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Weak Forms of L (θ-Generalized Closed) - Spaces

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Abstract

In this paper, we introduce some concepts namely θ-generalized L_i-spaces, where i=1,2,3,4, which are weaker forms of L(θ-generalized closed)-spaces, these are spaces whose Lindelof subsets are θ-generalized closed and study some of their properties and investigate their relationships with L(θ-generalized closed)- spaces as well as among themselves.

Keywords: Lindelof space, θ-closed sets, generalized-closed sets, θ-generalized closed sets.

Introduction

In 1968, Velick [2] introduced the concept of θ-closed sets in topological space. Levine [3] introduced the concept of generalized closed sets as a generalization of closed sets in topological spaces. Recently Dontchev and Maki [4] have introduced the concept of a θ- generalized closed set. This class of sets has been investigated also by Arockiarani [5].

Throughout this paper, a space X means topological space (X, τ) on which no separation axioms are assumed, unless explicitly stated. If A is a subset of a space X, then the closure and the interior of A are denoted by cl(A) and Int(A) respectively.

1. Preliminaries

In this section, we recall some basic definitions and example needed in this work.

Definition (1.1), [2]:

The θ-closure of A, denoted by $cl_\theta(A)$, is the set of all $x \in X$ for which every closed neighborhood of x intersects A nontrivially. A set A is called θ- closed if $A = cl_\theta(A)$.

Definition (1.2), [2]:

The θ-interior of A, denoted by $int_\theta(A)$, is the set of all $x \in X$ for which A contains a closed neighborhood of x. A set A is said to be θ-open provided that $A = int_\theta(A)$. Furthermore, the complement of a θ-closed set is θ-open and the complement of a θ-open set is θ-closed.

Definition (1.3), [3]:

A set A is called a generalized closed (or briefly g-closed) if $cl(A) \subseteq O$, whenever $A \subseteq O$, and O is open in X. The complement of generalized closed set is called generalized open set (or briefly g-open).

Definition (1.4), [4]:

A set A is said to be θ-generalized closed (or briefly θ-g closed) provided that $cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X. A set is called θ-generalized open (or briefly θ-g open) if its complement is θ-generalize closed.

From [4] it is easy to check that, every θ-closed set is θ-generalized closed and every θ-generalized closed set is g-closed. But the converse implication does not hold, see[4]. Also from[4] A countable union of θ-g closed sets need not be a θ-g closed set.

Definition (1.5):

Let A be a subset of X, then θ generalize-closure of A (or briefly $cl_{\theta g}(A)$) is the intersection of all θ-g closed sets which contain A, that is:

$cl_{\theta g}(A) = \cap \{F \subseteq X : F \text{ is } \theta\text{-gclosed}, A \subseteq F\}$.

So if A is θ-g closed, then $A = cl_{\theta g}(A)$.

Definition (1.6), [6]:

A set F in X is called Fσ-closed if it is the union of at most countably many closed sets.

Definition (1.7), [3]:

A space X is called $T_{1/2}$-space if every g- closed set is closed.

Definition (1.8), [7]:

A space X is called an Lc-space if every Lindelof subset of X is closed.

Definition (1.9), [1]:

A space X is called L (generalized-closed) (or briefly L (g-closed)) -space if every Lindelof subset of X is g-closed.

Definition (1.10), [1]:

A space is said to be L (θ-closed) -space if every Lindelof set in X is θ-closed.

Definition (1.11), [1]:

A space X is said to be L (θ-generalized closed) (or briefly L (θ-g closed))-space if every Lindelof subset of X is θ-g closed.

2. Generalization of L (θ-generalized closed) - spaces.

In this section, we give a generalization of L(θ-generalized closed)-spaces namely θ-generalized L_i (or briefly θ-gL_i), where i=1,2,3,4 and investigate their relationships with L(θ-g closed)- spaces as well as among themselves and study some of their properties.

Definition (2.1):

A set A is said to be Fσ-θ-generalized closed (or briefly Fσ-θ–g closed) if it is a countable union of θ-generalized closed sets. So every θ- g closed set is Fσ-θ- g closed set, but the converse is not true in general as in the following example.

Example (2.2):

Let X=R, be the real line with the usual topology. The sets G_n = $[\frac{1}{n},1]$ n=2,3,4,… are θ-closed sets which implies they are θ-g closed. But $G=\bigcup_{n=2}^{\infty}G_n=\bigcup_{n=2}^{\infty}[\frac{1}{n},1]=(0,1]$. Then G is Fσ-θ-g closed, but it is not closed, which implies it is not θ–closed.

To show that (0, 1] is not θ-g closed. For all $U\in\tau_U$ with $0\in U$, such that $U\cap(0,1]\neq\phi$ also $cl(U)\cap(0,1]\neq\phi$, which implies $cl_\theta(U)\cap(0,1]\neq\phi$, since $cl(U)\subset cl_\theta(U)$.

