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THE DISCRETE CLASSICAL OPTIMAL CONTROL PROBLEM OF A NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION (DCOCP)

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Abstract

In this paper, we consider a continuous classical optimal control for systems of nonlinear hyperbolic partial differential equations, with several equality and inequality state constraints. First, the considered continuous classical optimal control problem is discretized into a discrete classical optimal control problem by using the Galerkin finite element method in space and the implicit finite difference scheme in time. The classical continuous controls are approximated by picewise constants. Second the existence of a unique solution of the discrete state equations for fixed discrete classical control is studied. Third, we develop the existence theory for optimality of the discrete classical problem, and the discrete adjoint equations are developed corresponding to the discrete state equations. Finally the necessary conditions and a picewise minimum principle are developed for optimality of the discrete classical problem.

Introduction

During the last dictates, many researchers ([3], [5], [7], and many others), interested to study the discretization for the continuous relaxed optimal control problems for systems defined by ordinary and partial differential equations. At the beginning of this century the discretization for the continuous classical optimal control problem defined by semilinear parabolic partial differential equations and then the study of the obtained discrete classical optimal control problem was studied by [4].

Since many applications in physics as the problem of Electromagnetic waves, or the problem of Dynamical elasticity lead to a mathematical model represent by a classical optimal control problems governed by nonlinear hyperbolic partial differential equation, and since solving such problems numerically needs the discritization of the continuous optimal control problems to a discrete classical optimal control problems, so we interest in this paper to study the discretization of a classical optimal control problem for systems defined by nonlinear hyperbolic partial differential equations with several equality and inequality state constraints.

In this paper and in order to give a complete idea about our work, we saw it is important to give at the beginning a description for the continuous classical optimal control problem (CCOCP) which is studied in[1], then we discretize this continuous classical optimal control problem to a discrete classical optimal control problem (DCOCP). First we discretize the weak form of state equations in the continuous problem by using the Galerkin finite element methods in space and the implicit finite difference scheme in time (usually the Galerkin method with the finite difference scheme is used together to discretize such type of problems, cause there are suitable and are used successfully [1],[3], & [4]), while the continuous controls are approximated by picewise constants with respect to an independent partition of the space-time domain. Then the existence of a unique solution of the discrete state equations for fixed discrete classical control is proved. Also we prove the existence theory of optimal control for the discrete classical problem, and we derive the discrete adjoint-state equations corresponding to the discrete state equations. Finally the necessary conditions and a picewise minimum principle for optimality of the discrete classical optimal control problem are derived.

1.Description Of The Continuous Classical Optimal Control Problem:-

In this section we describe the continuous classical optimal control problem of a nonlinear hyperbolic partial differential equations which is studied by [1], in order to give a complete idea about how will descritize the indicated continuous classical optimal control problem (CCOCP) to a discrete classical optimal control problem (DCOCP) which is our aim in this work. So we begin with $\Omega \subset \mathbb{R}^d$ be an open and bounded region with Lipschitz boundary $\Gamma = \partial\Omega$, and let $I = (0,T)$, $0 < T < \infty$, $Q = \Omega \times I$. The nonlinear hyperbolic state equations are:-

$$y_{tt} + A(t)y = f(x,t,y,u), \text{in } Q \quad \text{..........(1)}$$

$$y(x,t) = 0, \text{in } \Sigma, \quad \text{..................................(2)}$$

$$y(x,0) = y^0(x), \text{in } \Omega, \Sigma = \Gamma \times [0,T] \quad \text{......(3)}$$

$$y_t(x,0) = y^1(x), \text{ in } \Omega \quad \text{.........................(4)}$$

where $u = u(x,t)$, $y = y_u(x,t)$ is the *state* which corresponds to the continuous *classical control* u, $A(t)$ is the 2nd order elliptic differential operator, i.e.

$$A(t)y = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[a_{ij}(x,t)\frac{\partial y}{\partial x_j}\right]$$

The set of ***continuous classical controls*** is $u \in W$, $W \subset L^2(Q)$, where

$$W = \{u \in L^2(Q) \mid u(x,t) \in U, \text{a.e.in } Q\},$$

where U is a compact and convex subset of $\mathbb{R}^v$ (usually $v = 1$ or $v = 2$),

the ***constraints*** on the state and control variables y and u are

$$G_m(u) = \int_Q g_m(x,t,y,u)dxdt = 0, 1 \le m \le p$$

$$G_m(u) = \int_Q g_m(x,t,y,u)dxdt \le 0, \quad p+1 \le m \le q$$

the ***cost function*** is

$$\textbf{Min. } G_0(u) = \int_Q g_0(x,t,y(x,t),u(x,t))dxdt$$

where $y = y_u$ is the solution of (1-4), for the control u, and $g_i(x,t,y,u)$, for $i = 1,2,...,q$ are defined on $Q \times R \times U$.

The continuous classical optimal control problem (CCOCP) is to minimize the cost function subject to $u \in W$ and the constraints equality constraints $G_m(u)$ (where $1 \le m \le p$), and the inequality constraints $G_m(u)$ (where $p+1 \le m \le q$). A control satisfying all the above constraints is called ***admissible*** and the set of admissible control is denoted by W_A.

Also here, we denote by $|.|$ the Euclidean norm in R^n, by $\|.\|_\infty$ the norm in $L^\infty(\Omega)$, by $(.,.)$ and $\|.\|_0$ the inner product and norm in $L^2(\Omega)$, by $(.,.)_1$ and $\|.\|_1$ the inner product and norm in Sobolev space $V = H_0^1(\Omega)$, by $<.,.>$ the duality bracket between V and its dual V^*, and by $\|.\|_Q$ the norm in $L^2(Q)$.

The weak form of the problems (1-4) is given $\forall v \in V$, $y(.,t) \in V$, , a.e. on I, by:

$$< y_{tt}, v > + a(t,y,v) = (f(t,y(t),u(t)),v), \quad \text{................................(5)}$$

$$y(0) = y^0, \text{ in } \Omega, \quad \text{.............................. (6)}$$

$$y_t(0) = y^1, \text{ in } \Omega \quad \text{...............................(7)}$$

where the initial conditions make sense if $y^0 \in V$, $y^1 \in L^2(\Omega)$, and $a(t,.,.)$ is the usual bilinear form associated with $A(t)$, we suppose that $a(t,v,w)$ is symmetric and for some α_1, α_2, $\forall v, w \in V$, and $t \in \bar{I}$, satisfies $|a(t,v,w)| \le \alpha_2 \|v\|_1 \|w\|_1$, and $a(t,v,v) \ge \alpha_1 \|v\|_1^2$,

2. Descritization And Description Of The Discrete Classical Optimal Control Problem:-

In this section we discritize the continuous classical optimal control problem which is considered in the pervious section. We suppose for simplicity the operator $a(t,.,.)$ is independent of t, the domain Ω is a polyhedron. For every integer n, let $\{S_i^n\}_{i=1}^{M(n)}$ be an admissible regular triangulation of $\bar{\Omega}$ into closed d-simplices [8], $\{I_j^n\}_{j=0}^{N(n)-1}$ be a subdivision of the interval $\bar{I}$ into $N(n)$ intervals, where $I_j^n := [t_j^n, t_{j+1}^n]$ of equal lengths $\Delta t = T/N$. Set $Q_{ij} := S_i^n \times I_j^n$. Let $V_n \subset V = H_0^1(\Omega)$ be the space of continuous piecewise affine mapping in Ω. Let W^n be the set of discrete (blockwise constants) classical controls (piecewise constants classical controls), i.e.

