ON QUASI-MAXIMAL MODULES

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Abstract

Let R be a commutative ring with identity, and let M be a unitary R-module. In this paper we introduce the concept of quasi-maximal module, some properties and characterizations of quasi-maximal modules are given. Also, various basic results about quasi-maximal modules and regular modules are considered. Moreover, some relations between quasi-maximal modules and other types of modules are considered.

1. Introduction

Every ring considered in this paper will be assumed to be commutative with identity and every module is unitary. We introduce the following: An R-module M is called a quasi-maximal module if and only if $\sqrt{ann_R M}$ is a semimaximal ideal of R, where $ann_R M=\{r: r\in R \text{ and } rM=0 \text{ for all } m\in M\}$, [1].

Our concern in this paper is to study quasi-maximal modules and look for any relation between quasi-maximal modules and certain types of well-known modules specially with semiprime modules.

This paper consists of three sections. Our main concern in section one, is to define and study quasi-maximal modules, and we give some characterizations for this concept. In section two, we study the relation between quasi-maximal and regular modules. In section three, we study the relationships between quasi-maximal modules and onother types of modules specially with the semiprime modules.

2.Basic Properties of Quasi-maximal modules

In this section, we introduce the concept of a quasi-maximal module and give some characterizations and establish some basic properties of this concept.

First, we introduce the following definition.

2.1 Definition:

A non-zero R-module M is called quasi-maximal module if and only if $\sqrt{ann_R M}$ is semimaximal ideal of R.

An ideal I of a ring R is called semimaximal if I is an intersection of finitely many maximal ideals of R, see [1].

2.2 Remarks and Examples:

1. Every maximal ideal is semimaximal ideal, but the converse is not true in general. For example: 6Z is a semimaximal ideal of a ring Z which is not maximal, see [2].
2. Let $M=\underset{p}{\oplus} Z_p$ as a Z-module be a quasi-maximal module, where p is prime number. Since $\sqrt{ann_Z(\underset{p}{\oplus} Z_p)} = \sqrt{\underset{p}{\cap} ann_Z(Z_p)} = \sqrt{\underset{p}{\cap}(pZ)} = \sqrt{pZ} = pZ$ is a semimaximal ideal of Z.
3. For each positive integer n, the Z-module $Z\oplus Z_n$ is not quasi-maximal module. Since $\sqrt{ann_Z(Z\oplus Z_n)} = (0)$ is not semimaximal ideal of Z.
4. Z as a Z-module is not quasi-maximal module.
5. Every submodule of quasi-maximal R-module is quasi-maximal R-module.

Proof:

Let N be a non-zero proper submodule of M, to show that $\sqrt{ann_R N}$ is semimaximal ideal of R, since $N\subseteq M$, which implies that $ann_R M\subseteq ann_R N$ and hence $\sqrt{ann_R M} \subseteq \sqrt{ann_R N}$. But $\sqrt{ann_R M}$ is semimaximal ideal of R because M is quasi-maximal R-module. Therefore, $\sqrt{ann_R N}$ is semimaximal ideal of R by [2,proposition (1.2.11),p.20]. Hence N is quasi-maximal R-module.

6. Q as a Z-module is not quasi-maximal module.

   Now, we state and prove the following result.

2.3 Proposition:

Z_m as a Z-module is a quasi-maximal module if and only if $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_n^{\alpha_n}$, where p_i is a distinct prime number and $\alpha_i \geq 1$, i=1,2,…,n.

Proof:

Suppose that Z_m is a quasi-maximal Z-module, to show that $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_n^{\alpha_n}$, where p_i is a distinct prime number and $\alpha_i \geq 1$, i=1,2,…,n. Thus $\sqrt{\text{ann}_Z Z_m}$ is seemimaximal ideal of Z, since $\sqrt{\text{ann}_Z Z_m} = \sqrt{mZ} = (p_1 \cdot p_2 \cdot \ldots \cdot p_n)$. Therefore $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdot \ldots \cdot p_n^{\alpha_n}$, where p_i is a distinct prime number and $\alpha_i \geq 1$, i=1,2,…,n.