Now we introduce the following definitions.

Definitions (2.3):

A topological space X is called

1. θ-generalized L_1 (or briefly θ-gL_1) -space if every Lindelof Fσ-θ-g closed set is θ-g closed.
2. θ-generalized L_2 (or briefly θ-gL_2) -space if L is Lindelof subset of X, then $cl_{\theta g}(L)$ is Lindelof.
3. θ-generalized L_3 (or briefly θ-gL_3) -space if every Lindelof subset of X is an Fσ-θ- g closed.
4. θ-generalized L_4 (or briefly θ-gL_4) -space if L is Lindelof subset of X, then there is a Lindelof Fσ- θ-gclosed set F with $L\subseteq F\subseteq cl_{\theta g}(L)$.

Proposition (2.4):

If X is an L (θ-g closed) - space, then X is a θ-g L_i-space, where i=1,2,3,4.

Proof:

For i=1, let $K\subseteq X$ be a Lindelof Fσ-θ-g closed set. Since X is L(θ-g closed)- space, then K is θ-g closed. Therefore X is θg-L_1-space.

For i=2, let A be a Lindelof subset of X, then A is θ-g closed, that is, A= $cl_{\theta g}$ (A), so $cl_{\theta g}$ (A) is also Lindelof. Hence X is θg-L_2-space.

For i=3, let B be a Lindelof subset of X, then B is θ-g closed and so B is Fσ- θ-g closed set. Hence X is θ-g L_3-space.

For i=4, let L be a Lindelof subset of X, then L is θ-g closed, so it is Fσ- θ -g closed set. Therefore

$L\subseteq L\subseteq cl_{\theta g}(L)$ and this implies that X is θ-gL_4-space.

Proposition (2.5):

Every space which is θ-gL_1 and θ-gL_4 is θ-gL_2- space.

Proof:

Let A be Lindelof subset of θ-gL_1 and θ-gL_4 space X. Then there is a Lindelof Fσ-θ-g closed set F such that $A\subseteq F\subseteq cl_{\theta g}(A)$. Since X is θ-g$L_1$-space, then F is θ- g closed, that is, F= $cl_{\theta g}$ (F), but $A\subseteq F$, then $cl_{\theta g}(A)\subseteq cl_{\theta g}(F)$ =F, which implies that F = $cl_{\theta g}(A)$. But F is Lindelof so $cl_{\theta g}$ (A) is also Lindelof. Hence X is θ-gL_2- space.

Proposition (2.6):

Every θ -gL_3-space is θ-gL_4-space.

Proof:

Let A be Lindelof subset of X, then A is Fσ- θ-g closed set. Since $A\subseteq A\subseteq cl_{\theta g}(A)$.

Put A=F then, $A\subseteq F\subseteq cl_{\theta g}(A)$.Therefore X is θ-g$L_4$-space.

Definition (2.7):

A subset A of a space X is called θ-g dense if $cl_{\theta g}(A)=X$.

Proposition (2.8):

Every θ-gL_2-space having a θ-g dense Lindelof subset is Lindelof.

Proof:

Let K be a Lindelof θ-g dense subset of θ-g L_2-space X, so X= $cl_{\theta g}$ (K) is also Lindelof.

Definition (2.9):

Let A be a subset of X, a point $x \in A$ is said to be θ-generalized interior (or briefly θ-g int) point of A, if there exists a θ-g open set U such that $x \in U \subseteq A$. The set of all θ-g interior points of A is denoted by $int_{\theta g}$ (A). Also A is called θ-g open if A= $int_{\theta g}$ (A).

Definition (2.10):

A space X is said to be θ-gT_1 if for every distinct points x and y there are two θ-g open sets U and V such that $x \in U, y \notin U$ and $x \notin V, y \in V$.

Theorem (2.11):

A space X is θ-gT_1, if and only if every singleton set is θ-g closed.

Proof:

Assume every singleton subset {x} of X be θ-g closed. We have to show X is θ-gT_1.
Let x and y be two distinct points of X. But X-{x} is a θ-g open set which contains y but does not contain x, similarly X-{y} is a θ-g open set which contains x but not contain y. Hence X is θ-g T_1.

Conversely let X be θ-gT_1 and let x be any point of X. To show that X-{x} is a θ-g open, let $y \in X$-{x}, then $y \neq x$. Since X is θ-gT_1, so there exists a θ-g open U_y such that $y \in U_y$ and $x \notin U_y$. It follows that $y \in U_y \subseteq X - \{x\}$, that is y is an θ-g interior point of X-{x}. Hence X-{x} is θ-g open set, therefore, {x} is θ-g closed set.

Proposition (2.12):

Every θ-gL_3-space is θ-gT_1-space.