$$W^n = \{w = w^n \in W \mid w(x,t) = w_{ij} \text{ in } Q_{ij}\}$$

The discrete state equations, for each $v \in V_n$ is written in the form

$$(z_{j+1}^n - z_j^n, v) + \Delta t\, a(y_{j+1}^n, v) = \Delta t (f(t_j^n, y_{j+1}^n, u_j^n), v),\ j = 0,1,\dots,N-1 \quad \dots(8)$$

$$y_{j+1}^n - y_j^n = \Delta t\, z_{j+1}^n,\ j = 0,1,\dots,N-1,\ \dots.(9)$$

$$(y_0^n, v) = (y^0, v),\ \dots\dots(10)$$

$$(z_0^n, v) = (y^1, v),\ \dots\dots(11)$$

where $y^0 \in V$, $y^1 \in L^2(\Omega)$ are given, and $y_j^n, z_j^n \in V_n$, for $j = 0,1,\dots,N$.

The discrete constraints on the control is $u^n \in W^n$, and the constraints on the discrete state and control are

$\left|G_m^n(u^n)\right| \le \varepsilon_{1m}^n$, for each $m = 1,2,\dots,p$

and

$G_m^n(u^n) \le \varepsilon_{2m}^n$, for each $m = p+1, p+2,\dots,q$

where ε_{1m}^n and ε_{2m}^n are given numbers, tend to zero as n goes to infinity.

The discrete cost is $G_0^n(u^n)$, where the discrete functionals $G_m^n(u^n)$ is defined by:

$$G_m^n(u^n) = \Delta t \sum_{j=0}^{N-1} \int_\Omega g_m(x, t_j^n, y_{j+1}^n, u_j^n)dx,$$

for each $m = 0,1,2,\dots,q$.

The set of all **discrete admissible classical controls** for the discrete optimal problem is given by

$$W_A^n = \{u^n \in W^n : \left|G_m^n(u^n)\right| \le \varepsilon_{1m}^n, (1 \le m \le p), G_m^n(u^n) \le \varepsilon_{2m}^n, (p+1 \le m \le q)\}$$

The discrete classical optimal control problem is to find (if it exists) $u^n \in W_A^n$, such that

$$G_0^n(u^n) = \min_{w^n \in W_A^n} G_0^n(w^n)$$

Now, suppose the function f is defined on $S_i^n \times I_j^n \times \square \times W^n$ ($i = 1,2,\dots,M$) continuous w.r.t. (with respect to) y_j^n, and u_j^n satisfies:-

$$\left|f(x, t_j^n, y_{j+1}^n, u_j^n)\right| \le F_j(x) + \beta \left|y_{j+1}^n\right|,$$

and

$$\left|f(x, t_j^n, y_{j+1}^n, u_j^n) - f(x, t_j^n, y_j^n, u_j^n)\right| \le L\left|y_{j+1}^n - y_j^n\right|$$

where, $j = 0,1,\dots,N-1, x \in \Omega$, $F_j(x) = F(x, t_j^n) \in L^2(\Omega)$ and L represents the Lipschtiz constant for any j.

Form here and up and for brevity we will drop sometimes the arguments t_j^n or /and x of the dependents variable y_j^n, z_j^n, u_j^n and any other terms which contained this independent variable, and the terms y_j^n, z_j^n, u_j^n of the functions which contained their.

The following theorem plays an important rule in the study of the continuity of the discrete functionals $G_m^n(u^n)$ and also in the existence of a discrete optimal control.

Theorem2.1:

For any fixed j $(0 \le j \le N-1)$, and for each control $u^n \in W^n$, the discrete state equations (8-11) for sufficiently small Δt, has a unique solution $y_{u^n}^n = y^n = (y_0^n, y_1^n, \dots, y_N^n)$.

***Proof*:**

For any fixed j $(0 \le j \le N-1)$, let $\{v_i, i = 1,2,\dots,M(n)\}$ be a finite basis of V_n (where $v_i(x)$, for $i = 1,2,\dots,n$ are continuous piecewise affine mapping in Ω, with $v_i(x) = 0$ on the boundary Γ), then equations (8-11) for any $i = 1,2,\dots,M$, and $y_j^n, z_j^n, y_{j+1}^n, z_{j+1}^n \in V_n$, can be written in the form

$$(z_{j+1}^n - z_j^n, v_i) + \Delta t\, a(y_{j+1}^n, v_i) = t(f(t_j^n, y_{j+1}^n, u_j^n), v_i), \quad \dots\dots(12)$$

$$y_{j+1}^n - y_j^n = \Delta t\, z_{j+1}^n,\ \dots\dots(13)$$

$$(y_0^n, v_i) = (y^0, v_i),\ \dots\dots(14)$$

$$(z_0^n, v_i) = (y^1, v_i),\ \dots\dots(15)$$

Rewriting (13), in the form

$$z_{j+1}^n = \frac{y_{j+1}^n - y_j^n}{\Delta t},\ \dots\dots(16)$$

Substituting (16) in (12), we have

$$(y_{j+1}^n, v_i) + (\Delta t)^2 a(y_{j+1}^n, v_i) = (y_j^n, v_i) + \Delta t (z_j^n, v_i) + (\Delta t)^2 (f(t_j^n, y_{j+1}^n, u_j^n), v_i),\ (17)$$

Now, form the basis of V_n, using the Galerkin method [9], we write

$$y_0^n = \sum_{k=1}^{M} c_k^0 v_k \ , \ y_j^n = \sum_{k=1}^{M} c_k^j v_k \ ,$$

$$y_{j+1}^n = \sum_{k=1}^{M} c_k^{j+1} v_k \ , \ z_0^n = \sum_{k=1}^{M} d_k^0 v_k \ ,$$

$$z_j^n = \sum_{k=1}^{M} d_k^j v_k \ , \text{ and } z_{j+1}^n = \sum_{k=1}^{M} d_k^{j+1} v_k$$

where, $c_k^j = c_k(t_j^n)$, and $d_k^j = d_k(t_j^n)$, for each $j = 0,1,...,N$ are unknown constants.

Substituting $y_0^n, y_j^n, y_{j+1}^n, z_0^j, z_j^n$, and z_{j+1}^n in equations (16,17,14,&15) we get (respectively) the following nonlinear system of ordinary differential equations

$$(A + (\Delta t)^2 B)C^{j+1} = AC^j + \Delta t A D^j + (\Delta t)^2 \vec{b}(\vec{V}^T C^{j+1}, u_j^n) \quad (18)$$

$$D^{j+1} = \tfrac{1}{\Delta t}(C^{j+1} - C^j) \ , \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

$$AC^0 = e^0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

$$AD^0 = e^1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

where

$A = (a_{ik})_{M\times M}$, $a_{ik} = (v_k, v_i)$,

$B = (b_{ik})_{M\times M}$, $b_{ik} = a(t, v_k, v_i)$,

$C_{M\times 1}^{j+r} = (c_1^{j+r}, c_2^{j+r}, ..., c_M^{j+r})^T$,

$D_{M\times 1}^{j+r} = (d_1^{j+r}, d_2^{j+r}, ..., d_M^{j+r})^T$, for $r = 0,1$,

$\vec{b} = (b_i)_{M\times 1}$, $b_i = (f(\vec{V}^T C^{j+1}, u_j^n), v_i)$,

$\vec{V} = (v_i)_{M\times 1}, e^0 = (e_j^0)_{M\times 1}$, $e^1 = (e_i^1)_{M\times 1}$,

($e_i^0 = (y^0, v_i)$ and $e_i^1 = (y^1, v_i)$, for each $i, k = 1,2,...,M$).

