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ON g, g^*, g^{**}-COMPACT FUNCTIONS

Haider J. Ali and Arkan J. Mohammed
Department of Mathematics, College of Science, Al-Mustansiriah University, Baghdad-Iraq.

Abstract

In this paper we introduce and study g , g^*, g^{**}- compact functions and we study the relation of compact functions with this types of the functions. Finally, we study further theorems and properties on g , g^*, g^{**}- compact functions.

Keywords: g- Compact set, Generalized closed set, Compact function.

1-Introduction

The objective of present paper is to introduce certain classes of sets namely g-compact sets and certain types of g, g^*, g^{**}-compact functions. Various properties of such functions have been discussed.

A space X means a topological spaces (X,τ) on which no separation axioms are assumed, unless explicitly stated. The interior and the closure of any subset A of X will be denoted by Int(A) and cl(A) respectively. A function $f:X\rightarrow Y$ is said to be a compact if $f^{-1}(K)$ is a compact subset of X, whenever K is a compact subset of Y [1]. A subset F of a space X is called generalized closed (briefly g-closed) if cl(F)$\subseteq$O , whenever F$\subseteq$O , and O is open in X [2]. Also, a subset O of a space X is said to be generalized open (briefly g-open) if O^c is g-closed set. It easy to show that every closed (open) set is g-closed (g-open). The author in [3] introduced the following definitions:

A function $f:X\rightarrow Y$ is said to be g-closed if f (F) is g-closed subset of Y ,whenever F is closed subset of X, and is said to be g^*-closed if f (F) is closed subset of Y ,whenever F is g- closed subset of X, also is said to be g^{**} -closed if f (F) is g-closed subset of Y, whenever F is g- closed subset of X . Also f is said to be g^{**}- continuous if $f^{-1}(F)$ is g- closed (g-open) whenever F is g-closed (g-open) subset of Y. Also, a subset O of X is g-open if and only if $F\subseteq O^o$ for every closed set $F\subset O$.

2-Certain Types of Compact Functions:

Definition (2.1), [4]:

A subset K of a space X is said to be generalized compact, (briefly g-compact) if for every g-open cover of K has a finite subcover.

Every g-compact set is compact, but the converse is not true in general as in the following example illustrate.

Example (2.2):

Let R be the real line , N be the subset of R and $\zeta=\{U\subseteq R \mid U=R \text{ or } U\cap N=\phi\}$.It is clear that (R,ζ) is a topological space.

Put $U_i=N^c\cup\{i\}=\{R-N\}\cup\{i\}$, $i=1,2,....$

U_i is not open subset of R ,where $i\in N$. Since $U_i\cap N=\{i\}$, $i=1,2,....$

Now to show that U_i is g- open subset of R. Since the only closed subset of R which is contained in U_i is ϕ, and so $\phi\subset U^0$ this implies to U_i is g-open for each $i=1,2,....$ Hence the family $\{U_i\}_{i=1}^{\infty}$ forms a g- open cover to R; that is, $\bigcup_{i=1}^{\infty}U_i=\bigcup_{i=1}^{\infty}(\{R-N\}\bigcup\{i\})=R$, but this cover can not reducible into finite subcover. Therefore R is not g – compact.

To show that R is compact. Since the only open set which is cover N is U=R and so every open cover to R must be contains U=R. This means every open cover to R ,we can choose finite subfamily {R} cover to R. Therefore R is compact.

Now we introduce the following definitions:

Definition (2.3):

A function f :X$\rightarrow$Y is said to be g-compact if $f^{-1}(K)$ is g-compact subset in X, whenever K is a compact subset in Y.

Every g-compact function is compact, but the converse is not true in general as in the following example illustrate.

Example (2.4):

The identity function $I_R : (R,\zeta) \rightarrow (R,\zeta)$ is compact but not g-compact function.
The compact sets in R is the set that contain N. Since any covering to this sets must be contain R, then we can choose {R} to cover this sets. Hence I_R is compact function, since for any compact subset K of R , $I_R^{-1}(K) = K$.

But I_R is not g- compact function, since R is compact but $I_R^{-1}(R) = R$ is not g-compact (example (2.2)) .

