ARTIN EXPONENT OF $U(4,\mathbb{Z}_p)$ USING BRAUER COEFFICIENT THEOREM

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Abstract

In this paper, we consider the Artin exponent of the Groups of unitriangular matrices $U(n,F)$ from the principal character of its cyclic subgroups and denoted by $A(U(n,F))$ where $n=4$, $F=\mathbb{Z}_p$, p is prime number, and we found that $A\left(U(4,\mathbb{Z}_p)\right)=p^3$.Furthermore, we found that, the order of this group $|U(4,\mathbb{Z}_p)|=p^4$, its exponent, $exp\ (U(4,\mathbb{Z}_p))=p$ and found general forms of all conjugacy classes

Introduction

Let G be a finite group and let f be an integral valued class function on G, Artin induction theorem[6] states that $|G|f$ is an integral linear combination of characters of G induced from characters of principle representations of cyclic subgroups of G.

In (1968), Lam [6] proved a sharp form of Artins theorem, he determined the least positive integer $A(G)$ such that $A(G)\chi$ is an integral linear combination of induced principal characters of cyclic subgroups for all rational valued characters χ of G.

In this work, the group G under consideration is Groups of unitriangular matrices $U(n,F)$, where $n=4$and $F=\mathbb{Z}_p$, p is prime number, The main results will be stated in section 2, as follows : in theorem(2.8) we give the general forms of all conjugacy classes of G, Furthermore, we found the order of G and its exponent in theorem(2.3) and theorem(2.4) respectively.

1-Basic Concepts and Theorems

In this section we will introduce the basic notations and definitions for the later work.

Definition(1.1), [8]:

Let F be a field. Then the general linear group $GL(n,F)$ is the group of all invertible (n × n) matrices with entries in F under matrix multiplication .

Definition(1.2), [5]:

Let V be a vector space over any field F, GL(V) denotes the group of all linear isomorphism of V onto itself.

Definition(1.3), [1]:

A representation of a group G is a homomorphism T : G ⟶GL(V).

Definition(1.4), [1]:

A matrix representation of a group G is a homomorphism T : G ⟶GL(n,F), where n is called the degree of the matrix representation.

Definition(1.5), [4]:

A representation T : $G\longrightarrow GL(1,\mathbb{C})$ such that T(x)=1 , $\forall\ x\in G$, it is called the linear representation or principle representation of G.

Definition(1.6), [2]:

A class function on a group G is a function $f:G\longrightarrow\mathbb{C}$ which is constant on conjugacy classes ,that is, $f(x^{-1}yx)=f(y)\quad\forall\ x,y\in G$.

If all value of f are in $\mathbb{Z}$, then it is called $\mathbb{Z}-$ valued class function.

Definition(1.7), [3]:

Let T be a matrix representation of a finite group G over a field F, the character χ of T is the mapping $\chi:G\longrightarrow F$ defined by $\chi(g)=tr\left(T(g)\right)$,$\forall\ g\in G$, where $tr\left(T(g)\right)$ refers to the trace of the matrix $T(g)$.

Clearly, $\chi(1)=n$, which is called the degree of χ ,also character of degree 1 is called linear character.

Definition(1.8), [3]:

The function 1_G with constant value 1 on G, is a linear character, it is called the principle or unit character of G.

Lemma(1.9):

Characters of a group G are class functions on G. Proof:see[3].

Definition(1.10),[4]:

Let G be a finite group and let $H \leq G$. Then the normalize of H in G :

$N_G(H)=\{x \in G | xHx^{-1} = H\}$.

Lemma(1.11):

Let G be a finite group, and let $h \in G$. Then the number of elements in the conjugacy class of h is equal to the index $[G:C_G(h)]$ of the centralizer $C_G(h)$ of h in G.

Proof:see [7].

Lemma(1.12):

Let χ be a rational valued character of G , then, for all $g \in G$, $\chi(g) \in \mathbb{Z}$.

Proof:see [3].

Lemma(1.13):

Let χ be a rational valued character of G, and let $x, y \in G$ with $\langle x \rangle = \langle y \rangle$,

Then $\chi(x) = \chi(y)$.

Proof:see[3].

