ON COPRIMARY MODULES

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Abstract

　　Let R be commutative ring with identity. In this paper, we give the dual of the concept primary modules which is called coprimary modules with some properties Also the concept of carational modules with some properties is given.

1. Introduction

　　Let R be commutative ring with identity and M be an R-modules. A proper submodule N of M is said to be prime if $rx \in N$ ($r \in R$ and $x \in M$) implies either $x \in N$ or $r \in [N:M] = \{r \in R, rM \subseteq N\}$ [1]. And M is called prime module if the zero submodule of M is prime submodule [2]. A proper submodule N of M is said to be primary if $rx \in N$ for $r \in R$ and $x \in M$, then either $x \in N$ or $r \in \sqrt{[N:M]} = \{r \in R, r^n M \subseteq N$ for some $n > 0\}$, [3].

　　In [2] an R-module M is called coprime modules if $\forall r \in R$ the homomorphism. $f_r : M \to M$ defined by $f_r(m) = rm$ is either zero or epimorphism.

　　In this paper we introduce the notion of coprimary and corational modules with some of these properties

2. Primary Modules

　　This section is devoted to describe primary modules.

　　The following proposition is useful for characterization for primary module.

Proposition (2.1):

　　A proper submodule N of an R-module M is primary submodule if and only if for every $r \in R$, the homomorphism $f_r : \frac{M}{N} \to \frac{M}{N}$ defined by $f_r(m+N) = r^n m + N$, $\forall m \in M$ for some $n \in Z^n$ is either zero or monomorphism.

Proof:

　　Let $r \in R$, $f_r : \frac{M}{N} \to \frac{M}{N}$ defined by $f_r(m + N) = r^n m + N$ where $n \in Z^+$. Suppose that $f \neq 0$ and let $m_1 + N \in Ker(f_r)$, then $f_r(m_1+N)=N$ i.e. $r^n m_1 \in N$ but N is primary submodule, then either $m_1 \in N$ or $r^n \in \sqrt{[N:M]}$. If $m_1 \in N$, thus $Ker(f_r) = N$ i.e. f_r is a monomorphism. If $r^n \in \sqrt{[N:M]}$ i.e. $r^\ell M \subseteq N$ ($n \le \ell$), then $f_r(m_1+N) = r^n m_1 + N = N$. then f_r is zero a

Contradiction. To prove the other direction. Assume $rx \in N$ for $r \in R$ and $x \in M$ and $f_r : \frac{M}{N} \to \frac{M}{N}$ such that $f_r(m+N) = r^n m + N$ is either zero or monomorphism if $x \notin N$ but $f_r(x + N) = r^{n-1} rx + N$, $rx \in N$, i.e. $f_r(x + N) = N$, i.e. f is not monomorphism, thus f is zero, then $r^n m + N = N$ $\forall m \in M$, then $r \in \sqrt{[N:M]}$.

Proposition (2.2):

　　A non zero R-module M is primary module if and only if $\sqrt{annM}=\sqrt{annN}$ for every non zero submodule N of M.

Proof :

　　Let $r \in \sqrt{annN}$, i.e. $r^n m = 0$, $\forall m \in N$, $n \in Z^+$ but M is primary, then there exist $f_r : M \to M$ defined by $f_r(m) = r^n m$, for some $n \in Z^+$, $m \in M$ is either zero or monomorphism. If $f_r(m) = r^n m$ is zero, then $r \in \sqrt{annM}$, if f_r is monomorphism, then $Ker(f_r) = \{0\}$, $\forall m \in M$ but $r^n m = 0$ for every $m \in N$ and N is non zero. This is a contradiction.

　　Thus $\sqrt{annN} \subseteq \sqrt{annM}$, then $\sqrt{annN} = \sqrt{annM}$. Let $r \in R$ if $f_r(m)=r^n m$ is not zero and let $m \neq 0$, such that $m \in Ker(f_r)$, thus $r^n m=0$, i.e., $r \in \sqrt{ann <m>} = \sqrt{annM}$ i.e. $r^n m = 0$, $\forall m \in M$, thus f = 0 a contradiction, then f is a monomorphism. Thus (by Prop.2.1) M is primary.

3. Coprimary Module

　　In this section, we introduce the concept of coprimary module as a dual of primary module.

Definition (3.1):

　　Let M be an R-module, then M is Coprimary module if $\forall r \in R$, such that the homomorphism $f_r : M \to M$ defined by $f_r(m)=r^n m$ for some $n \in Z^n$ is either zero or epimorphism.

It is proved in [2] that an R-module M is Coprime if $\forall$ r $\in$ R, the homomorphism f_r:M→M defined by $f_r(m)$ = rm is either zero or epimorphism, therefore it is clear that every Coprime module is coprimary module but the converse is not true as the following example.