Definition (2.5):

A function f :X$\rightarrow$Y is said to be g^*-compact
if $f^{-1}(K)$ is compact subset in X, whenever K is g- compact subset in Y.

Every compact function is g^*- compact but the converse is not true in general as in the following example illustrate.

Example (2.6):

Let $I:(R,\tau_D) \rightarrow (R,\zeta)$, where I be the identity function. I is g^*-compact function since the g-compact sets in ζ are the only finite sets which their inverse images are compact sets in τ_D. But I is not compact function since (R, ζ) is compact (example(2.2))hence $I^{-1}(R) = R$, but (R,τ_D) is not compact, which implies I is not compact function.

Definition (2.7):

A function f :X$\rightarrow$Y is said to be g^{**}- compact if $f^{-1}(K)$ is g- compact subset in X, whenever K is g- compact subset in Y.

Every g-compact function is g^{**}- compact, which they g^*- compact, but the converses is not true in general.

Definition (2.8), [5]:

A function f :X$\rightarrow$Y is said to be point inversely compact (briefly p.i.compact) if $f^{-1}(y)$ is compact subset in X, for every $y \in Y$.

We introduce the following concept.

Definition (2.9):

A function f :X$\rightarrow$Y is said to be point inversely generalized compact (briefly p.i.g-compact) if $f^{-1}(y)$ is g- compact subset in X, for every $y \in Y$.

Remarks (2.10):

Every compact function is p.i.compact but the converse is not true in general, every p.i.g-compact function is p.i.compact but the converse is not true in general and every g-compact function is p.i.compact and p.i.g-compact but the converse is not true in general.

Example (2.11):

Let I_R: (R, τ_D) $\rightarrow$ (R, τ_u),where I_R (x)= x, for all $x \in R$.

To show that I_R is p.i.compact and p.i.g-compact. Since $I_R^{-1}(\{x\}) = \{x\}$, for all $x \in R$ and every finite set in every topological space is compact and g- compact.
But it is clear [0,1] is compact in usual topology, while $I_R^{-1}([0,1]) = [0,1]$ is not compact in the discrete topology and hence it is not g-compact.
Therefore I_R is neither compact nor g- compact function

Theorem (2.12):

Let f :X$\rightarrow$Y be g-closed function and p.i.compact, then f is g^*-compact.

Proof:

Let K be g- compact subset of Y and $\{U_\alpha\}_{\alpha\in\Omega}$ be an open cover of $f^{-1}(K)$, where Ω is the index set. T be a family of all finite subset of Ω set. $U_t = \bigcup_{\alpha\in t} U_\alpha$, where $t \in T$ .Since f is p.i.compact, for every $k \in K$, implies $f^{-1}(\{k\})$ is compact set and contained in U_t ,where $t \in T$ .Hence $K \in Y - f(X - U_t)$.
So $K \subset \bigcup_{t\in T}(Y - f(X - U_t))$.

But $Y-f(X-U_t)$ is g-open, then there exist $t_1, t_2, \ldots, t_n \in T$ s.t $K \subset \bigcup_{i=1}^{n}(Y-f(X-U_{t_i}))$ (since K is g-compact) and so

$$f^{-1}(K) \subset \bigcup_{i=1}^{n} f^{-1}(Y-f(X-U_{t_i}))$$
$$=\bigcup_{i=1}^{n}(X-f^{-1}f(X-U_{t_i})),$$
since $f^{-1}(Y)=X$
$$\subset \bigcup_{i=1}^{n}(X-(X-U_{t_i}))$$
$$=\bigcup_{i=1}^{n}U_{t_i}=\bigcup_{\alpha\in\Omega_0}U_\alpha,$$

where $\Omega_0 = t_1 \cup t_2 \cup \ldots \cup t_n$.

So $f^{-1}(K)$ is compact in X. Therefore f is g^*-compact function.

Theorem (2.13):

Every g- closed subset of g-compact space is g- compact.

Proof:

Let K be a g- closed subset of g-compact space X. Let $\{G_\alpha\}_{\alpha\in\Omega}$ be a g-open cover of K, that is; $K \subset \bigcup_{\alpha\in\Omega} G_\alpha$. But X –K is g-open so, $X=(X-K)\cup(\bigcup_{\alpha\in\Omega}G_\alpha)$. Since X is g- compact then $X=(X-K)\cup(\bigcup_{i=1}^{n}G_{\alpha_i})$,and $K \subset (\bigcup_{i=1}^{n}G_{\alpha_i})$.Therefore K is g- compact.