Definition(1.14), [3]:

Let H be a subgroup of a group G and ψ be a class function of H, then $\psi \uparrow^G$, the induced class function on G is given by :

$\psi \uparrow^G (g) = \frac{1}{|H|}\sum_{x\in G}\psi^{\circ}(xgx^{-1})$

Where $\begin{cases} \psi^{\circ}(h) = \psi(h) & if \ h \in H \\ \psi^{\circ}(h) = 0 & if \ h \notin H \end{cases}$

Clearly $\psi \uparrow^G$ is a class function on G and $\psi \uparrow^G (1) = [G:H]\psi(1)$.

Another useful formula for computing $\psi \uparrow^G (y)$ explicitly is to choose representatives $x_1, x_2, \ldots, x_m$ for the m classes of H contained in the conjugacy class C_y in G which is given by

$\psi \uparrow^G (y) = \frac{|C_G(y)|}{|C_H(x_i)|}\sum_{i=1}^{m}\psi(x_i)$ .....(1-1)

Where $\psi \uparrow^G (y) = 0$ if $H \cap C_y = \emptyset$. This formula is immediate from the definition of $\psi \uparrow^G$ since as x runs over G , $xyx^{-1} = x_i$ for exactly $|C_G(y)|$ values of x.

Proposition(1.15):

Let H be a subgroup of G, and ψ to be a character of H, then $\psi \uparrow^G$ is a character.

Definition(1.16), [6]:

The character induced from the unit character of a cyclic subgroups of G is called *Artin character*, and denoted by $\psi(x)$

Example(1.17):

The three conjugacy classes of the symmetric group S_3 are

$C_{(1)} = (1)$, $C_{(12)} = \{(12), (13), (23)\}$ and $C_{(123)} = \{(123), (132)\}$, We calculate the Artin characters (induced characters) of S_3 from the unit characters of the cyclic subgroups H_i , i=1,2,3 by using formula (1-1)

The orders of the three classes are $|C_{(1)}| = 1$, $|C_{(12)}| = 3$, $|C_{(123)}| = 2$

and the orders of the centralizers are $|C_{S_3}(1)| = 6$, $|C_{S_3}(12)| = 2$, $|C_{S_3}(123)| = 3$

Thus

1) (1^3) : $1_{H_1} \uparrow^{S_3} (1) = \frac{6}{1}\sum 1 = 6$, $1_{H_1} \uparrow^{S_3} (12) = 0$ and $1_{H_1} \uparrow^{S_3} (123) = 0$

$\psi_1(x) = (6 \quad 0 \quad 0)$ Since, $(1) \notin C_{(12)}$ and $(1) \notin C_{(123)}$

2)(12): $1_{H_2} \uparrow^{S_3} (1) = \frac{6}{2}\sum 1 = 3$, $1_{H_2} \uparrow^{S_3} (12) = \frac{2}{2}\sum 1 = 1$, and $1_{H_2} \uparrow^{S_3} (123) = 0$

$\psi_2(x) = (3 \quad 1 \quad 0)$ Since, $\langle (12) \rangle \cap C_{(123)} = \emptyset$

3)(123): $1_{H_3} \uparrow^{S_3} (1) = \frac{6}{3}\sum 1 = 2$, $1_{H_3} \uparrow^{S_3} (12) = 0$

and $1_{H_3} \uparrow^{S_3} (123) = \frac{3}{3}\sum 1 + 1 = 2$

$\psi_3(x) = (2 \quad 0 \quad 2)$

Since, $\langle (123) \rangle \cap C_{(12)} = \emptyset$.

Table (1-1)

Artin characters of S_3.

C_g	(1^3)	(12)	(123)
$\lvert C_g \rvert$	1	3	2
$\lvert C_{S_3}(g) \rvert$	6	2	3
ψ_1	6	0	0
ψ_2	3	1	0
ψ_3	2	0	2

Definition(1.18), [6]:

The Artin exponent, $A(G)$, of a group G is the smallest positive integer $A(G)$ such that $A(G)\psi$ is an integer linear combination of the induced principle characters of the cyclic subgroups of G, for all rational valued characters ψ of G.

Remark(1.19), [6]:

Let $H_1 = \{1\}, H_2, \ldots, H_q$ be the full set of non-conjugate cyclic subgroups of G. We write 1_j, for the principle character on H_j and denote the Artin character (induced character) on G by ψ_j , which is the character afforded by the rational representation of G and it is clearly depends only on the conjugacy class of the cyclic subgroup H_j.