Now, to solve the above nonlinear system, we use the method which knows by the predictor and corrector method in the numerical analysis, as following. For predictor step we set $C^{j+1} = C^j$ in the components b_i of the vector $\vec{b}$ in the right hand side of (18), hence equations (18-21) become a linear system.

Form the assumptions on the operator $a(.,.)$ we have the matrices A and B are positive definite, then $A + (\Delta t)^2 B$ is positive definite, hence it is regular, then the above system has a unique solution, i.e. solving equations (18,20,21) w.r.t. $j+1$, for fixed j we get $\bar{C}^{j+1}$ which substitutes in (19) to get $\bar{D}^{j+1}$, (this procedure can be repeat it for each $j = 0,1,...,N-1$).

Second, in the corrector step we substitute again $C^{j+1} = \bar{C}^{j+1}$ in the components b_i of the vector $\vec{b}$ in the right hand side (R.H.S.)of (18), hence equations (18-21) become again a linear system, and by the same above way the indicated equations has a unique solution C^{j+1}, which substitutes in (19) to get D^{j+1}, (also this procedure can be repeat it for each $j = 0,1,...,N-1$). The above steps can be expressed by the following iterative method to solve the corrector step, i.e. equations (17& 16) can be expressed respectively

$$(\overset{(l+1)}{y}{}_{j+1}^{n}, v_i) + (\Delta t)^2 a(\overset{(l+1)}{y}{}_{j+1}^{n}, v_i) = (y_j^n, v_i) + \Delta t(z_j^n, v_i) + (\Delta t)^2 (f(t_j^n, \overset{(l+1)}{y}{}_{j+1}^{n}, u_j^n), v_i), \quad (22)$$

and

$$\overset{(l+1)}{z}{}_{j+1}^{n} = \frac{\overset{(l+1)}{y}{}_{j+1}^{n} - y_j^n}{\Delta t} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

From equation (23) we see that $\overset{(l+1)}{z}{}_{j+1}^{n}$ depends on $\overset{(l+1)}{y}{}_{j+1}^{n}$ which is obtained from (22), i.e. the above iterative method is just for $\overset{(l+1)}{y}{}_{j+1}^{n}$, hence (22) also can be expressed as

$$\overset{(l+1)}{y} = h(\overset{(l)}{y})$$

where $l = 0,1,...$ refer to the numbers of the iterations and

$$\overset{(l+1)}{y}{}^{n} = (\overset{(l+1)}{y}{}_0^n, \overset{(l+1)}{y}{}_1^n, ..., \overset{(l+1)}{y}{}_N^n),$$

and

$$\overset{(l+1)}{\bar{y}}{}^{n} = (\overset{(l+1)}{\bar{y}}{}_0^n, \overset{(l+1)}{\bar{y}}{}_1^n, ..., \overset{(l+1)}{\bar{y}}{}_N^n).$$

Now, let $\overset{(l+1)}{y}{}^{n}$ and $\overset{(l+1)}{\bar{y}}{}^{n}$ are the solutions of (22) ,i.e.

$$(\overset{(l+1)}{y}{}^{n}_{j+1}, v_i) + (\Delta t)^2 a(\overset{(l+1)}{y}{}^{n}_{j+1}, v_i)$$
$$= (y^n_j, v_i) + \Delta t (z^n_j, v_i) +$$
$$(\Delta t)^2 (f(t^n_j, \overset{(l)}{y}{}^{n}_{j+1}, u^n_j), v_i), \quad (24a)$$

and

$$(\overset{(l+1)}{\bar{y}}{}^{n}_{j+1}, v_i) + (\Delta t)^2 a(\overset{(l+1)}{\bar{y}}{}^{n}_{j+1}, v_i)$$
$$= (y^n_j, v_i) + \Delta t (z^n_j, v_i) +$$
$$(\Delta t)^2 (f(t^n_j, \overset{(l)}{\bar{y}}{}^{n}_{j+1}, u^n_j), v_i), \quad (24b)$$

Subtracting (24a) from (24b), substituting $v_i = \overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1}$, in the obtained equation, we get

$$(\overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1}, \overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1}) + (\Delta t)^2$$
$$a(\overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1}, \overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1}) = (\Delta t)^2$$
$$(f(\overset{(l)}{\bar{y}}{}^{n}_{j+1}, u^n_j) - f(\overset{(l)}{y}{}^{n}_{j+1}, u^n_j), \overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1})$$

The 2nd term in the L.H.S. (left hand side) of the above equations is positive (from the assumptions on $a(.,.)$), using the Lipschitz condition on the f in the R.H.S. of the equation, then using the Caschy-Schwarz inequality in R.H.S. of the obtained equation, the above equation becomes

$$(\overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1}, \overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1}) \le (\Delta t)^2 L$$
$$\left\| \overset{(l)}{\bar{y}}{}^{n}_{j+1} - \overset{(l)}{y}{}^{n}_{j+1} \right\|_0 \left\| \overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1} \right\|_0$$

$\Rightarrow$

$$\left\| \overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1} \right\|_0^2 \le \frac{(\Delta t)^2}{\alpha_2} L$$
$$\left\| \overset{(l)}{\bar{y}}{}^{n}_{j+1} - \overset{(l)}{y}{}^{n}_{j+1} \right\|_0 \left\| \overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1} \right\|_0$$

$\Rightarrow$

$$\left\| \overset{(l+1)}{\bar{y}}{}^{n}_{j+1} - \overset{(l+1)}{y}{}^{n}_{j+1} \right\|_0 \le \alpha \left\| \overset{(l)}{\bar{y}}{}^{n}_{j+1} - \overset{(l)}{y}{}^{n}_{j+1} \right\|_0,$$

where $\alpha = \frac{(\Delta t)^2}{\alpha_2} L$

$\Rightarrow$

$$\left\| h(\overset{(l)}{\bar{y}}{}^{n}_{j+1}) - h(\overset{(l)}{y}{}^{n}_{j+1}) \right\|_0 \le \alpha \left\| \overset{(l)}{\bar{y}}{}^{n}_{j+1} - \overset{(l)}{y}{}^{n}_{j+1} \right\|_0$$

i.e. the function h is contractive, and since the sequence $\{\overset{(l)}{y}{}^{n}\}$ contain in □, i.e. $h(\overset{(l+1)}{y}) = \overset{(l+1)}{y} \in$ □, for each l, then the sequence $\{\overset{(l)}{y}{}^{n}\}$ of the above iterative method converges to some point $y \in$ □ [2].

