لتكن R حلقة ذات عنصر محايد غير صفري ولـيكن M مقاسا أيمنا أحاديا معرفا على R. في هذا البحث درسنا مفهوم المقاسات شبه الأغمارية من النمط s. تناولنا بعـض خصائص هذا المفهوم وأعطينا بعض الشروط التي تجعـل كل مقاس شبه أغماري من النمط s مقاسا شبه أغمـاري. كذلك أعطينا بعض الشروط التـي تجعـل هـذه الـصفة تورث إلى المقاسات الجزئية من مقاس شبه أغمـاري مـن النمط S.