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ON SEMI GENERALIZED CARTAN G-SPACE

Anmar Hashim Jasim Al-Sheikhly
Department of Mathematics, College of Science, Al-Mustansiriyah University
Baghdad-Iraq.

Abstract

A semi generalized Cartan G-space is the main aim in this paper.We get the following results:

(i) A Cartan G-space is semi generalized Cartan.

(ii) A semi Cartan G-space is semi generalized Cartan.

(iii) We study this space (semi generalized Cartan G-space) and give enough examples and theorems about it, where we study its properties, subspace, and the equivariant homeomorphic image.

(iv) We prove that a G-space X is semi generalized Cartan if X has a star thin semi generalized open set.

1. Introduction

A Cartan G-space is introduced by Palais in [12]. A semi Cartan G-space is introduced in [2]. In this paper, we study a semi generalized Cartan G-space in section 3. On the other hand, we introduce the definition by depending on a semi generalized neighborhood which itself depends on the concept of a semi generalized open set in [13].

Recalling that a G-space X is compact if each open cover of X has a finite subcover, and X is a locally compact if each point of X is contained in a compact neighborhood. Besides, its completely regular whenever A is a closed set in X and $x\notin A$, there is a continuous function $f:X\to I$ such that $f(x)=0$ and $f(A)=1$.

Throughout this paper we take X to be a completely regular and Hausdorff space and G to be locally compact but not compact toplological group.

2. Preliminaries

In this section, some definitions and theorems are given which are used in this work.

Definition 2.1 [3]:

A topological transformation group is a triple (G, X, π) where G is a topological group, X is a topological space and $\pi : G \times X \to X$ is a function satisfies :

(i) π is continuous.

(ii) $\pi(e,x) = x$ for each $x\in X$, where e is the identity for G.

(iii) $\pi(g_1, \pi(g_2,x)) = \pi(g_1 g_2, x)$ for all $x\in X$ and $g_1, g_2 \in G$.

The function π is called an action of G on X. The space X togather with π is called a G-space.(or more precisely a left G-space).

Definition 2.2 [8]:

Let X be a G-space. A subset A of X is invariant under a subset S of G if $SA \subseteq A$ where $SA=\{sa \mid s\in S, a\in A\}$.

Definition 2.3 [8]:

A subset A of a topological group G is syndetic in G if there is a compact subset K of G such that G = AK.

Definition 2.4 [3]:

The subgroup $G_x = \{ g\in G \mid gx = x\}$ of G is called the isotropy subgroup (or the stability subgroup) of G at x.

Definition 2.5 [8]:

Let X be a G-space and $x \in X$. Then the point x is said to be :

(i) Fixed point if $gx = x$ for each $g\in G$.

(ii) Periodic point if G_x is syndetic in G.

Definition 2.6 [1]:

Let X be a G-space.A subset S of X with $S \neq X$ is called a star if for each $x\in X$ there exists $g\in G$ such that $gx\in S$.

Definition 2.7 [3]:

Let (G,X,π_1) and (G,Y,π_2) be G-spaces. A continuous function $\lambda:X\to Y$ is called an equivariant function if λ satisfies :

For each $g\in G$, $x \in X$, $\lambda(\pi_1(g, x)) = \pi_2(g, \lambda(x))$. Or simply, $\lambda(gx) = g\lambda(x)$.

Definition 2.8 [12]:

If U and V are subsets of a G-space X,then U is thin relative to V if the set ((U,V)) = { g∈ G | gU ∩ V ≠ ϕ }is relatively compact in G.If U is thin relative to itself, then it is said that U is thin.

Definition 2.9 [12]:

A G-space X is a Cartan G-space if every point of X has a thin neighborhood.

Definition 2.10 [9]:

A subset A in a topological space X is semi open if there exists an open set O such that $O \subseteq A \subseteq \overline{O}$.

S.O(X) will denote the class of all semi open sets in X.

Definition 2.11 [11]:

The union of all semi open sets containing in a subset A of a topological space X is called the semi interior of A,denoted by A^{os}

Definition 2.12 [4]:

A subset A in a topological space X is called a semi neighborhood of a point x in X if there exists a semi open set U in X such that $x \in U \subseteq A$.