3. Existence Of A Discrete Classical Optimal Control:-

In order to study the existence of a discrete classical optimal control, we suppose in addition to the above assumptions, that the function $g^n_m(x, t^n_j, y^n_j, u^n_j)$, $(\forall m = 0,1,...,q)$ is defined on $\Omega \times I^n_j \times S^n_i \times W^n$, $(\forall i = 1,...,M$, $\& \forall j = 0,1,...,N)$, continuous w.r.t. y^n_j & u^n_j for fixed x and j, measurable w.r.t. x for fixed y^n_j & u^n_j, and $\forall x \in \Omega$, $j = 0,1...,N$ satisfies

$$\left| g^n_m(x, t^n_j, y^n_j, u^n_j) \right| \le G^n_{jm}(x) + \gamma_{jm}(y^n_j)^2,$$

where $G^n_{jm}(x) = G^n_m(x, t^n_j) \in L^2(\Omega)$, $\gamma_{jm} \ge 0$,.

Lemma 3.1:

The operator $u^n \mapsto y^n = y^n_{u^n}$ is continuous.

***Proof*:**

Let

$$u^n = (u^n_0, u^n_1, ..., u^n_{N-1}),$$
$$u^{nk} = (u^{nk}_0, u^{nk}_1, ..., u^{nk}_{N-1}),$$
$$y^n = (y^n_0, y^n_1, ..., y^n_{N-1}),$$
$$y^{nk} = (y^{nk}_0, y^{nk}_1, ..., y^{nk}_{N-1}),$$
$$z^n = (z^n_0, z^n_1, ..., z^n_{N-1}),$$

and

$$z^{nk} = (z^{nk}_0, z^{nk}_1, ..., z^{nk}_{N-1}).$$

We want to prove that if $u^{nk} \to u^n$ as $k \to \infty$, then $y^{nk}_{u^{nk}} = y^{nk} \to y^n = y^n_{u^n}$ as $k \to \infty$, i.e. we want to prove if $u^{nk}_j \to u^n_j$, $\forall j$, as $k \to \infty$, then $y^{nk}_j \to y^n_j, \forall j$, as $k \to \infty$. We will prove it by mathematical induction, as follows:-

First, and from the initial conditions of the (14), and (15), and the projection theory that $y_0^{nk} \to y_0^n$ and $z_0^{nk} \to z_0^n$, as $k \to \infty$.

Second: assume for any fixed j, that $y_j^{nk} \to y_j^n$, and $z_j^{nk} \to z_j^n$ as $k \to \infty$, and we shall prove that $y_{j+1}^{nk} \to y_{j+1}^n$, as $k \to \infty$.

Let $y_{j+1}^n = h(y_j^n, z_j^n, u_j^n, y_{j+1}^n)$, and

$y_{j+1}^{nk} = h(y_j^{nk}, z_j^{nk}, u_j^{nk}, y_{j+1}^{nk})$, then

$$\left\|y_{j+1}^{nk} - y_{j+1}^n\right\|_0 =$$

$$\left\|h(y_j^{nk}, z_j^{nk}, u_j^{nk}, y_{j+1}^{nk}) - h(y_j^n, z_j^n, u_j^n, y_{j+1}^n)\right\|_0$$

$$\leq$$

$$\left\|h(y_j^{nk}, z_j^{nk}, u_j^{nk}, y_{j+1}^{nk}) - h(y_j^{nk}, z_j^{nk}, u_j^{nk}, y_{j+1}^n)\right\|_0$$

$$+\left\|h(y_j^{nk}, z_j^{nk}, u_j^{nk}, y_{j+1}^n) - h(y_j^n, z_j^n, u_j^n, y_{j+1}^n)\right\|_0$$

Since

$h(y_j^{nk}, z_j^{nk}, u_j^{nk}, y_{j+1}^n) \to h(y_j^n, z_j^n, u_j^n, y_{j+1}^n)$,

as $k \to \infty$, i.e.

$$\left\|h(y_j^{nk}, z_j^{nk}, u_j^{nk}, y_{j+1}^n) - h(y_j^n, z_j^n, u_j^n, y_{j+1}^n)\right\|_0 \leq \varepsilon_k \to 0, \text{ as } k \to \infty,$$

Then

$$\left\|y_{j+1}^{nk} - y_{j+1}^n\right\|_0 \leq \alpha \left\|y_{j+1}^{nk} - y_{j+1}^n\right\|_0 + \varepsilon_k$$

$$\Rightarrow$$

$$\left\|y_{j+1}^{nk} - y_{j+1}^n\right\|_0 \leq \frac{\varepsilon_k}{1-\alpha} \to 0, \text{ as } k \to \infty$$

$$\Rightarrow$$

$$y_{j+1}^{nk} \to y_{j+1}^n, \text{ as } k \to \infty$$

$$\Rightarrow$$

$$y_j^{nk} \to y_j^n, \forall j,$$

i.e. the operator $u^n \mapsto y^n = y_{u^n}^n$ is continuous**.**

Lemma 3.2:

The functional $G_m^n(u^n)$ (for each $m = 0,1,...,q$) is continuous w.r.t. u^n on $L^2(Q)$.

Proof:

For each m ($m = 0,1,...,q$)**,** we have

$$G_m^n(u^n) = \Delta t \sum_{j=0}^{N-1} \int_\Omega g_m^n(x, t_j^n, y_j^n, u_j^n)dx$$

From the assumptions on g_m^n, Lemma 3.1, and Proposition 1.2 in [3], the functional $\int_\Omega g_m^n(x, t_j^n, y_j^n, u_j^n)dx$ is continuous w.r.t. y_j^n and u_j^n, for each j, and for each $x \in \Omega$.

Then $G_m^n(u^n)$ is continuous w.r.t. u^n on $L^2(Q)$.

Lemma 3.3:

If the function f is Lipschitizian w.r.t. y^n and u^n, the discrete controls u^n and u'^n are bounded in $L^2(Q)$, y_j^n, and $(y_\varepsilon)_j^n = y_{\varepsilon j}^n = y_j^n + \delta_\varepsilon y_j^n$ (with $\varepsilon > 0$) are the corresponding discrete states solutions to the discrete controls u_j^n and $(u_\varepsilon)_j^n = u_{\varepsilon j}^n = u_j^n + \varepsilon\delta u_j^n$, respectively (for each $j = 0,1,...,N$), then

$$\| \delta_\varepsilon \mathrm{y}_l^n \|_1^2 \leq c\varepsilon^2 \left\|\delta u^n\right\|_Q^2$$

and

$$\| \delta_\varepsilon \mathrm{z}_l^n \|_0^2 \leq c\varepsilon^2 \left\|\delta u^n\right\|_Q^2,$$

Or

$$\| \delta_\varepsilon \mathrm{y}_l^n \|_1^2 \leq c \text{ , and } \| \delta_\varepsilon \mathrm{z}_l^n \|_0^2 \leq c$$

Proof:

From the discrete state equations (8-11), we get (for $j = 0,1,...,N-1$)

$$(\delta_\varepsilon z_{j+1}^n - \delta_\varepsilon z_j^n, v) + \Delta t\, a(\delta_\varepsilon y_{j+1}^n, v)$$
$$= \Delta t (f(y_{j+1}^n + \delta_\varepsilon y_{j+1}^n, u_j^n + \varepsilon\delta u_j^n), v)$$
$$+\Delta t (f(y_{j+1}^n, u_j^n), v), \text{ .... } (25)$$

$$\delta_\varepsilon y_{j+1}^n - \delta_\varepsilon y_j^n = \Delta t\, \delta_\varepsilon z_{j+1}^n, \text{ ....}(26)$$

$$\delta_\varepsilon y_0^n = \delta_\varepsilon z_0^n = 0, \text{ ....}(27)$$

In the all next steps we will use c, L', and L for various constants.