Conversely, if $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_n^{\alpha_n}$, where p_i is a distinct prime number and $\alpha_i \geq 1$, i=1,2,…,n, to show Z_m is a quasi-maximal Z-module, $\sqrt{\text{ann}_Z Z_m} = \sqrt{mZ} = \sqrt{p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_n^{\alpha_n}} =$ $(p_1 \cdot p_2 \cdot \ldots \cdot p_n) = \bigcap_{i=1}^{n} p_i$.

Hence Z_m is a quasi-maximal Z-module.

Aspecial case of proposition (2.3), we give the following corollary.

2.4 Corollary:

Z_p as a Z-module is quasi-maximal module, where p is prime number.

The following theorem gives some characterizations for quasi-maximal modules.

2.5 Theorem:

Let M be an R-module. Then (1) $\Rightarrow$(2), (2) $\Rightarrow$(3), (3) $\Rightarrow$(4), (4) $\Rightarrow$(1) if M is finitely generated :

1. M is a quasi-maximal R-module.
2. $[\sqrt{\text{ann}_R M} : A]$ is a semimaximal ideal of R for every ideal A of R such that $A \not\subseteq \sqrt{\text{ann}_R M}$.
3. $[\sqrt{\text{ann}_R M} : r]$ is a semimaximal ideal of R for every element $r \in R$ such that $r \notin \sqrt{\text{ann}_R M}$.
4. $\sqrt{\text{ann}_R (m)}$ is a semimaximal ideal of R for every non-zero element $m \in M$.

Proof:

(1) $\Rightarrow$ (2) Suppose that M is quasi-maximal. Then $\sqrt{\text{ann}_R M}$ is semimaximal ideal of R. Assume that A is an ideal of R such that $A \not\subseteq \sqrt{\text{ann}_R M}$. Since $\sqrt{\text{ann}_R M} \subseteq [\sqrt{\text{ann}_R M} : A]$. Thus by [2,proposition (1.2.11),p.20], we get $[\sqrt{\text{ann}_R M} : A]$ is a semimaximal ideal of R.

(2) $\Rightarrow$ (3) By taking A=R and from (2), we get the result.

(3) $\Rightarrow$ (4) let $0 \neq m \in M$.

Because $1 \notin \sqrt{\text{ann}_R (m)}$, $[\sqrt{\text{ann}_R (m)} : R]$ is semimaximal by (3).

But $[\sqrt{\text{ann}_R (m)} : R] = \sqrt{\text{ann}_R (m)}$, so $\sqrt{\text{ann}_R (m)}$ is semimaximal ideal of R.

(4) $\Rightarrow$ (1) since M is finitely generated, $M = \sum_{i=1}^{n} Rxi$, $xi \in M$. Thus $\sqrt{\text{ann}_R M} = \bigcap_{x \in M} \sqrt{\text{ann}_R (x)}$, by (4), $\sqrt{\text{ann}_R (x)}$ is semimaximal ideal of R. Thus $\bigcap_{x \in M} \sqrt{\text{ann}_R (x)}$ is semimaximal ideal of R by [2,corollary (1.2.15),p.21]. Therefore $\sqrt{\text{ann}_R M}$ is semimaximal ideal of R. Hence M is quasi-maximal R-module.

The following proposition shows a direct sum of quasi-maximal R-modules is a quasi-maximal R-module.

2.6 Proposition:

Let M_1 and M_2 be two R-modules. Then $M_1 \oplus M_2$ is a quasi-maximal R-module if and only if M_1 and M_2 are quasi-maximal R-modules.

Proof:

Suppose that M= $M_1 \oplus M_2$ is a quasi-maximal R-modules by remarks and examples (2.2), (5)), M_1 and M_2 are quasi-maximal R-module.

Conversely, Assume that M_1 and M_2 are quasi-maximal R-modules, let $0 \neq m \in M$, $m=(m_1,m_2)$ and $\text{ann}_R(m) = \text{ann}_R(m_1) \cap \text{ann}_R(m_2)$. Thus $\sqrt{\text{ann}_R (m)} =$

$\sqrt{ann_R(m_1) \cap ann_R(m_2)} = \sqrt{ann_R(m_1)} \cap \sqrt{ann_R(m_2)}$. Since M_1 and M_2 are semimaximal ideals of R.

Thus $\sqrt{ann_R(m_1)} \cap \sqrt{ann_R(m_2)}$ is a semimaximal ideal of R by [2, proposition (1.2.14), p.21]. Then $\sqrt{ann_R(m)}$ is semimaximal ideal of R and hence $M= M_1 \oplus M_2$ is a quasi-maximal R-modules.