Definition 2.13:

(i) A subset A of a topological space X is called a semi closed if X-A is semi open. [5]

(ii)The intersection of all semi closed sets containing A is called the semi closure of A. It is denoted by $\overline{A}^{s}$. [11]

Theorem 2.14 [14]:

Let F be semi closed subset of a topological space X and let Y be an open subset of X. Then F ∩ Y is semi closed in Y.

Theorem 2.15 [14]:

Let Y be a subspace of a topological space X. If A∈ S.O(Y), then there is a subset K of X such that K ∈ S.O(X) and A = K ∩ Y.

Definition 2.16:

Let f be a function from a topological space X into a topological space Y. Then f is called :

(i) Irresolute if $f^{-1}(O)$ is S.O in X for each S.O subset O of Y. [7]

(ii) Semi open if f(O) is S.O in Y for each open set O in X. [11]

(iii) Semi closed if f(F) is semi closed in Y for each closed set F in X. [11]

Theorem 2.17 [6]:

Let f be a continuous semi open function from a topological space X into a topological space Y. If A is semi open in X, then f(A) is semi open in Y.

Definition 2.18 [10]:

A function f from a topological space X into a topological space Y is a semi homeomorphism if f satisfies:

(i) f is one-one and onto.

(ii) f is continuous .

(iii) f is semi open (or semi closed).

Definition 2.19 [2]:

A G-space X is called a semi Cartan G-space if every point of X has a thin semi neighborhood.

Definition 2.20 [13]:

A subset A of a topological space X is called a semi generalized closed (written sg-closed) if $\overline{A}^{s} \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.

Definition 2.21 [13]:

A subset A of a topological space X is called a semi generalized open (written sg-open) if X-A is sg-closed.

Theorem 2.22 [13]:

A subset A of a topological space X is sg-open in X iff $F \subseteq A^{os}$ whenever F is a semi closed subset of X and $F \subseteq A$.

Definition 2.23:

A subset A of a topological space X is called a semi generalized neighborhood (written sg-neighborhood) of a point x in X if there exists an sg-open set U in X such that $x \in U \subseteq A$.

Theorem 2.24:

Let f be a semi homeomorphism and an irresolute function from a topological space X into a topological space Y. If A is an sg-open set in X, then f(A) is an sg-open set in Y.

Proof:

Let F be a semi closed subset of Y such that $F \subseteq f(A)$.

So $f^{-1}(F) \subseteq f^{-1} f(A)$.

Since f is 1-1, then $f^{-1}(F) \subseteq A$.

Because f is irresolute, then $f^{-1}(F)$ is semi closed in X.

We have A is sg-open in X. So by 2.22 we get $f^{-1}(F) \subseteq A^{os}$.
Hence $f f^{-1}(F) \subseteq f(A^{os})$.
Since f is onto, then $F \subseteq f(A^{os})$.
Now to show that $f(A^{os}) \subseteq [f(A)]^{os}$.
Since $A^{os} \subseteq A$, then $f(A^{os}) \subseteq f(A)$.
Since f is semi open and continuous, then by [2.17] $f(A^{os})$ is semi open in Y.
Because $[f(A)]^{os} \subseteq f(A)$ and since $[f(A)]^{os}$ is the largest semi open set which is contained in f(A).
Hence $f(A^{os}) \subseteq [f(A)]^{os}$, which leads to $F \subseteq [f(A)]^{os}$.
Thus by 2.22 f(A) is sg-open in Y.

Theorem 2.25:

Let X be a topological space and Y be an open subspace of X. If A is sg-open in X, then $A \cap Y$ is sg-open in Y.

***Proof*:**

Let F_Y be a semi closed subset of Y such that $F_Y \subseteq A \cap Y$.

So $F_Y \subseteq A$.
Since F_Y is semi closed in Y, then by [2.15] $F_Y = F \cap Y$ where F is semi closed in X.
Hence $F \cap Y \subseteq A$.
Since F is semi closed and Y is open in X, then by [2.14] is $F \cap Y$ is semi closed in Y.
Since A is sg-open in X, then by [2.22] $F \cap Y \subseteq A^{os}$.
That is $F_Y \subseteq A^{os}$.
Then $F_Y \cap Y \subseteq A^{os} \cap Y$.
Which leads to $F_Y \subseteq A^{os} \cap Y$.
Hence $A^{os} \cap Y \subseteq A \cap Y$.
But we have $(A \cap Y)_Y^{os} \subseteq A \cap Y$, and since $(A \cap Y)_Y^{os}$ is the largest semi open set which is contained in $A \cap Y$.
So $A^{os} \cap Y \subseteq (A \cap Y)_Y^{os}$, and that leads to $F_Y \subseteq (A \cap Y)_Y^{os}$.
By 2.22 we get $A \cap Y$ is sg-open in Y.