Now, substituting $v = \delta_\varepsilon z_{j+1}^n$ in (25), using Lipschitiz property w.r.t. y^n and u^n, then rewriting the first term in the obtained equation in another way, we get

$$\| \delta_\varepsilon \mathrm{z}_{j+1}^n \|_0^2 - \| \delta_\varepsilon \mathrm{z}_j^n \|_0^2 + \| \delta_\varepsilon \mathrm{z}_{j+1}^n - \delta_\varepsilon \mathrm{z}_j^n \|_0^2 +$$
$$+\Delta t\, a(\delta_\varepsilon \mathrm{y}_{j+1}^n, \delta_\varepsilon \mathrm{z}_{j+1}^n) \leq L'\varepsilon^2 \Delta t \left\|\delta u_j^n\right\|_0^2 + L\Delta t$$
$$(\left\|\delta_\varepsilon \mathrm{y}_{j+1}^n\right\|_0^2 + \left\|\delta_\varepsilon \mathrm{z}_{j+1}^n\right\|_0^2)$$

$$\le L'\varepsilon^2\Delta t\left\|\delta u_j^n\right\|_0^2 + L\Delta t\left(\left\|\delta_\varepsilon y_{j+1}^n\right\|_1^2 + \left\|\delta_\varepsilon z_{j+1}^n\right\|_0^2\right) \quad (28)$$

Since

$$a(\delta_\varepsilon y_{j+1}^n - \delta_\varepsilon y_j^n, \delta_\varepsilon y_{j+1}^n - \delta_\varepsilon y_j^n) = (\Delta t)^2 a(\delta_\varepsilon z_{j+1}^n, \delta_\varepsilon z_{j+1}^n),$$

and

$$a(\delta_\varepsilon y_{j+1}^n, \delta_\varepsilon y_{j+1}^n) - a(\delta_\varepsilon y_j^n, \delta_\varepsilon y_j^n) = -(\Delta t)^2 a(\delta_\varepsilon z_{j+1}^n, \delta_\varepsilon z_{j+1}^n) + 2\Delta t\, a(\delta_\varepsilon y_{j+1}^n, \delta_\varepsilon z_{j+1}^n)$$

Then

$$\Delta t\, a(\delta_\varepsilon y_{j+1}^n, \delta_\varepsilon z_{j+1}^n) = \tfrac{1}{2}[a(\delta_\varepsilon y_{j+1}^n, \delta_\varepsilon y_{j+1}^n) - a(\delta_\varepsilon y_j^n, \delta_\varepsilon y_j^n) + a(\delta_\varepsilon y_{j+1}^n - \delta_\varepsilon y_j^n, \delta_\varepsilon y_{j+1}^n - \delta_\varepsilon y_j^n)] \quad \text{............}(29)$$

By substituting (29) in the L.H.S. of (28), summing both sides of the obtained equation from $j=0$ to $j=l-1$, using the assumptions on $a(.,.)$, and the initial conditions (27), we get

$$\left\|\delta_\varepsilon z_l^n\right\|_0^2 + \sum_{j=0}^{l-1}\left\|\delta_\varepsilon z_{j+1}^n - \delta_\varepsilon z_j^n\right\|_0^2 + \tfrac{\alpha_2}{2}\left\|\delta_\varepsilon y_l^n\right\|_1^2 + \tfrac{\alpha_2}{2}\sum_{j=0}^{l-1}\left\|\delta_\varepsilon y_{j+1}^n - \delta_\varepsilon y_j^n\right\|_1^2$$
$$\le L'\varepsilon^2\Delta t\sum_{j=0}^{l-1}\left\|\delta u_j^n\right\|_0^2 + L\Delta t\sum_{j=0}^{l-1}\left(\left\|\delta_\varepsilon y_{j+1}^n\right\|_1^2 + \left\|\delta_\varepsilon z_{j+1}^n\right\|_0^2\right) \quad (30)$$

But

$$\left\|\delta_\varepsilon y_{j+1}^n\right\|_1^2 \le 2\left\|\delta_\varepsilon y_{j+1}^n - \delta_\varepsilon y_j^n\right\|_1^2 + 2\left\|\delta_\varepsilon y_j^n\right\|_1^2$$

and

$$\left\|\delta_\varepsilon z_{j+1}^n\right\|_0^2 \le 2\left\|\delta_\varepsilon z_{j+1}^n - \delta_\varepsilon z_j^n\right\|_0^2 + 2\left\|\delta_\varepsilon z_j^n\right\|_0^2$$

Substituting these inequalities in (30), we have

$$\left\|\delta_\varepsilon z_l^n\right\|_0^2 + (c - L\Delta t)\sum_{j=0}^{l-1}\left\|\delta_\varepsilon z_{j+1}^n - \delta_\varepsilon z_j^n\right\|_0^2 + \tfrac{\alpha_2}{2}\left\|\delta_\varepsilon y_l^n\right\|_1^2 + (c - L\Delta t)\sum_{j=0}^{l-1}\left\|\delta_\varepsilon y_{j+1}^n - \delta_\varepsilon y_j^n\right\|_1^2 \le$$
$$L'\varepsilon^2\Delta t\sum_{j=0}^{N-1}\left\|\delta u_j^n\right\|_0^2 + L\Delta t\sum_{j=0}^{l-1}\left(\left\|\delta_\varepsilon y_j^n\right\|_1^2 + \left\|\delta_\varepsilon z_j^n\right\|_0^2\right)$$
$$\le \varepsilon^2 L'\left\|\delta u^n\right\|_Q^2 + L\Delta t\sum_{j=0}^{l-1}\left(\left\|\delta_\varepsilon y_j^n\right\|_1^2 + \left\|\delta_\varepsilon z_j^n\right\|_0^2\right), (31)$$

where $c = \min(1, \frac{\alpha_2}{2})$ in the L.H.S. of (31).

Now, choosing $\Delta t < \frac{c}{L}$, then the 2nd & the 4th terms in the L.H.S. of (31) become positive , and (31) becomes

$$b\left(\left\|\delta_\varepsilon z_l^n\right\|_0^2 + \left\|\delta_\varepsilon y_l^n\right\|_1^2\right) \le \left\|\delta_\varepsilon z_l^n\right\|_0^2 + c\left\|\delta_\varepsilon y_l^n\right\|_1^2$$
$$\le \varepsilon^2 L'\left\|\delta u^n\right\|_Q^2 + L\Delta t\sum_{j=0}^{l-1}\left(\left\|\delta_\varepsilon y_j^n\right\|_1^2 + \left\|\delta_\varepsilon z_j^n\right\|_0^2\right)$$

$$\Rightarrow$$

$$\left\|\delta_\varepsilon z_l^n\right\|_0^2 + \left\|\delta_\varepsilon y_l^n\right\|_1^2 \le \varepsilon^2 L'\left\|\delta u^n\right\|_Q^2 + L\Delta t\sum_{j=0}^{l-1}\left(\left\|\delta_\varepsilon y_j^n\right\|_1^2 + \left\|\delta_\varepsilon z_j^n\right\|_0^2\right)$$