Definition(1.20), [6]:

Let G be a finite group, an integer $m \in \mathbb{Z}$ is said to be an Artin exponent for G if, given any rational character χ on G such that $:m\chi = \sum_{k=1}^{q} a_k \psi_k$ is solvable for integer unknowns $a_k \in \mathbb{Z}$ and for any given rational character χ on G.

Remark(1.21), [6]:

All Artin exponents form an ideal in the integers and [G:1] is in this ideal We pick the (unique) positive generator $A(G)$ for this ideal and we shall call it the Artin exponent of G, $A(G)$ divides $|G|$.

Proposition(1.22):

Let 1_G denote the principal character of G and $d \in \mathbb{Z}$, then d is an Artin exponent of G if it has the following property:

There exist (unique) integers $a_k \in \mathbb{Z}$ such that $d.1_G = \sum_{k=1}^{q} a_k \psi_k$

Where $\psi_1, \psi_2, \ldots, \psi_q$ are the Artin characters.

If, $a_1, a_2, \ldots, a_q$ have no common factor, then $d = A(G)$ and conversely.

Proof : see[6].

Proposition(1.23):

Let G be an arbitrary finite group, and $H = \{H_1, H_2, \ldots, H_q\}$ be a full set of non -conjugate cyclic subgroups of G, then $A(G)$ is the smallest positive integer m such that:

$m.1_G = \sum_{H_k \in H} a_k .1_{H_k} \uparrow^G$..............(1-2)

With each $a_k \in \mathbb{Z}$.

Proof:see[6].

Remark(1.24), [6]:

1) If m is a positive integer , and (1-2) holds for some set of integers $\{a_k\}$ with greatest common divisor =1, then necessarily $m = A(G)$.
2) Given a group G, We can compute the characters $\{1_{H_k} \uparrow^G\}$ explicitly, and then use proposition (1.22) to determine $A(G)$.

Theorem(1.25):

$A(G) = 1$ iff G is cyclic.

Proof:see[6].

Remark(1.26), [6]:

$A(G)$ gives an interesting numerical measure of the deviation of G from being a cyclic group. The invariant $A(G)$ is, therefore, merely a measure of noncyclicity.

Example(1.27):

Consider $G = S_3$, Let $H = \{H_1, H_2, H_3\}$ with H_i cyclic subgroups of order i.According to example(1.17) and its table, if we multiply ψ_1 by -1 , ψ_2 by 2 , and ψ_3 by 1,

then we have :

$2.1_{S_3} = -(1_{H_1} \uparrow^{S_3}) + 2(1_{H_2} \uparrow^{S_3}) + (1_{H_3} \uparrow^{S_3})$

and therefore $A(S_3) = 2$.

Definition(1.28), [4]:

Let G be a group, then the exponent of G is the least common multiple of the orders of its elements, and denoted by exp(G)

Definition(1.29):

$$For\ n \in \mathbb{Z}^+, \mu(n) = \begin{cases} & \text{If } n=1 \\ & \text{If n is not square free} \\ & \text{If } n=p_1.p_2\ldots p_r \text{ where the } p_i \text{ are distinct primes.} \end{cases}$$

This function is called the Mobius function Then $\mu(n_1 n_2) = \mu(n_1)\mu(n_2), if (n_1, n_2) = 1.$

Theorem(1.30): [Brauer Coefficient Theorem]

For any finite group G

$\mathbf{1} = \sum_{j=1}^{q} b_j \, 1_{c_j} \uparrow^G$, where $b_j = \frac{1}{[N(c_j):c_j]} \sum_{c \supseteq c_j} \mu([c:c_j])$,

The summation being over all cyclic subgroups c of G over c_j.

2- Artin Exponent of $U(4,\mathbb{Z}_p)$

This section concerns with some members of an important class of groups; the finite linear groups, groups of unitriangular matrices $U(n,F)$, with n=4 and $F = \mathbb{Z}_p$, p is prime number. After describing important features of groups and investigating their conjugacy classes we move on to evaluate its Artin Exponent.

Definition(2.1), [8]:

Let $U(n,F) = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ be the group of $n \times n$ upper unitriangular matrices with entries in F under matrix multiplication, that is, $U(n,F)$ consists of matrices such that $x_{ij} = 0$ for all $i > j$ and $x_{ii} = 1$ for all i.