Remark 2.26 :

(i) If U and V are relatively thin, then so are any translates g_1U and g_2V. In particular if U is thin,then any two translates of U are relatively thin. [12]
(ii) A closed subset of a locally compact space is locally compact. [15]
(iii) A closed subset of a compact space is compact. [15]

3. Main Results

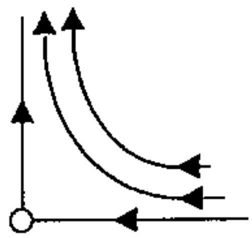
Here we introduce a new G-space, which we call a semi generalized Cartan G-space, which is weaker than a Cartan G-space and a semi Cartan G-space. Besides, we give examples and theorems.

Definition 3.1:

A G-space X is called a semi generalized Cartan (written sg-Cartan) G-space if every point of X has a thin sg-neighborhood.
i.e. $\forall x \in X\ \exists\ U$ (sg-neighborhood of x) s.t. the set $((U,U)) = \{ g \in G \mid gU \cap U \neq \phi \}$ is relatively compact in G.

Examples 3.2:

(i) (R,+) with the relative usual topology is a locally compact but not compact topological group and the set :
$D=\{(x,y) \in R^2 \setminus \{(0,0)\} \mid x \geq 0, y \geq 0\}$ with the relative usual topology is a completely regular T_2 space. Let R acts on D as follows:
$\pi : R \times D \rightarrow D$ such that $\pi(t, (x,y)) = (xe^{-t}, ye^{t})$ for each $t \in R$ and $(x,y) \in D$.
Clear that D is an R-space.See the figure below.
[2] proves that D is a semi Cartan R-space, and because every semi neighborhood is sg-neighborhood,then D is an sg-Cartan R-space.



(ii) $(R \setminus \{0\}, \cdot)$ with the usual topology is a locally compact non-compact topological group. Besides, R^2 with the usual topology is a completely regular Hausdorff space.
Then $R \setminus \{0\}$ acts on R^2 as follows :
$\pi : R \setminus \{0\} \times R^2 \rightarrow R^2$ is defined by $\pi(r, (x,y)) = (rx, ry)$ for each $r \in R \setminus \{0\}$ and $(x,y) \in R^2$.
Clear that R^2 is $R \setminus \{0\}$ -space.

But R^2 is not sg-Cartan $R\backslash\{0\}$-space, since $(0,0) \in R^2$ has no thin sg-neighborhood since for any sg-neighborhood U of (0,0) the set $((U,U)) = R\backslash\{0\}$ is not relatively compact in $R\backslash\{0\}$.

Theorem 3.3:

(i) A Cartan G-space is sg-Cartan.

(ii) A semi Cartan G-space is sg-Cartan.

***Proof*:**

(i) Let $x \in X$.

Since X is Cartan, then there exists U a thin neighborhood of x.

Since every neighborhood is sg-neighborhood.

So U a thin sg-neighborhood of x.

Hence X is an sg-Cartan G-space.

(ii) By the same way in (i),and since every semi neighborhood is sg-neighborhood.

We get X to be an sg-Cartan G-space.

Proposition 3.4:

If X is an sg-Cartan G-space, then for each $x \in X$ the isotropy subgroup G_x at x is compact.

***Proof* :**

Let $x \in X$.

Since X is sg-Cartan, then there exists U a thin sg-neighborhood of x.

The next step is to show that $G_x \subseteq ((U,U))$.

Let $g \in G_x$ then $gx = x$ which leads to $gU \cap U \neq \phi$.

Then $g \in ((U,U))$.

Hence $G_x \subseteq ((U,U))$ which is relatively compact in G.

By [2] G_x is closed in G.

Then by 2.26(iii) G_x is compact.

Theorem 3.5:

If X is an sg-Cartan G-space, then :

(a) There is no fixed point.

(b) There is no periodic point.

***Proof* :**

(a) Let $x \in X$ such that x is a fixed point.

Since X is an sg-Cartan G-space, then x has U as a thin sg-neighborhood in X.

Because x is a fixed point, then $gx = x$ for each $g \in G$.