By using the discrete Gronwall's inequality [6], we get that

$$\left\|\delta_\varepsilon z_l^n\right\|_0^2 + \left\|\delta_\varepsilon y_l^n\right\|_1^2 \le c\varepsilon^2 L'\left\|\delta u^n\right\|_Q^2 \le c\varepsilon^2\left\|\delta u^n\right\|_Q^2$$

$$\Rightarrow$$

$$\left\|\delta_\varepsilon y_l^n\right\|_1^2 \le c\varepsilon^2\left\|\delta u^n\right\|_Q^2 \ \ \&\ \ \left\|\delta_\varepsilon z_l^n\right\|_0^2 \le c\varepsilon^2\left\|\delta u^n\right\|_Q^2$$

Since u^n and u'^n are bounded in $L^2(Q)$, then the above inequality become

$$\left\|\delta_\varepsilon y_l^n\right\|_1^2 \le c \quad \text{..}(32a)$$

and

$$\left\|\delta_\varepsilon z_l^n\right\|_0^2 \le c \quad \text{..}(32b)$$

Theorem 3.1:

Assume that $W_A^n \ne \phi$, W^n is compact, f is defined by

$$f(x, t_j^n, y_{j+1}^n, u_j^n) = f_1(x, t_j^n, y_{j+1}^n) + f_2(x, t_j^n, y_{j+1}^n)u_j^n$$

such that

$\left|f_l(x,t_j^n,y_{j+1}^n)\right| \le F_l(x,t_j^n)+\beta_l\left|y_{j+1}^n\right|, \beta_l \ge 0$,
for $l=1,2$, with $F_l \in L^2(Q)$,
g_m^n (for $m=1,2,...,p$) is independent on u^n, g_0^n and g_m^n (for $m=p+1,p+2,...,q$) are convex w.r.t. (y^n,u^n), then there exists a classical optimal control.

***Proof*:**

Since $W_A^n \ne \phi$, then there exists $u^n \in W^n$, such that

$\left|G_m^n(u^n)\right| \le \varepsilon_{1m}^n$, for $m=1,2,...,p$

and

$G_m^n(u^n) \le \varepsilon_{2m}^n$, for $m=p+1,p+2,...,q$

But the operator $u^n \mapsto y^n = y_{u^n}^n$ is continuous (Lemma 3.1), the functionals $G_m^n(u^n)$ are continuous w.r.t. u^n and y^n for each $m=0,1,...,q$ (Lemma 3.2), and W^n is compact, then W_A^n is compact.

So we get $G_0^n(u^n)$ is a real continuous function defined on the compact set W_A^n, then there exists an optimal control, i.e. there exists $\bar{u}^n \in W_A^n$, such that

$$G_0^n(\bar{u}^n) = \min_{u^n \in W_A^n} G_0^n(u^n)$$

4. The Necessary Conditions For Discrete Optimality:-

In order to state the necessary conditions for discrete classical optimal control, we suppose in addition that the functions f, f_u, f_y, g_m, g_{my}, g_{mu} are defined on $S_i^n \times I_j^n \times R \times W^n$, (where U' is an open set containing the compact set U), measurable w.r.t. x, and continuous w.r.t. y_j^n & u_j^n, and satisfy(for each $j=0,1,...,N$)

$\left|f_y(x,t_j^n,y_j^n,u_j^n)\right| \le L$

$\left|f_u(x,t_j^n,y_j^n,u_j^n)\right| \le L'$

$\left|g_{my}^n(x,t_j^n,y_j^n,u_j^n)\right| \le G_m^n(x,t_j^n)+\gamma_{1m}\left|y_j^n\right|$,
for $m=0,1,...,q$

and

$\left|g_{mu}^n(x,t_j^n,y_j^n,u_j^n)\right| \le G_m^n(x,t_j^n)+\gamma_{2m}\left|y_j^n\right|$,
for $m=0,1,...,q$

where

$G_m^n \in L^2(\Omega)$, $(x,t_j^n,y_j^n,u_j^n) \in S_i^n \times I_j^n \times R \times W^n$
with $\gamma_{1m} \ge 0$, and $\gamma_{2m} \ge 0$, $m=0,1,...,q$

Lemma 4.1:

Dropping the index m, the general discrete classical ***adjoint state*** $\phi_{u^n}^n = \phi^n = (\phi_0^n, \phi_1^n, ..., \phi_{N-1}^n)$ is given by (for $j=N-1,N-2,...,0$):

$(\psi_{j+1}^n - \psi_j^n, v) + \Delta t\, a(\phi_j^n, v)$
$= \Delta t(\phi_j^n f_y(y_{j+1}^n, u_j^n) +$ .... (33)
$g_y(y_{j+1}^n, u_j^n), v)$, $v \in V_n$

$\phi_{j+1}^n - \phi_j^n = \Delta t\, \psi_j^n$ ....(34)

$\phi_N^n = \psi_N^n = 0$ ....(35)

where $\phi_j^n, \psi_j^n \in V_n$, for each $j=0,1,...,N$

The ***directional derivative*** of G is given by:

$DG^n(u^n, u'^n - u^n)$

$$= \lim_{\varepsilon \to 0} \frac{G(u^n + \varepsilon\delta u^n) - G(u^n)}{\varepsilon}$$

$$= \Delta t \sum_{j=0}^{N-1} (H_u^n(t_j^n, y_{j+1}^n, \phi_j^n, u_j^n), \delta u_j^n) \quad \text{...........(36)}$$

where $u^n, u'^n \in W^n$, $\delta u_j^n = u_j'^n - u_j^n$, and the ***Hamiltonian*** H^n is defined by:

$H^n(x,t_j^n,y_{j+1}^n,\phi_j^n,u_j^n)$
$:= \phi_j^n f(x,t_j^n,y_{j+1}^n,u_j^n) + g^n(x,t_j^n,y_j^n,u_j^n)$,
for each $j=0,1,...,N-1$

***Proof*:**

By using equation (25), with $v=\phi_j^n$, and then using the Frechét derivative of the function f in R.H.S. of the obtained equation (which it exists from the assumptions on f [10]), then multiplying both sides by Δt, summing over j (from $j=0$ to $j=N-1$), using (32a) & (32b), we get

$$\sum_{j=0}^{N-1} \frac{(\delta_\varepsilon z_{j+1}^n - \delta_\varepsilon z_j^n, \phi_j^n)}{\Delta t} + \Delta t \sum_{j=0}^{N-1} a(\delta_\varepsilon y_{j+1}^n, \phi_j^n)$$

$$= \Delta t \sum_{j=0}^{N-1} (f_y \delta_\varepsilon y_{j+1}^n + \varepsilon f_u \delta u_j^n, \phi_j^n) +$$

$$+ O_1(\varepsilon)\left\|\delta u^n\right\|_Q \quad \text{............................(37)}$$

where $O_1(\varepsilon) \to 0$, as $\varepsilon \to 0$, and $O_1(\varepsilon) = \left\|\delta_\varepsilon y^n\right\|_Q + c\varepsilon\left\|\delta u^n\right\|_Q$

Substituting $v = \delta_\varepsilon y^n_{j+1}$ in (33), then multiplying both sides by Δt, summing over j (from $j = 0$ to $j = N-1$), we get

$$\sum_{j=0}^{N-1} \frac{(\psi^n_{j+1} - \psi^n_j, \delta_\varepsilon y^n_{j+1})}{\Delta t} + \Delta t \sum_{j=0}^{N-1} a(\phi^n_j, \delta_\varepsilon y^n_{j+1})$$
$$= \Delta t \sum_{j=0}^{n-1} (\phi^n_j f_y, \delta_\varepsilon y^n_{j+1}) + \Delta t \sum_{j=0}^{n-1} (g^n_y, \delta_\varepsilon y^n_{j+1}) \quad (38)$$