$U(n,F)$ is a subgroup of $GL(n,F)$

In this work we interested in the group $U(4,\mathbb{Z}_p) = \left\{ \begin{bmatrix} 1 & g_1 & g_2 & g_3 \\ 0 & 1 & g_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; g_1, g_2, g_3, g_4 \in \mathbb{Z}_p \right\}$, where p is prime number.

Theorem(2.2):

The order of the group $U(4,\mathbb{Z}_p)$ is $|U(4,\mathbb{Z}_p)| = p^4$

Proof 1:

$$U(4,\mathbb{Z}_p) = \left\{ \begin{bmatrix} 1 & g_1 & g_2 & g_3 \\ 0 & 1 & g_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; g_1, g_2, g_3, g_4 \in \mathbb{Z}_p \right\}$$

Order of the group $U(4,\mathbb{Z}_p)$ depending on choices number of $g_1, g_2, g_3, and g_4$.

$g_1, g_2, g_3, and g_4$ can be chosen arbitrary from $\mathbb{Z}_p$, i.e., $|\mathbb{Z}_p| = p$ choices for g_1, p choices for g_2, p choices for g_3, and p choices for g_4, thus $|U(4,\mathbb{Z}_p)| = p.p.p.p = p^4$

Theorem(2.3):

Every element, excepted identity element e, in the group $G = U(4,\mathbb{Z}_p)$ has order p

That is, $\forall\, g \in G$, we have

$$o(g) = \begin{cases} 1 & if \quad g = e \\ p & if \quad g \neq e \end{cases}$$

Proof 2:

If $g = e$, then $o(g) = 1$.

$\forall\, e \neq g \in G$ has the form

$$g = \begin{bmatrix} 1 & g_1 & g_2 & g_3 \\ 0 & 1 & g_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, where\ g_1, g_2, g_3, and\ g_4 \in \mathbb{Z}_p$$

and $g_1, g_2, g_3, and\ g_4$ are not all zero

$$g^3 = \begin{bmatrix} 1 & 3g_1 & 3g_2 + 3g_1g_4 & 3g_3 \\ 0 & 1 & 3g_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g^2 = \begin{bmatrix} 1 & 2g_1 & 2g_2 + g_1g_4 & 2g_3 \\ 0 & 1 & 2g_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

In general,

$$g^r = \begin{bmatrix} 1 & rg_1 & r(g_2 + \frac{r-1}{2} g_1g_4) & rg_3 \\ 0 & 1 & rg_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let m be the order of g, then $g^m = e$

$$\Rightarrow \begin{bmatrix} 1 & mg_1 & m(g_2 + \frac{m-1}{2} g_1g_4) & mg_3 \\ 0 & 1 & mg_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We get, $mg_1 \equiv 0 \mod p$

$mg_3 \equiv 0 \mod p$

$mg_4 \equiv 0 \mod p$

$m\left(g_2 + \frac{(m-1)}{2} g_1g_4\right) \equiv 0 \mod p$

Since, $\mathbb{Z}_p$ is a field and g_1, g_2, g_3, g_4 are not all zero, then $m = p$.

Theorem(2.4):

Exponent of the group $G = U(4,\mathbb{Z}_p)$ is, $exp(G) = p$.

Proof 3:

Let $l.c.m(a,b)$ be the least common multiple of a and b.

By theorem(2.3),
$exp(G) = l.c.m(1,p) = p.$

Theorem(2.5):

The center of the group $G = U(4, \mathbb{Z}_p)$ is the subgroup

$$Z(G) = \left\{ \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & j & k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| j \neq 0, i, j, k \in \mathbb{Z}_p \right\}$$

and $|Z(G)| = p^2$

Proof 4 :

Let $g, h \in G$, where $g = \begin{bmatrix} 1 & g_1 & g_2 & g_3 \\ 0 & 1 & g_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

and $h = \begin{bmatrix} 1 & h_1 & h_2 & h_3 \\ 0 & 1 & h_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$g.h = \begin{bmatrix} 1 & h_1 + g_1 & h_2 + g_1 h_4 + g_2 & h_3 + g_3 \\ 0 & 1 & h_4 + g_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$h.g = \begin{bmatrix} 1 & g_1 + h_1 & g_2 + h_1 g_4 + h_2 & g_3 + h_3 \\ 0 & 1 & g_4 + h_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $g_1 = g_4 = 0$, then $\forall h \in G$, we have
$g.h = h.g$