So, we have the following application of (2.6).

2.7 Corollary:

$\underset{\alpha \in \wedge}{\oplus} M_\alpha$ is a quasi-maximal R-module if and only if M_α is a quasi-maxiaml R-modules for all α.

3. Quasi-maximal and Regular Modules

In this section, we study the relationships between quasi-maximal modules, regular rings and regular modules.

Recall that a ring R is called regular (Von-Numann) if for each element $a \in R$, there exists an element $r \in R$ such that a=ara ($a=a^2r$ if R is commutative), see [3].

We start with the following proposition.

3.1 Proposition:

If M is a quasi-maximal R-module, then $\frac{R}{\sqrt{ann_R M}}$ is regular ring.

Proof:

Since M is quasi-maximal R-module, then $\sqrt{ann_R M}$ is semimaximal ideal of R. Thus by [2,proposition (1.3.1),p.26], we get $\frac{R}{\sqrt{ann_R M}}$ is regular ring.

The following corollary is an immediate consequence of proposition (3.1).

3.2 Corollary:

If $0 \neq x$ is an element of an R-module M such that $\sqrt{ann_R(x)}$ is semimaximal ideal of R, then $\frac{R}{\sqrt{ann_R(x)}}$ is regular ring.

Proof:

It is obvious according to theorem (2.5) and proposition (3.1).

Before we introduce other results we need the following definitions are needed:

A submodule N of an R-module M is said to be pure if $IM \cap N = IN$ for every ideal I of R, see [3]. And an R-module M is called regular module if every submodule of M is pure.

An ideal I of a ring R is called a semiprime ideal if for all $a \in R$, $a^2 \in I$, then $a \in I$, see [4].

A ring R is called a local ring if R contains only one maximal ideal, see [1].

An R-module M is called semisimple if every submodule of M is a direct summand of M. A ring R is said to be semisimple ring if and only if R is a semisimple R-module, see [4].

The following proposition shows that ($ann_R M$ is semiprime ideal of R) is a sufficient condition for quasi-maximal module to be regular module.

3.3 Proposition:

Let M be a quasi-maximal R-module and $ann_R M$ is semiprime ideal of R. Then M is regular R-module.

Proof:

Let M be a quasi-maximal R-module, then $\sqrt{ann_R M}$ is semimaximal idael of R, but $ann_R M$ is semiprime ideal of R, then $\sqrt{ann_R M} = ann_R M$. Thus $ann_R M$ is semimaximal ideal of R. Therefore $\frac{R}{ann_R M}$ is semisimple ring by [2,proposition (1.2.5),p.17], which implies that $\frac{R}{ann_R M}$ is regular ring. Let f: $\frac{R}{ann_R M} \longrightarrow \frac{R}{ann_R(x)}$ be a function that defined by $f(r+ ann_R M)=r+ ann_R(x)$ for all $r \in R$. It can be easily shown that f is well-defined and f is an epimophisim. Thus $\frac{R}{ann_R(x)}$ is a regular ring by [2,proposition (1.1.28)]. Therefore M is regular R-module by [2,definition (1.1.30)].

As an application of proposition (3.3) we give the following corollaries.

3.4 Corollary:

Let M be a quasi-maximal R-module and $ann_R M$ is semiprime ideal of R. Then J(M)=0.

Proof:

From proposition (3.3) and [2,proposition (1.1.65),p.13].

3.5 Corollary:

Let M be an R-module over a local ring R and $ann_R M$ is a semiprime ideal of R. Then M is semisimple R-module.

Proof:

It follows directly by proposition (3.3) and [2,proposition (1.1.29),p.6].

Recall that an R-module M is said to be flat if for each injective homomorphism $f:N' \longrightarrow N$ from R-module N into another R-module N_1, the homomorphisim $1_M \otimes f: M \underset{R}{\otimes} N' \longrightarrow M \underset{R}{\otimes} N$ is injective, where 1_M is the identity isomorphisim of M, see [1].

Now, we end this section by the following proposition.

3.6 Proposition:

Let M be a quasi-maximal R-module. Then M is flat $\dfrac{R}{\sqrt{ann_R M}}$-module.