Now, by subtracting (38) from (37), we get

$$\sum_{j=0}^{N-1} \frac{(\delta_\varepsilon z^n_{j+1} - \delta_\varepsilon z^n_j, \phi^n_j)}{\Delta t} - \sum_{j=0}^{N-1} \frac{(\psi^n_{j+1} - \psi^n_j, \delta_\varepsilon y^n_{j+1})}{\Delta t}$$
$$= \Delta t \sum_{j=0}^{N-1} (\varepsilon f_u \delta u^n_j, \phi^n_j) - \Delta t \sum_{j=0}^{N-1} (g^n_y, \delta_\varepsilon y^n_{j+1})$$
$$+ O_1(\varepsilon) \left\| \delta u^n \right\|_Q \quad \ldots\ldots\ldots\ldots (39)$$

Now, for given values $y^n_j, j = 0,1,\ldots,N$, in a vector space, we will define the following functions a.e. on $\bar{I}$,

$y^n_-(t) := y^n_j$, $t \in I^n_j$, $\forall j = 0,1,\ldots,N$,

$y^n_+(t) := y^n_{j+1}, t \in I^n_j$, $\forall j = 0,1,\ldots,N-1$,

$y^n_\wedge(t) :=$ the functions which is affine on each I^n_j, such that

$y^n_\wedge(t^n_j) := y^n_j$, for each $j = 0,1,\ldots,N$.

By using these notations for y, ϕ, z & ψ in the 1st and the 2nd terms of the L.H.S. of (39), we have

$$\sum_{j=0}^{N-1} \frac{(\delta_\varepsilon z^n_{j+1} - \delta_\varepsilon z^n_j, \phi^n_j)}{\Delta t}$$
$$= \int_0^T ((\delta_\varepsilon z^n_\wedge)', \phi^n_-) dt \qquad (40a)$$

and

$$\sum_{j=0}^{N-1} \frac{(\psi^n_{j+1} - \psi^n_j, \delta_\varepsilon y^n_{j+1})}{\Delta t}$$
$$= \int_0^T ((\psi^n_\wedge)', \delta_\varepsilon y^n_+) dt \qquad (40b)$$

Now, by using the discrete integration by parts twice to the integral in (40a), i.e.

$$\int_0^T ((\delta_\varepsilon z^n_\wedge)', \phi^n_-) dt$$
$$= -\int_0^T (\delta_\varepsilon z^n_+, (\phi^n_\wedge)') dt + (\delta_\varepsilon z^n_N, \phi^n_N) - (\delta_\varepsilon z^n_0, \phi^n_0)$$
$$= -\int_0^T (\delta_\varepsilon z^n_+, (\phi^n_\wedge)') dt \text{, (from (27) and (35))}$$
$$= -\int_0^T ((\delta_\varepsilon y^n_\wedge)', \psi^n_-) dt \text{, (from (26) and(34))}$$
$$= \int_0^T (\delta_\varepsilon y^n_+, (\psi^n_\wedge)') dt - (\delta_\varepsilon y^n_N, \psi^n_N) + (\delta_\varepsilon y^n_0, \psi^n_0)$$
$$= \int_0^T (\delta_\varepsilon y^n_+, (\psi^n_\wedge)') dt \text{, (from (27) \& (35))}$$
$= $ R.H.S. of (40b)

Substituting theses results in the L.H.S. of equation (39), then this side becomes zero, and equation (39) gives

$$\Delta t \sum_{j=0}^{N-1} (g^n_y, \delta_\varepsilon y^n_{j+1}) = \Delta t \sum_{j=0}^{N-1} (\varepsilon f_u \delta u^n_j, \phi^n_j) + O_1(\varepsilon) \left\| \delta u^n \right\|_Q \quad \ldots. (41)$$

On the other hand, we have that (since the Frechét derivative of the function g exists, from the assumptions on this function [10])

$$G^n(u^n_\varepsilon) - G^n(u^n)$$
$$= \Delta t \sum_{j=0}^{N-1} \int_\Omega (g^n_y \delta_\varepsilon y^n_{j+1} + \varepsilon g^n_u \delta u^n_j) dx \quad \ldots\ldots\ldots\ldots (42)$$
$$+ O_2(\varepsilon) \left\| \delta u^n \right\|_Q$$

where $O_2(\varepsilon) \to 0$, as $\varepsilon \to 0$

Now, by substituting (41) into (42), we have

$$G^n(u^n_\varepsilon) - G^n(u^n)$$
$$= \varepsilon \Delta t \sum_{j=0}^{N-1} \int_\Omega (\phi^n_j f_u + g^n_u) \delta u^n_j) dx \quad \ldots\ldots\ldots\ldots (43)$$
$$+ O_3(\varepsilon) \left\| \delta u^n \right\|_Q$$

where $O_3(\varepsilon) = O_1(\varepsilon) + O_2(\varepsilon) \to 0$, as $\varepsilon \to 0$

Dividing by ε the both sides of (43), taking the limit when $\varepsilon \to 0$, we get

$$DG^n(u^n, u'^n - u^n) = \Delta t$$
$$\sum_{j=0}^{N-1} (\phi^n_j f_u(t^n_j, y^n_{j+1}, u^n_j) + g^n_u(t^n_j, y^n_{j+1}, u^n_j), \delta u^n_j)$$
$$= \Delta t \sum_{j=0}^{N-1} (H^n_u(t^n_j, y^n_{j+1}, \phi^n_j, u^n_j), \delta u^n_j) \quad \ldots\ldots\ldots (44)$$

Lemma 4.2:

The operator $u^n \mapsto \phi^n = \phi^n_{u^n}$ is continuous.

Proof:

The proof is similar to Lemma 3.1.

Lemma 4.3 :

The operator $u^n \mapsto DG^n_m(u^n)$, is continuous w.r.t. u^n, for each $0 \le m \le q$.

Proof:

The proof is similar to Lemma 3.2.

Theorem 4.1:

If $u^n \in W^n$ is an optimal classical control of the considered problem, W^n is convex, then u^n (classical weakly) extremal, i.e. there exists multipliers $\lambda^n_m \in \mathbb{R}$, (for each $m = 0,1,...,q$) with $\lambda^n_0 \ge 0$, and $\lambda^n_m \ge 0$, (for $m = p+1, p+2,...,q$) satisfy $\sum_{m=0}^{q} |\lambda^n_m| = 1$, such that

$$\sum_{m=0}^{q} \lambda^n_m DG^n_m(u^n, u'^n - u^n) \ge 0, \forall u'^n_j \in W^n, (45)$$

and

$\lambda^n_m [G^n_m(u^n) - \varepsilon^n_m] = 0$, for $m = p+1, p+2,...,q$ (transversity condition) ....(46)

where $\phi^n_j = \sum_{m=0}^{q} \lambda^n_m \phi^n_{mj}$, and $g^n_u = \sum_{m=0}^{q} \lambda^n_m g^n_{mu}$ in the definition of $H^n_u = \sum_{m=0}^{q} H^n_{mu}$.