Hence, $g \in Z(G)$ and

$$Z(G) = \left\{ \begin{bmatrix} 1 & 0 & g_2 & g_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & g_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| g_2 \neq 0, g_2, g_3 \in \mathbb{Z}_p \right\}$$

Since, $\forall g_2, g_3 \in \mathbb{Z}_p$,

$g = \begin{bmatrix} 1 & 0 & g_2 & g_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in Z(G)$ and since we have

$(p-1) choises\ for\ g_2\ and\ p\ choises\ for\ g_3$

and $|\mathbb{Z}_p| = p$,and since $\forall g_3 \in \mathbb{Z}_p$,

$g = \begin{bmatrix} 1 & 0 & 0 & g_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in Z(G)$,also we have p choises for $g_3$and $|\mathbb{Z}_p| = p$, then
$|Z(G)| = (p-1)p$ +p= p^2-p+p=p^2.

Remark(2.6):

We classify the elements of the group $U(4, \mathbb{Z}_p)$ into four disjoint sets :

1) Let $U_x = \left\{ x_i = \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| i \in \mathbb{Z}_p \right\}$

we called U_x set of all elements of kind x,

2) Let

$$U_y = \left\{ y_j = \begin{bmatrix} 1 & 0 & j & k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| j \neq 0; j, k \in \mathbb{Z}_p \right\}$$

we called U_y set of all elements of kind y

We note that $U_x, U_y = Z(G)$

3) Let

$$U_z = \left\{ z_{mn} = \begin{bmatrix} 1 & l & 0 & m \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| n \neq 0;\ l, m, n \in \mathbb{Z}_p \right\}$$

we called U_z set of all elements of kind z

4) Let

$$U_w = \left\{ w_{r,t,u} = \begin{bmatrix} 1 & r & s & t \\ 0 & 1 & u & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| r \neq 0;\ r, s, t, u \in \mathbb{Z}_p \right\}$$

we called U_w set of all elements of kind w

$U_x \cap U_w = \emptyset U_x \cap U_z = \emptyset U_x \cap U_y = \emptyset$ are disjoint sets , i.e.,

5) U_x , U_y ,U_z ,and

$U_z \cap U_w = \emptyset U_y \cap U_z = \emptyset$, and

Proposition(2.7):

Let $1 \leq q \leq p-1$, then

1) $\forall i = 0,1, \ldots, p-1$; $(x_i)^q$ are elements of kind x , that is, $(x_i)^q \in U_x$.

2) $\forall j = 1,2, \ldots, p-1$; $(y_j)^q$ are elements of kind y , that is, $(y_j)^q \in U_y$.

Proof 5:

1) $x_i = \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and

$(x_i)^q = \begin{bmatrix} 1 & 0 & 0 & qi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ where qi$\in \mathbb{Z}_p$ then

$\forall\, x_i \in U_x$; $(x_i)^q \in U_x$

2) $y_j = \begin{bmatrix} 1 & 0 & j & k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and

$(y_j)^q = \begin{bmatrix} 1 & 0 & qj & qk \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $where\ qj, qk \in \mathbb{Z}_p$

Since, $q \neq 0$ and $j \neq 0$ then $qj \neq 0$, therefore $(y_j)^q \in U_y$.

Theorem(2.8):

The group $G = U(4, \mathbb{Z}_p)$ has exactly $(p^3 + p^2 - p)$ conjugacy classes

1) $\forall\, i = 0,1,\dots,p-1$; We have classes of the form $C_{x_i} = x_i = \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and

$|C_{x_i}| = 1$

2) $\forall\, j1,2,\dots,p-1$;We have classes of the form

$$C_{y_j} = \left\{ y_j = \begin{bmatrix} 1 & 0 & j & k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; r = 0,1,\dots,p-1 \right\}$$