Example (3.2):

Z_4 as Z-module is not coprime since if r = 2 s.t f_2 : Z_4→ Z_4, $f_2(m)$ = 2m, f_2 = {$\bar{0},\bar{2}$ }, i.e. f ≠ 0 and f is not epimorphism but Z_4 is coprimary module since $\forall$ r $\in$ Z either f_r is zero or epimorphism.

The following Remark gives another characterization for coprimary modules.

Remark (3.3):

A non zero R-module M is coprimary module iff for every r $\in$ R, for some n $\in$ Z^+ such that r^nM = M or r^nM = 0.

Proposition (3.4) :

Let M be an R- module, then M is coprimary module iff $\sqrt{ann\frac{M}{N}}=\sqrt{annM}$ for every proper submodule N of M.

Proof :

Let N be a proper submodule of R-module M and r$\in\sqrt{ann\frac{M}{N}}$, then $r^nM\subseteq N$, n $\in$ Z^+ but M ≠ N and M coprimary, then r^nM = 0, thus r $\in \sqrt{annM}$, clearly $\sqrt{annM} \subseteq \sqrt{ann\frac{M}{N}}$. Let r $\in$ R If r^nM ≠ M $\forall$ n$\in$ Z^+. Let r^nM = N, N is a proper submodule of M. r^n $\in$ ann$\frac{M}{N}$, thus r $\in \sqrt{ann\frac{M}{N}} = \sqrt{annM}$. i.e., r^sM = 0 for some s $\in$ Z^+, thus M is coprimary module.

Proposition (3.5):

If M is coprimary R-module, then $\sqrt{annM}$ is primary ideal.

Proof:

Let r_1, r_2 be non zero elements of R such that $r_1.r_2$ $\in \sqrt{annM}$, then for some n $\in$ Z^n, $(r_1.r_2)^nM$ = 0, then $r_1^nr_2^nM$ = 0 if $r_2 \notin \sqrt{annM}$ i.e r_2^kM ≠ 0 $\forall$ k$\in$ Z^+ then r_2^nM ≠ 0 thus r_2^nM=M and r_1^nM = 0 this implies that $r_1 \in \sqrt{annM}$
Then $\sqrt{annM}$ is primary ideal.

Proposition (3.6):

Any homomorphic image of coprimary module is coprimary module.

Proof:

Let M and N be R-modules with M is coprimary and f : M→N be epimorphism and let r $\in$ R, then $\exists$ n $\in$ Z^+, r^nM = M or r^nM = 0. r^nM = $r^nf(M)$ = $f(r^nM)$.
If r^nM = M, then r^nM = N or r^nN = 0 and if r^nM = 0, then r^nN=$r^nf(N)$.i.e r^nN = f(0) = 0. Then N is coprimary.

4. Corational modules

A submodule N of an R-module M is called corational in M if Hom(M,$\frac{N}{K}$) = 0 for all submodule K of all submodule K of M such that K $\subseteq$ N $\subseteq$ M, [2]. Now, we introduce the concept of corational module and study some properties of this modules.

Definition (4.1):

Let M be an R-module, then M is called corational module if every proper submodule of M is corational.

Example (4.2):

Z_p^{∞} as Z-module is corational module, since every proper module of Z_p^{∞} is corational.

Remark(4.3):

A submodule of corational module need not be corational module. Z_p^{∞} is corational module but $< \frac{1}{p^2} + Z> \cong Z_p^2$ is not corational module, since Hom $(Z_p^2, \frac{Zp}{\{0\}}) \neq 0$
Thus Z_p is not corational submodule of Z_p^2.

Proposition (4.4):

If M is corational module, then M is Hollow module.

Proof:

Let N be a proper submodule of M, then N is corational submodule. Thus N is small submodule of M, [4]. Thus M is hollow.

The converse of this is not true, since Z_4 is Hollow module but is not corational module since Hom(Z_4,{0,2}/{0} ≠ 0).

The following remark follows from the definition.

Remark (4.5):

If M is corational module, then M is indecomposable module.

Proposition (4.6):

Let M be corational R-module, then $\frac{M}{L}$ is corational module for any L submodule of M.

Proof:

Let $\frac{N}{L}$ be submodule of R-module $\frac{M}{L}$ and K $\subseteq$ N. Let g:$\frac{M}{L} \to \frac{N/L}{K/L}$ be a homomorphism where $\frac{K}{L} \subseteq \frac{N}{L}$. It is known that there exists an isomorphism $: \frac{N/L}{K/L} \to \frac{N}{K}$. Let $:$ M$\to \frac{M}{L}$ since N $\subseteq$ M. Thus N is corational submodule, then $\psi \circ g \circ \pi = 0$, That is g$\circ$ (M) = 0, hence g = 0. Then $\frac{N}{L}$ is corational submodule of $\frac{M}{L}$, then $\frac{M}{L}$ is corational module.

Reference

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الخلاصة

لتكن R حلقة ابدالية احادية. في هذه الدراسة قمنا بتعريف المقاس الابتدائي المضاد ودراسة خواصه كذلك اعطي تعريف المقاس النسبي المضاد مع بعض خواصه.