Proof:

Assume that M is quasi-maximal R-module. Then $\sqrt{ann_R M}$ is semimaxiimal ideal of R. Thus $\dfrac{R}{\sqrt{ann_R M}}$ is semisimple ring by [2,proposition (1.2.5),p.17]. Hence M is flat $\dfrac{R}{\sqrt{ann_R M}}$-module by [2,proposition (1.1.26),p.6].

4.Some Relations Between Quasi-maximal modules and Other Modules

In this section, we study the relationships between quasi-maximal modules and maximal, semisimple and semiprime modules.

We begin with following proposition.

4.1 Proposition:

Every module M over a Boolean ring is quasi-maximal module.

Proof:

Since R is a Boolean ring, then every ideal of R is semimaximal by [2,corollary (1.2.7),p.18]. In particular $\sqrt{ann_R M}$ is semimaximal ideal of R and hence M is quasi-maximal R-module.

4.2 Proposition:

If R is a semisimple ring. Then every module M over R is quasi-maximal module.

Proof:

Assume that R is a semisimple ring. Then every proper ideal of R is semimaximal by [2,corollary (1.2.6),p.18]. In particular $\sqrt{ann_R M}$ is semimaximal ideal of R.

Now, we have the following proposition.

4.3 Proposition:

Let M be a finitely generated semisimple R-module and $ann_R(x)$ is semiprime ideal of R for all $0 \neq x \in M$. Then M is quasi-maximal R-module.

Proof:

We have M is semisimple R-module, then $ann_R(x)$ is semimaximal ideal of R for all $0 \neq x \in M$ by [2,proposition (1.2.26),p.25]. But $ann_R(x) = \sqrt{ann_R(x)}$ because $ann_R(x)$ is semiprime ideal of R. Thus $\sqrt{ann_R(x)}$ is semimaximal ideal of R for all $0 \neq x \in M$ and hence by theorem ((2.5),4), M is quasi-maximal R-module.

An R-module M is called a Max-module if $\sqrt{ann_R N}$ is a maximal ideal of R, for each non-zero submodule N of M, see [5].

By using this concept, we have the following.

4.4 Proposition:

Every Max-module is quasi-maximal module.

Proof:

Let M be a max-module, then $\sqrt{ann_R M}$ is maximal ideal of R by [5,remarks and examples (2.2),(6),p.5].

Note that, the converse of proposition (4.4) is not true in general. For example, the Z-module Z_{12} is a quasi-maximal by remark (2.3), but it is not a Max-module since $N = <\bar{2}>$ is a submodule of Z_{12} and

$\sqrt{ann_Z <\bar{2}>} = \sqrt{6Z} = 6Z$ is not maximal ideal of Z.

Next, we study quasi-maximal modules and semiprime modules. But first the following definitions are needed.

An R-module M is called semiprime if and only if $ann_R(N)$ is a semiprime ideal of R for each non-zero R-submodules N of M, see [6].

A submodule N of an R-module M is called a primary submodule if N≠M and whenever rx∈N for r∈R and x∈M we have either x∈N or $r^n \in [N \underset{R}{:} M]$ for some $n \in Z_+$, see [7]. And an R-module M is said to be a primary if (0) is a primary R-submodule of M.

A submodule N of an R-module M is called essential in M for each non-zero R-submodule L of M, N∩L≠0, see [4]. And R-module M is called uniform if every non-zero R-submodule of M is essential.

4.5 Remark:

Let M be a quasi-maximal R-module. Then it is not necessary to be semiprime R-module. For example , the Z-module Z_{30} is quasi-maximal Z-module by proposition (2.3) but it is not semiprime Z-module, since $<\bar{5}>$ is a submodule of Z_{30} and $\sqrt{ann_Z <\bar{5}>} = \sqrt{6Z} = 6Z$ is not semiprime ideal of Z.

In the following proposition, we give necessary conditions under which remark (4.5) holds.

4.6 Proposition:

Let M be a quasi-maximal primary R-module such that $\sqrt{ann_R N} = ann_R N$ for each non-zero submodule N of M. Then M is semiprime R-module.