And $|C_{y_j}| = 1$

3) $\forall\, n = 1,2,\dots,p-1$ and $\forall\, m = 0,1,\dots,p-1$; We have classes of the form

$$C_{z_{m,n}} = \left\{ z_{m,n} = \begin{bmatrix} 1 & l & 0 & m \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; l = 0,1,\dots,p-1 \right\}$$

and $|C_{z_{m,n}}| = p$

4) $\forall\, r = 1,2,\dots,p-1$ and $\forall\, t = 0,1,\dots,p-1$; We have classes of the form

$$C_{w_{r,t,u}} = \left\{ w_{r,t,u} = \begin{bmatrix} 1 & r & s & t \\ 0 & 1 & u & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; u = 0,1,\dots,p-1 \right\}$$

and $|C_{w_{r,t,u}}| = p$

Proof 6:

1),2) By theorem(2.5), $\forall\, i = 0,1,\dots,p-1, j = 1,\dots,p-1$; the elements $x_i, y_j \in Z(G)$, then these elements form a conjugacy classes of their own , and $|C_{x_i}| = 1$, $|C_{y_j}| = 1$

3) To find a conjugacy classes of $z_{m,n}$, we consider an arbitrary element

$g = \begin{bmatrix} 1 & g_1 & g_2 & g_3 \\ 0 & 1 & g_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in G$

and its inverse $g^{-1} =$

$\begin{bmatrix} 1 & -g_1 & g_1 g_4 - g_2 & -g_3 \\ 0 & 1 & -g_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, Then

$$g.z_{m,n}.g^{-1} = \begin{bmatrix} 1 & l & g_1 g_4 - (l + g_1).g_4 + g_1.n & m \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $m_1 \neq m_2, n_1 \neq n_2$; and $z_{m1,n1}$ is conjugate to $z_{m2,n2}$, then $g.z_{m1,n1}.g^{-1} = z_{m2,n2}$

$$\Rightarrow \begin{bmatrix} 1 & l & g_1 g_4 - (l + g_1).g_4 + g_1.n_1 & m_1 \\ 0 & 1 & n_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & l & 0 & m_2 \\ 0 & 1 & n_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow m_1 = m_2 \text{ and } n_1 = n_2$$

Thus, $\forall\, n = 1,2,\dots,p-1$ and $\forall\, m = 0,1,\dots,p-1$; $C_{z_{m,n}}$ are all distinct .

In $C_{z_{m,n}}$, $l = 0,1,\dots,p-1$, then $|C_{z_{m,n}}| = p$

4) To find a conjugacy classes of $w_{r,t,u}$, we consider an arbitrary element

$$g=\begin{bmatrix}1 & g_1 & g_2 & g_3\\ 0 & 1 & g_4 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\in G$$ and its inverse

$$g^{-1}=\begin{bmatrix}1 & -g_1 & g_1g_4-g_2 & -g_3\\ 0 & 1 & -g_4 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$,Then

$$g.w_{r,t,u}.g^{-1}=\begin{bmatrix}1 & r & g_1g_4-(r+g_1)g_4+s+g_1.u & t\\ 0 & 1 & u & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

If $r_1\neq r_2, t_1\neq t_2, and u_1\neq u_2$ and $z_{r1,t1,u1}$ is conjugate to $z_{r2,t2,u2}$, then

$g.w_{r1,t1,u1}.g^{-1}=w_{r2,t2,u2}$

$$\Longrightarrow \begin{bmatrix}1 & r_1 & g_1g_4-(r_1+g_1)g_4+s+g_1.u_1 & t_1\\ 0 & 1 & u_1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & r_2 & s & t_2\\ 0 & 1 & u_2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\Longrightarrow\ r_1=r_2\ ,\ t_1=t_2$$

and $u_1=u_2$

Thus, $\forall\ r=1,2,\dots,p-1$

$\forall\ t=0,1,2,\dots,p-1$, and

$\forall\ u=0,1,\dots,p-1$; $C_{w_{r,t,u}}$ are all distinct

In $C_{w_{r,t,u}}$, $s=0,1,\dots,p-1$, then

$|C_{w_{r,t,u}}|=p$

To show that the conjugacy classes C_{x_i} , C_{y_j} and $C_{z_{m,n}}$ and $C_{w_{r,t,u}}$,are disjoint:

We have $C_{x_i}\subseteq U_x$, $C_{y_j}\subseteq U_y$ and $C_{z_{m,n}}\subseteq U_z$, then $C_{x_i}\cap C_{y_j}=\emptyset$,

$C_{x_i}\cap C_{z_{m,n}}=\emptyset$

$C_{x_i}\cap C_{w_{r,t,u}}=\emptyset$,

$C_{y_j}\cap C_{z_{m,n}}=\emptyset, C_{y_j}\cap C_{w_{r,t,u}}=\emptyset,$

$C_{z_{m,n}}\cap C_{w_{r,t,u}}=\emptyset$ Hence C_{x_i} , C_{y_j} ,$C_{z_{m,n}}$and $C_{w_{r,t,u}}$ are disjoint .