Proof:

Let X be θ-gL_3-space and $x \in X$, to prove X is θ-gT_1-space, it is sufficient to prove $\{x\}$ is θ–g closed set. Since $\{x\}$ is countable, then it is Lindelof in X. But X is θ-gL_3 space, then $\{x\}$ is Fσ-θ-gclosed set, that is, $\{x\} = \bigcup_{i \in I} U_i$, where U_i is θ-g closed for each $i \in I$ and I is a countable set, this implies $\{x\}$ is θ-g closed. Therefore X is θ-gT_1-space.

Definition (2.13):

A topological space X is said to be θ-gp-space if every Fσ- θ-g closed set is θ-g closed.

Remark (2.14):

Every θ- gp-space is a θ-gL_1-space.

Proposition(2.15):[4]

Let $A \subseteq Y \subseteq X$.

(i) if A is θ- g closed relative to Y, Y is θ - g closed and open subspace of X, then A is θ - g closed in X.

(ii) if A is θ-g closed in X, then A is θ - g closed relative to Y.

Theorem (2.16):

The property of space being θ-gL_3 is a hereditary property.

Proof:

Let X be a θ-g L_3 and Y is a subspace of X. To show Y is also θ-gL_3. Given L is a Lindelof subset of Y and so L is Lindelof subset of X, then L is Fσ- θ-g closed in X, that is, there exists a family $\{F_i\}_{i \in I}$ of θ- g closed sets in X such that $L = \bigcup_{i \in I} F_i$, where I is a countable set.

By setting $F_i^* = Y \cap F_i$, one can get F_i^* is θ-g closed in Y for each i.

$$L \subseteq L \bigcap Y = (\bigcup_{i \in I} F_i) \bigcap Y = \bigcup_{i \in I} (F_i \bigcap Y) = \bigcup_{i \in I} F_i^* .$$

So L is Fσ- θ-g closed in Y. Hence Y is θ-g L_3 -space.

Theorem (2.17):

The property of space being θ-g L_1 is a hereditary on an open and θ-gclosed set.

Proof:

Let Y be a θ-closed subspace of θ-gL_1 space X. To show that Y is θ-gL_1.
Suppose that L is a Lindelof Fσ-θ–g closed subset of Y, that is, there exists a family $\{F_i\}_{i \in I}$ of θ–g closed sets in Y, such that L=$\bigcup_{i \in I} F_i$. So F_i is θ-g closed set in X for each i (by proposition)

(2.15(i)). Hence $L=\bigcup_{i\in I} F_i$ is Lindelof Fσ-θ-g closed in X, Since X is θ-gL_1, then L is θ-g closed in X. But $L \subseteq Y \subseteq X$, then L is θ-g closed in Y by proposition (2.15(ii)). Hence Y is θ-g L_1.

Definition (2.18):

Let A be a subset of space X. A point $x \in X$ is said to be θ-g adherent point of A if $cl_\theta(U)\bigcap A \neq \phi$, where U is open set containing x. The set of all θ-g adherent points of A is θ-generalize- closure of A.

Proposition (2.19)

Let Y be open subset of X and $K \subseteq Y$, then $cl_{\theta g}(K)_{inY} = cl_{\theta g}(K)_{inX}\bigcap Y$.

Proof:

Clearly:

$cl_{\theta g}(K)_{inY} \subseteq cl_{\theta g}(K)_{inX}\bigcap Y$ .... (1)

Now, to show that:

$cl_{\theta g}(K)_{inX}\bigcap Y \subseteq cl_{\theta g}(K)_{inY}$ .If $x \in cl_{\theta g}(K)_{inX}\bigcap Y$, then $x \in cl_{\theta g}(K)_{inX}$ and $x \in Y$. If $x \in cl_{\theta g}(K)_{inX}$, then $cl_\theta(U)_{inX}\bigcap K \neq \phi$ where U is open subset of X and $x \in U$, but $K = Y \bigcap K$.

So $cl_\theta(U)_{inX}\bigcap Y)\bigcap\ K \neq \phi$, but $cl_\theta(U)_{inX}\bigcap Y = cl_\theta(U)_{inY}$ (since if Y be open subset of X and $K \subseteq Y$, then $cl_\theta(K)_{inY} = cl_\theta(K)_{inX}\bigcap Y$ [4]). Hence $cl_\theta(U)_{inY}\bigcap K \neq \phi$, which implies

$x \in cl_{\theta g}(K)_{inY}$ $cl_{\theta g}(K)_{inX}\bigcap Y \subseteq cl_{\theta g}(K)_{inY}$ .... (2)

From (1) and (2) we get:

$cl_{\theta g}(K)_{inY} = cl_{\theta g}(K)_{inX}\bigcap Y$.

Lemma (2.20), [4]:

A space X is $T_{1/2}$ if and only if every θ-g closed set is closed.