If W^n has the form $W^n = \{u' = u'^n_j : u'^n_j \in U, j = 0,1,...,N-1\}$, with $U \subset \mathbb{R}$, then the above relations are equivalent to the following minimum principle in blockwise form:

$$(\phi^n_j f_u(y^n_{j+1}, u^n_j) + g^n_u(y^n_j, u^n_j), u^n_{ij})_{T_i}$$
$$= \min_{u'^n \in U} (\phi^n_j f_u(y^n_{j+1}, u^n_j) + g^n_u(y^n_j, u^n_j), u'^n)_{T_i}$$

$\forall j = 0,1,...,N-1$, and $\forall i = 1,2,...,M$, ...(47)

Proof:

From Lemma(3.2), the functionals $G^n_m(u^n)$, for each $m = 0,1,...,q$ is continuous w.r.t. $u^n \in W^n$. From the above assumptions and Lemma 4.3, the functionals $DG^n_m(u^n) = DG^n_m(u^n, u'^n - u^n)$, for each $m = 0,1,...,q$ is continuous w.r.t. (u^n, u'^n) and linear w.r.t. $u'^n - u^n$, then the functionals $G^n_m(u^n)$ is k-differentiable for every integer k, then by ***Khun-Tanger-Lagrange Theorem*** [10], there exist multipliers $\lambda^n_0 \ge 0, \lambda^n_m \in \mathbb{R}$, (for $m = 1,2,...,p$), and $\lambda^n_m \ge 0$, (for $m = p+1, p+2,...,q$), with $\sum_{m=0}^{q} |\lambda^n_m| = 1$, such that inequality (45), and equality (46) are satisfy.

Let we use Lemma 4.1, then inequality (45) for each $u'^n_j \in W^n$ becomes:

$$\Delta t \sum_{j=0}^{N-1} \sum_{m=0}^{q} \int_\Omega [\lambda^n_m \phi^n_{mj} f_u(t^n_j, y^n_{j+1}, u^n_j)$$
$$+ g^n_{mu}(t^n_j, y^n_j, u^n_j)](u'^n_j - u^n_j) dx \ge 0$$

$\Rightarrow$

$$\Delta t \sum_{j=0}^{N-1} \int_\Omega [(\sum_{m=0}^{q} \lambda^n_m \phi^n_{mj}) f_u(t^n_j, y^n_{j+1}, u^n_j)$$
$$+ (\sum_{m=0}^{q} \lambda^n_m g^n_{mu}(t^n_j, y^n_j, u^n_j))](u'^n_j - u^n_j) dx \ge 0$$

Set

$$\phi^n_j = \sum_{m=0}^{q} \lambda^n_m \phi^n_{mj}, \ \& \ g^n_u = \sum_{m=0}^{q} \lambda^n_m g^n_{mu}(t^n_j, y^n_j, u^n_j),$$

then

$$\Delta t \sum_{j=0}^{N-1} \int_\Omega [\phi^n_j f_u(t^n_j, y^n_{j+1}, u^n_j) + g_u(t^n_j, y^n_j, u^n_j)]$$
$$(u'^n_j - u^n_j) dx \ge 0, \quad \forall u'^n_j \in W^n \quad (48)$$

$\Rightarrow$

$$\Delta t \sum_{j=0}^{N-1} (\phi^n_j f_u(t^n_j, y^n_{j+1}, u^n_j) +$$
$$g_u(t^n_j, y^n_j, u^n_j), u'^n_j - u^n_j) \ge 0, \forall u'^n_j \in W^n, (49)$$

To prove (49) is equivalent to the minimum principle blockwise form (47), we define $W^n = \{u' = u'^n_j : u'^n_j \in U, j = 0,1,...,N-1\}$, with $U \subset \mathbb{R}$.

Let $u'^n_j = u^n_j$, for all j except once say k, i.e. $u'^n_k \ne u^n_k$, then (49) becomes

$$(\phi^n_k f_u(t^n_k, y^n_{k+1}, u^n_k) + g_u(t^n_k, y^n_k, u^n_k), u^n_k)$$
$$= \min_{u' \in U} (\phi^n_k f_u(t^n_k, y^n_{k+1}, u^n_k) + g_u(t^n_k, y^n_k, u^n_k), u')$$

Since k is arbitrary, then for each $j = 0,1,...,N-1$, we have

$$(\phi_j^n f_u(t_j^n, y_{j+1}^n, u_j^n) + g_u(t_j^n, y_j^n, u_j^n), u_j^n)$$
$$= \min_{u' \in U}(\phi_k^n f_u(t_k^n, y_{k+1}^n, u_k^n) + g_u(t_j^n, y_j^n, u_j^n), u')$$

Then for each $j = 0,1,...,N-1$, and $i = 1,2,...,M$, we have

$$(\phi_j^n f_u(t_j^n, y_{j+1}^n, u_j^n) + g_u(t_j^n, y_j^n, u_j^n), u_{ij}^n)_{T_i}$$
$$= \min_{u' \in U}(\phi_k^n f_u(t_k^n, y_{k+1}^n, u_k^n) + g_u(t_j^n, y_j^n, u_j^n), u')_{T_i}$$

The proof of the minimum principle blockwise form (47) is equivalent to (49), follows conversely form the above steps.

5.Conculusions

The finite element method associated with the implicit finite difference scheme used successfully to discretize the continuous state and its adjoint equations in the continuous classical optimal control problem to a discrete state and adjoint equations, while the Grank-Nicolson finite difference method or the θ-finite difference method with $0 < \theta < 1$ failed to give a suitable discretization for the adjoint state equations.

The Galerkin method is suitable to solve the nonlinear hyperbolic partial differential equations (discrete state equations) associated with fixed discrete classical controls.

The existence theory for optimality of a discrete classical optimal control problem is developed so as the necessary conditions and a picewise minimum principle for optimality.

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الخلاصة

يتناول هذا البحث دراسة مسألة السيطرة الامثلية التقليدية من النمط المقسم (discrete) لنظام من المعادلات التفاضلية الجزئية غير الخطية من النوع الزائدي بوجود قيود عديدة متباينة. أولا" قمنا بتقسيم (discretize) مسألة السيطرة الامثلية التقليدية من النمط المستمر الى مسألة سيطرة امثلية تقليدية من النمط المقسم وذلك بتجزئة مجال تعريف المتغيرات المستقلة الى مناطق صغيرة مستقلة حيث استخدمنا طريقة العناصر الثابته بالنسبة للفضاء وطريقة الفروقات المنتهية بالنسبة لمتغير الزمن. قمنا بايجاد حلول معادلات تفاضلية جزئية مقسمة (discrete solutions) غير خطية زائدية لسيطرة تقليدية مقسمة (discrete control) وثابتة باستخدام طريقة كاليركن لتقريب الحلول المضبوطة.

ثانيا" قمنا باشتقاق نظرية الوجود للحصول على سيطرة أمثلية تقليدية مقسمة تحت قيود عديدة ومتباينة.

ثالثا" ايجا الصيغة المتقطعة للمعادلات المرافقة (adjoint equations) التفاضلية الجزئية المصاحبة للمعادلات التفاضلية الجزئية المقسمة غيرالخطية الزائدية.

اخيرا" قمنا باشتقاق الشروط الضرورية وصيغة مبدأ الحزم الاصغري لمناطق ثابتة (Minimum principle blockwise form) لمسالة السيطرة الامثلية التقليدية المقسمة.