To find the total number of the conjugacy classes

Number of conjugacy classes in (1), (2)= p^2

Number of conjugacy classes in (3) = p$(p-1)$

Number of conjugacy classes in (4)= $p^2(p-1)$.

Then the total number of the conjugacy classes

$=p^2+p(p-1)+p^2(p-1)-p^2+p^2-p+p^3-p^2-p^3+p^2-p$

To show that these are all conjugacy classes of the group $G=U(4,\mathbb{Z}_p)$, we add up the elements contained in those conjugacy classes, we get

$p^2(1)+[p(p-1)](p)+[p^2(p-1)](p)=p^4=|G|$ Thus, this theorem gives all conjugacy classes of the group $U(3,\mathbb{Z}_p)$

Proposition(2.9):

Order of the centralizers, $|C_G(g)|$, of g in the group $G=U(4,\mathbb{Z}_p)$ are :

1) $\forall\ i=0,1,\dots,p-1;\quad |C_G(x_i)|=p^4$

2) $\forall\ j=1,2,\dots,p-1;\quad |C_G(y_j)|=p^4$

3) $\forall\ n=1,2,\dots,p-1$ and $\forall\ m=0,1,\dots,p-1$; $|C_G(z_{m,n})|=p^3$

4) $\forall\ r=1,2,\dots,p-1$,$t=0,1,2,\dots,p-1$,and $\forall m=0,1,\dots,p-1$; $|C_G(z_{m,n})|=p^3$

Proof 7:

By lemma (1.11), $|C_G(g)|=\frac{|G|}{|C_g|}$ and by theorem(2.2), $|G|=p^4$

1) By theorem(2.8), $\forall\ i=0,1,\dots,p-1;\quad |C_{x_i}|=1$, then

$$|C_G(x_i)|=\frac{|G|}{|C_{x_i}|}=\frac{p^4}{1}=p^4$$

2) By theorem(2.8), $\forall\ j=1,2,\dots,p-1;\quad |C_{y_j}|=1,$ then

$$|C_G(y_j)|=\frac{|G|}{|C_{y_j}|}=\frac{p^4}{1}=p^4$$

3) By theorem(2.8), $\forall\ n=1,2,\dots,p-1$ and $\forall\ m=0,1,\dots,p-1$; $|C_{z_{m,n}}|=p$,

Then $$|C_G(z_{m,n})|=\frac{|G|}{|C_{z_{m,n}}|}=\frac{p^4}{p}=p^3$$

4) By theorem(2.8), $\forall\ r=1,2,\dots,p-1$, $\forall\ t=0,1,2,\dots,p-1$and $\forall\ u=0,1,\dots,p-1$; $|C_{w_{r,t,u}}|=p$,

Then $$|C_G(w_{r,t,u})|=\frac{|G|}{|C_{w_{r,t,u}}|}=\frac{p^4}{p}=p^3$$

Proposition (2.10):

Let G=U(4,$\mathbb{Z}_p$) then we have the following:-

1) (p+1) cyclic subgroups of order p which generated by elements of the classes of the form C_{x_i}, C_{y_j} *with normalizer equal to* p^4

p^2(p+1) cyclic subgroups of order p which generated by elements of the classes of the form

$C_{z_{m,n}}$,$C_{w_{r,t,u}}$ *with normalizer equal to* p^3

Proof 8:

1)By theorem(2.3),(2.8),all elements of the conjugacy classes of the form C_{x_i}, C_{y_j}have order p(except the identity element),and each class contains only one element which is of the formU_x, U_y then we have p^2-1 elements of order p,since every cyclic subgroup of order p contains p-1 elements of order p ,then we have $\frac{p^2-1}{p-1}=p+1$ cyclic subgroup of order p. Since every cyclic subgroup of order p generated by elements of the form U_x or U_y contains p-1 classes of the form C_{x_i} *or* C_{y_j},then the normalizer of these cyclic subgroups is equal to p^4.