So $gU \cap U \neq \phi$ for each $g \in G$.

That is $((U,U)) = G$.

Since ((U,U)) is relatively compact in G, then G is compact.

But G is not compact, which leads to a contradiction.

Hence X has no fixed point.

(b) Let $x \in X$ such that x is a periodic point.

Then G_x is a syndetic subgroup in G.

That is there is a compact subset K of G such that $G = G_x K$.

By 3.4 G_x is compact in G for each $x \in X$.

Thus G is compact

But that leads to a contradiction since G is not compact.

Hence X has no periodic point.

Theorem 3.6:

Let X & Y be G-spaces and Let $\lambda: X \to Y$ be an equivariant,irresolute and a semi homeomorphism function. If X is an sg-Cartan G-space, then so is Y.

***Proof* :**

Let $y \in Y$.

Since λ is onto, then there exists $x \in X$ such that $\lambda(x) = y$.

Since X is an sg-Cartan G-space and $x \in X$, then x has U as a thin sg-neighborhood.

Since λ is semi homeomorphism and irresolute, then by 2.24 we have $\lambda(U)$ is a sg-neighborhood of y in Y.

To show that $\lambda(U)$ is thin we have to prove that $((U,U)) = ((\lambda(U), \lambda(U)))$.

Since λ is 1-1 and equivariant function, then

$g \in ((U,U)) \leftrightarrow gU \cap U \neq \phi \leftrightarrow \lambda(gU \cap U) \neq \phi \leftrightarrow \lambda(gU) \cap \lambda(U) \neq \phi \leftrightarrow g\lambda(U) \cap \lambda(U) \neq \phi \leftrightarrow g \in ((\lambda(U),\lambda(U)))$.

Hence $((U, U)) = ((\lambda(U),\lambda(U)))$.

Because ((U, U)) is relatively compact, then so is $((\lambda(U),\lambda(U)))$.

Hence Y is sg-Cartan.

Theorem 3.7:

If a G-space X has a star thin sg-open set U, then X is an sg-Cartan G-space.

***Proof*:**

Let $x \in X$.

Since U is a star set,then there is $g \in G$ such that $gx \in U$.

Hence $x \in g^{-1} U$.

Since $\pi_g : X \to X$ is semi homeomorphism and irresolute for each $g \in G$,then by 2.24 $g^{-1}U$ is a sg-open set of x.

Since U is thin,then by 2.26(i) we get $((g^{-1}U, g^{-1}U))$ is relatively compact in G.
That is $g^{-1}U$ is a thin sg-neighborhood of x in X
Thus X is an sg-Cartan G-space.

Theorem 3.8:

If X is an sg-Cartan G-space, H is a closed subgroup of G and Y is an open subspace of X which is invariant under H, then Y is an sg-Cartan H-space.

***Proof*:**

By [8], (H,Y) is a topological transformation group.
Since Y is a subspace of X and X is a completely regular Hausdorff space,then so is Y.
Since G is locally compact and H is a closed subgroup of G, then by 2.26(ii) H is locally compact.
Hence Y is an H-space.
We are going to prove that Y is sg-Cartan.
Let $y \in Y$. Then $y \in X$.
Since X is an sg-Cartan G-space then y has U as a thin sg-neighborhood in X.
Let $U' = U \cap Y$.
Since Y is an open subspace of X, then by 2.25 we have U' to be an sg-neighborhood of y in Y
Since $((U', U')) \subseteq ((U,U))$ and because ((U,U)) is relatively compact in G, then so is $((U', U'))$.
Since H is a closed subgroup of G, then $((U', U'))$ is relatively compact in H.
Hence Y is an sg-Cartan H-space.
This complete the prove of the theorem.

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الخلاصة

إن فضاء-G شبه المعمم لكارتان هو هدفنا الأساسي في هذا البحث. وقد حصلنا على النتائج التالية:

1- إن فضاء-G لكارتان هو فضاء- G شبه المعمم لكارتان.

2- إن فضاء- G شبه كارتان هو فضاء- G شبه المعمم لكارتان.

3- درسنا هذا الفضاء و وضعنا امثلة ومبرهنات كافية عنه، حيث درسنا خصائصه، فضائه الجزئي و صورة التكافؤ التبولوجي المتساوي التغير.

4- برهنا بأنه يكون X فضاء-G شبه المعمم لكارتان اذا كان لـــ X مجموعة معممة واهية شعاعية.