Remark (2.14):

Every finite set is g- compact.

Theorem (2.15):

A continuous function from g-compact space into T_2 space is g- closed.

Proof:

Let F be a closed subset of X which is g-compact, so X is compact. Then F is compact in X.

Since f is continuous function, then f (F) is compact in Y which is T_2 –space, then f (F) is closed, so it is g-closed in Y.

Therefore f is g-closed function.

Theorem (2.16):

Let f :X→Y be g-compact function and A is a closed subset of X , then $f|A: A\to Y$ is also g- compact.

Proof:

Let K be a compact subset of Y, then $f^{-1}(K)$ is g- compact subset in X, but A is closed in X, so $A \cap f^{-1}(K)$ is closed in $f^{-1}(K)$. Hence it is g-closed in $f^{-1}(K)$.

Therefore by theorem (2.13) $A \cap f^{-1}(K)$ is g-compact. But $f^{-1}|_A(K)=A\cap f^{-1}(K)$, then $f|A$ is g- compact.

Definition (2.17),[5]:

Let f :X→Y be a function and T be a subset of Y ,we define $f_T : f^{-1}(T)\to T$ by: $f_T(x)=f(x)$,for all $x\in f^{-1}(T)$.

Theorem (2.18):

If f :X→Y is g-compact continuous function and T is closed subset of Y, then $f_T : f^{-1}(T)\to T$ is also g- compact.

Proof:

Let G be a compact subset of T, then it is compact in Y and so $f^{-1}(G)$ is g-compact in X. Since $f^{-1}(T)$ is closed in X, then $f^{-1}(T)\cap f^{-1}(G)$ is closed in $f^{-1}(G)$, which implies it is g-closed, then by theorem (2.13) $f^{-1}(T)\cap f^{-1}(G)$ is g-compact. But $f_T^{-1}(G)=f^{-1}(T)\cap f^{-1}(G)$; that is, f_T is g-compact.

Theorem (2.19):

Let f :X→Y be bijective function. Then the g^{**}- continuous image of g-compact set is g-compact.

Proof:

Let K be a g- compact subset of X, and $\{V_\alpha\}_{\alpha\in\Omega}$ be a g- open cover of f (K);that is, $f(K)=\bigcup_{\alpha\in\Omega}V_\alpha$.

So K= $f^{-1}f(K)=f^{-1}\bigcup_{\alpha\in\Omega}V_\alpha=\bigcup_{\alpha\in\Omega}f^{-1}(V_\alpha)$.

Since f is g**-continuous, then $f^{-1}(V_\alpha)$ is g-open set for all $\alpha \in \Omega$, so $\{f^{-1}(V_\alpha)\}_{\alpha\in\Omega}$ is a g-open cover of K ,which is g- compact , so

$K = \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$,then

$$f(K) = f(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})) = \bigcup_{i=1}^{n} ff^{-1}(V_{\alpha_i})$$

$$= \bigcup_{i=1}^{n} V_{\alpha_i} .$$

Therefore *f*(K) is g-compact.

Theorem (2.20):

Let f_1 :X→Y and f_2 :Y→Z be functions then

1. If f_1 is g-compact and f_2 is g *- compact, then $f_2 \circ f_1$ is g **- compact.
2. If f_1 is g*-compact and f_2 is g- compact, then $f_2 \circ f_1$ is compact.
3. If f_1 is g*-compact and f_2 is g **- compact then, $f_2 \circ f_1$ is g* -compact.
4. If f_1 and f_2 is g**- compact, then $f_2 \circ f_1$ is g** -compact.
5. If f_1 is g-compact and f_2 is compact, then $f_2 \circ f_1$ is g- compact.
6. If f_1 is compact and f_2 is g*- compact, then $f_2 \circ f_1$ is g*compact.
7. If f_1 is g** -compact and f_2 is g – compact, then $f_2 \circ f_1$ is g- compact.