Proof:

Assume that M is primary R-module, then $\sqrt{ann_R M} = \sqrt{ann_R N}$ for each non-zero submodule N of M by [8,theorem (2.1.3),p.22]. But $\sqrt{ann_R M}$ is semimaximal ideal of R because M is quasi-maximal R-module. Thus $\sqrt{ann_R N}$ is semimaximal ideal of R, which implies that $ann_R(N)$ is semimaxiaml ideal of R for each non-zero submodule N of M (since $\sqrt{ann_R N} = ann_R N$ for each non-zero submodule N of M). Then by [6, definition (4.1.1), p.62], M is semiprime R-module.

The condition ($\sqrt{ann_R N} = ann_R N$ for each non-zero submodule N of M) can not be dropped from proposition (4.6) as the following example shows:

4.7 Example:

The Z-module Z_{16} is primary [8,corollary (2.1.8),p.24]. Also it is quasi-maximal, since $\sqrt{ann_Z Z_{16}} = \sqrt{16Z} = 2Z$ is semimaximal ideal of R, but $\sqrt{ann_Z N} \neq ann_Z N$ for each non-zero submodule N of Z_{16} since $<\bar{4}>$ is a submodule of Z_{16}, $\sqrt{ann_Z <\bar{4}>} = 2Z \neq ann_Z <\bar{4}> = 4Z$. Thus Z_{16} is not semiprime Z-module.

The converse of proposition (4.5) is not true in general, as the following examples shows.

4.8 Example:

Z as a Z-module is primary and semiprime module. Also $\sqrt{ann_Z N} = ann_Z N$ for each non-zero submdoule N of M. But Z is not quasi-maximal Z-module.

In the following proposition, we introduce a sufficient condition under which the converse of proposition (4.5) is true.

4.9 Proposition:

Let M be a uniform and semiprime module over a PID and $\sqrt{ann_R N} = ann_R N$ for each non-zero submodule N of M. Then M is quasi-maximal R-module.

Proof:

Since M is uniform and semiprime R-module. Then by [6,proposition (4.2.3),p.73], M is prime. This is means for each non-zero submodule N of M, $ann_R N = ann_R M$, [9]. Thus $\sqrt{ann_R M} = \sqrt{ann_R N} = ann_R N$ which implies that $\sqrt{ann_R M} = ann_R N = ann_R M$ (since M is prime). Therefor $\sqrt{ann_R M} = ann_R M$. Also we get M is quasi-prime module by [9,remark

(1.2.2),p.10]. Hence $ann_R M$ is prime ideal of R by [9,corollary (1.2.7),p.14]. So, $ann_R M$ is semimaximal ideal of R (since R is PID). Then $ann_R M$ is semimaximal ideal of R, which implies that $\sqrt{ann_R M}$ is semimaximal ideal of R and hence M is quasi-maximal R-module.

The following results are another consequences of proposition (4.9), but first we need to recall some definitions.

An R-module M is called F-regular if every submodule of M is pure, see [3].

An R-module M is called divisible if rM=M for all $0\neq r\in R$, see [1].

An R-module M is called quasi-Dedekind if every non-zero R-submodule of M is quasi-invertible, where a submodule N of M is called quasi-invertible if $Hom(\frac{M}{N},M)=0$, see [1].

Now, we can easily obtain the following.

4.10 Corollary:

Let R be a PID. If M is an F-regular divisible and uniform R-module such that $\sqrt{ann_R N}=ann_R N$ for each non-zero submodule N of M, then M is quasi-maximal R-module.

Proof:

It follows directly by [6, proposition (4.2.7), p.74] and proposition (4.9).

According to the fact, that every uniform and quasi-Dedekind R-module M is semiprime [6,proposition (4.2.4),p.72] the following is an immediately consequence of proposition (4.9).

4.11 Corollary:

Let R be a PID. If M is a uniform quasi-Dedekind R-module such that $\sqrt{ann_R N}=ann_R N$ for each non-zero submodule N of M, then M is quasi-maximal R-module.

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الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايد، وليكن M مقاساً احاديا أيسراً على الحلقة R في هذا البحث قدمنا مفهوم مقاس شبه اعظمي، بعض الخواص والتميزات قد اعطيت وكذلك دُرست العديد من النتائج الاساسية حول المقاسات شبه أعظمي والمقاسات المنتظمة، بالاضافة الى هذا دُرست بعض العلاقات بينه وبين انواع اخرى من المقاسات.