Theorem (2.21):

Let X be a Lindelof, $T_{1/2}$ and θ-gL_2- space. Then any closed and open subspace of X is also θ-gL_2.

Proof:

Suppose Y is a closed, and open subspace of X. If K is Lindelof set in Y, so it is Lindelof in X, which is θ-gL_2-space, then $cl_{\theta g}$ (K) in X is Lindelof in X. Also $cl_{\theta g}$ (K) is closed in X, since X is $T_{1/2}$, so $cl_{\theta g}(K)_{in X} \bigcap Y$ is closed in Y. Since:

$cl_{\theta g}(K)_{inY} = cl_{\theta g}(K)_{inX}\bigcap Y$

by proposition(2.19), then $cl_{\theta g}(K)_{in Y}$ is closed in Y, But Y is Lindelof, hence $cl_{\theta g}(K)_{in Y}$ Lindelof in Y. Therefore Y is θ-gL_2 space.

Theorem (2.22):

The property θ-gL_4 is hereditary on θ-closed property.

Proof:

Let Y be a θ-closed subspace of θ -gL_4-space X, so Y be a closed subspace of X, since every θ-closed set is closed. And L be a Lindelof in Y, then L is Lindelof in X, which is θ-g L_4-space, then there exists Lindelof Fσ-θ- g closed set F in X such that $F = \bigcup_{j\in I} T_j$, where T_j is θ-g closed in X with $L \subseteq F \subseteq cl_{\theta g}(L)_{in X}$.

Set $K = Y\bigcap F$, Y is closed in X, then K is closed in F, but F is Lindelof in X, so K is Lindelof in F and so it is Lindelof in Y. To show that K is Fσ-θ-g closed in Y. Since $K=\bigcup_{j\in I}(T_j\bigcap Y)$, and $T_j\bigcap Y$ is θ-g closed in X, since the intersection of a θ-g closed set and a θ-closed set is always θ-g closed [4]. Hence $T_j\bigcap Y$ is θ-g closed in Y by proposition (2.15(ii)). Therefore K is Fσ-θ-g closed in Y, but $L=L\bigcap Y \subset K \subset cl_{\theta g}(L)_{in X} \bigcap Y = cl_{\theta g}(L)_{in Y}$.

Therefore Y is θ-gL_4.

Theorem (2.23):

Let X be a T_2 space, then X is L(θ-g closed) –space if and only if it is θ-gL_1 -space and θ-gL_2 -space.

Proof:

Assume if X is an L (θ-g closed), then it is θ-gL_1 and θ-gL_2 by definition.

Conversely, suppose L be a Lindelof set in X and $x \notin L$. But X is T_2, so for each $y \in L$, there exists an open set V_y containing y with $x \notin cl(V_y)$. Therefore $L \subseteq \bigcup\{V_y : y \in L\}$, but L is Lindelof, then there exists a countable set $C \subseteq L$, such that:

$L \subseteq \bigcup\{V_y : y \in C\} \subseteq \bigcup\{cl(V_y) : y \in C\}$.

For each $y \in C$, $cl(V_y)$ is closed so $L \cap cl(V_y)$ is closed in L, but L is Lindelof in X, then $L \cap cl(V_y)$ is Lindelof in L, which implies it is Lindelof in X . But X is θ-gL_2 , then $cl_{\theta g}(L \cap cl(V_y))$ is Lindelof.

Now if $W = \bigcup\{cl_{\theta g}(L \cap cl(V_y)) : y \in C\}$. Then W is a countable union of Lindelof sets which implies it is also Lindelof, also W is a countable union of θg-closed sets which implies it is Fσ-θ-g closed. But X is θ-gL_1- space, then W is θ-g closed. But $x \notin L \cap cl(V_y)$, $x \notin \{cl_{\theta g}(L \cap cl(V_y) : y \in C\}$,

that is, $x \notin W$.

Since $L \subseteq \bigcup\{cl(V_y) : y \in C\}$,

so $L \subseteq \bigcup\{L \cap cl(V_y) : y \in C\}$,

$cl_{\theta g}(L) \subseteq \bigcup\{cl_{\theta g}(L \cap cl(V_y) : y \in C\} = W$.

Therefore $x \notin cl_{\theta g}(L)$, that is x is not θ-g adherent point of L. Hence L is θ-gclosed**.**

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الخلاصة

في هذا البحث قدمنا بعض المفاهيم والتي تسمى الفضاءات θ-gL_i حيث ان i=1,2,3,4 والذي هي فضاءات أضعف من الفضاءات L(θ-g closed) هذه الفضاءات التي فيها المجاميع الجزئيه اللندلوفيه تكون المجاميع θ-g closed.

ثم درسنا بعض من خصائص وصفات الفضاءات θ-gL_i والعلاقات فيما بينها وبين الفضاءات L(θ-g closed).