Proofs:

1.Let K be a g- compact subset of Z , then $f_2^{-1}(K)$ is compact subset of Y , so $f_1^{-1}(f_2^{-1}(K))$ is g-compact in X. But $f_1^{-1}(f_2^{-1}(K)) = f_1^{-1} \circ f_2^{-1}(K) = (f_2 \circ f_1)^{-1}(K)$.

Therefore $f_2 \circ f_1$ is g**- compact.

In the same way we can prove the others.

Theorem (2.21):

Let f_1 :X→Y and f_2 :Y→Z be functions then

1. If $g \circ f$ is g- compact and f is surjective g**-continuous function , then g is g- compact.
2. If $g \circ f$ is g*-compact and g is one to one g**-continuous function , then f is g*- compact.
3. If $g \circ f$ is g**-compact and g is one to one g**-continuous function , then f is g**- compact.

Proofs:

1.Let M be a compact subset of Z, then $(g \circ f)^{-1}(M)$ is g-compact in X, so $f(g \circ f)^{-1}(M)$ is g-compact in Y. also

$f(g \circ f)^{-1}(M) = f(f^{-1} \circ g^{-1})(M) =$

$f(f^{-1}(g^{-1}(M))) = g^{-1}(M)$.

Therefore g is g- compact.

2. Let M be g- compact subset of Y, then by theorem (2.19) we obtain $g(M)$ is g-compact in Z.

Hence $(g \circ f)^{-1}(g(M))$ is compact in X. But

$(g \circ f)^{-1}(g(M)) = (f^{-1} \circ g^{-1})(g(M)) =$

$f^{-1}(g^{-1}(g(M)) = f^{-1}(M)$.

Therefore f is g*- compact.

In the same way we can prove (3).

Theorem (2.22):

If A is a closed subset of a space X, then the inclusion function of A is g**-compact.

Proof:

Let $i: A \to X$ be an inclusion function and let K be a g-compact subset of X.

Since $i^{-1}(K) = A \cap K$ is closed in K , so it is g- closed. Hence $A \cap K$ is g -compact that is, $i^{-1}(K)$ is g-compact.

Theorem (2.23):

Let $f: X \to Y$ be a homeomorphism then if K is a g-compact subset of X so f (K) is also g-compact.

Proof:

Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a g-open cover of f (K) and let F be closed subset of $f^{-1}(V_{\alpha_0})$, for some $\alpha_0 \in \Lambda$, $f(F) \subset V_{\alpha_0}$ which is g-open set. Then $f(F) \subset V_\alpha^0$, so $F \subset f^{-1}(V_\alpha^0) = (f^{-1}(V_\alpha))^0$.

So $f^{-1}(V_\alpha)$ is a g-open set, but $K = \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$

so $K = \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$, implies

$$f(K) = f(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})) = \bigcup_{i=1}^{n} ff^{-1}(V_{\alpha_i}) = \bigcup_{i=1}^{n} V_{\alpha_i} .$$

Therefore $f(K)$ is g-compact set.

Theorem (2.24):

Let $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ be functions, then if

$f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is g^*-compact then, f_1 and f_2 are also g*- compact.

Proof:

To prove f_1 is g^*- compact.

Let K be g- compact subset of Y_1 , also $\{y_2\}$ is a g-compact subset of Y_2 ,where $y_2 \in Y_2$.

But $K \times \{y_2\} \cong K$ (by theorem 2.23), which is g-compact.

$(f_1 \times f_2)^{-1}(K \times \{y_2\})$ is compact subset of $X_1 \times X_2$, but

$$(f_1 \times f_2)^{-1}(K \times \{y_2\}) = (f^{-1}{}_1 \times f_2^{-1})(K \times \{y_2\})$$
$$= (f_1^{-1}(K) \times f_2^{-1}(\{y_2\}).$$

Hence $f_1^{-1}(K)$ is compact subset of X_1, therefore f_1 is g^*- compact.

In the same way, we can prove f_2 is g^*-compact.

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الخلاصة

في هذا البحث قدمنا ودرسنا الدوال المتراصة – g, g*, g** وكذلك درسنا العلاقة بين الدوال المتراصة مع هذه الانماط من الدوال المتراصة.وأخيراً درسنا مبرهنات وخصائص تلك الدوال.