2)By theorem(2.3),(2.8),since we have p(p-1) classes of the form $C_{z_{m,n}}$, p^2(p-1) classes of the form $C_{w_{r,t,u}}$,which each class contains p elements ,then we have $p^2(p-1)$ elements of the form

U_z $p^3(p-1)$elements of the form U_w ,

then we have $p^3(p-1)+p^2(p-1)$ = (p-1)(p^3+p^2)=

(p-1)p^2(p+1) elements of order p of the form$U_z$or $U_w$each cyclic subgroup of order p contains p-1 elements of order p then we have $\frac{(p-1)p^2(p+1)}{(p-1)}$=$p^2$(p+1) cyclic subgroups of order p.

Since every cyclic subgroup of order p which generated by element of the form$U_z$or U_w fixed (by conjugation) only by p^2elements (which is form the centre of the group), and $p^2(p-1)$ elements of the form$U_z$or U_w then the normalizer of these cyclic subgroups is equal to $p^2+p^2(p-1)=$ p^2(1+p-1)=p^3.

Theorem(2.11):

For any prime number p the Artin exponent of the group $U(4,\mathbb{Z}_p)$ is equal to p^3

Proof 9:

According to the Brauer coefficients theorem, we calculate Brauer's coefficients using the formula in theorem (1.30)

$$b_1=\frac{1}{\frac{p^4}{1}}[1+\sum\mu[c:1]]$$

$$b_1=\frac{1}{\frac{p^4}{1}}[1+(p+1)\mu(\tfrac{p}{1})+p^2(p+1)\,\mu(\tfrac{p}{1})]$$

$$b_1=\frac{1}{\frac{p^4}{1}}[1+(p+1)(-1)+p^2(p+1)(-1)]$$

$$=\frac{1}{p^4}[1-p-1-p^3-p^2]=\frac{-p}{p^4}[p^2+p+1]=\frac{-1}{p^3}[p^2+p+1]$$

$$b_2=\frac{1}{\frac{p^4}{p}}[\mu(\tfrac{p}{p})]=\frac{1}{p^3}$$

$$b_3=\frac{1}{\frac{p^3}{p}}[\mu(\tfrac{p}{p})]=\frac{1}{p^2}$$

$$\rightarrow \chi_1=\frac{1}{p^2}\emptyset_3+\frac{1}{p^3}\emptyset_2-\frac{1}{p^3}(p^2+p+1)\emptyset_1$$

∴ Artin exponent of U(4, $\mathbb{Z}_p$)=p^3.

References

[1] Burrow ,M. "Representation Theory of Finite Group", Academic Press, New York, 1965.

[2] Feit, W. "Characters of Finite Groups", W.A.Benjan, Inc, New York, 1967.

[3] Isaacs,I.M. " Character Theory of Finite Groups", Academic Press, New York, 1976 .

[4] J.L.Alperin with Rowen B.Bell, "Groups and Representations", Spring-Verlag, New York, Inc, 1995.

[5] Keown, R. "An Introduction To Group Representation Theory ", Academic Press, New York, 1976.

[6] Lam, T. "*Artin Exponent of Finite Groups*", Colombia University, New York, Journal of Algebra, Vol.9, 1968,pp.94-119.

[7] Surowski, D. "Workbook in Higher Algebra", Department of Mathematics, Kansas State University, Manhattan, USA.

[8] Vavilov, Nikolai, "*Linear Groups Over Rings*", 2008.

الخلاصة

الغرض الرئيسي لهذا البحث هو تحديد أس أرتن لزمرة المصفوفات المثلثية الاحادية $U(n,f)$ عنــدما $n=4$ و $f=Z_p$ وقد وجدت بأن أس أرتن لهذه الزمرة مساويا الــى p^3 أي ان $A(U(4,Z_p))=p^3$ وان رتبتهـــــا هـــــي p^4 أي ان $|U(4,Z_p)| = p^4$ وأســـــــها هـــــو p أي ان $\exp(U(4,Z_p))=p$ وقد اوجدت صـــيغة عامـــة لحـــساب صفوف الترافق للزمرة اعلاه وبشكل عام لكل p.