ADOMAIN DECOMPOSITION METHOD FOR SOLVING SYSTEMS OF MULTI-DIMENSIONAL LINEAR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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Abstract
The aim of this work is to use Adomian decomposition method to solve Systems of multi-dimensional linear Fredholm integral equations of the second kind.

1-Introduction

Adomain decomposition method was first introduced by Adomain G. in 1980. This method is used to solve differential equations, [1], [2]. The convergence of Adomian decomposition method applied to the one-dimensional integral equations is discussed in [5]. Moreover this method is used to solve systems of the one-dimensional Volterra integral equations of first kind, [4], systems of linear equations and systems of the one-dimensional Volterra integral equations of second kind, [3], systems of the one-dimensional Fredholm integral equations of the second kind, [7] and systems of fractional differential equations, [6]. Here we use this method to solve systems of the multi-dimensional linear Fredholm integral equations of the second kind:

$$u_i(x_1, x_2, ..., x_n) = f_i(x_1, x_2, ..., x_n) + \int_{a_1}^{b_1}\int_{a2}^{b_2}\cdots\int_{an}^{bn}\sum_{j=1}^{n} k_{i,j}(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n). u_j(y_1, y_2, ..., y_n)\,dy_n\,dy_{n-1}\ldots dy_1,\ i = 1, 2, ..., n$$

Where f_i is a known function of $x_1, x_2, ..., x_n, k_{i,j}$ is a known function of $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n, \{a_i\}_{i=1}^{n}, \{b_i\}_{i=1}^{n}$ are known constants such that $a_i \le x_i \le b_i$, i=1,2,…,n and $u_1, u_2, \ldots, u_n$ are the unknown functions that must be determined.

2- Adomain Decomposition Method Applied to System(1)

Consider the system of the multi-dimensional linear Fredholm integral equations of the second kind given by equation (1).

We rewrite this equation as a canonical form of Adomain's equation by letting

$$N_i(x_1, x_2, ..., x_n) = \int_{a_1}^{b_1}\int_{a2}^{b_2}\cdots\int_{an}^{bn}\sum_{j=1}^{n} k_{i,j}(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n) u_j(y_1, y_2, ..., y_n)\,dy_n\,dy_{n-1}..dy_1 \quad (*)$$

to get

$$u_j(x_1, x_2, ..., x_n) = f_i(x_1, x_2, ..., x_n) + N_i(x_1, x_2, ..., x_n)$$

To solve equation(2) by Adomain's decomposition method, we let

$$u_j(x_1, x_2, ..., x_n) = \sum_{m=0}^{\infty} u_{i,m}(x_1, x_2, ..., x_n)$$

and

$$N_i(x_1, x_2, ..., x_n) = \sum_{m=0}^{\infty} A_{i,m}$$

where

$A_{i,m}, m = 0,1,...$ are polynomial depending on $u_{1,0}, u_{1,1}..., u_{1,m}, ..., u_{n,0}, u_{n,1}, ..., u_{n,m}$ and they are called Adomian polynomials. Hence, equation (2) can be rewritten as:

$$\sum_{m=0}^{\infty} u_{i,m}(x_1, x_2, ..., x_n) = f_i(x_1, x_2, ..., x_n) + \sum_{m=0}^{\infty} A_{i,m}(u_{1,0}, u_{1,1}..., u_{1,m}, ..., u_{n,0}, u_{n,1}, ..., u_{n,m}) \quad ....(3)$$

From equation (3) we define :

$$u_{i,0}(x_1, x_2, ..., x_n) = f_i(x_1, x_2, ..., x_n),$$
$$u_{i,m+1}(x_1, x_2, ..., x_n) = A_{i,m}(u_{1,0}, u_{1,1}..., u_{1,m}, ..., u_{n,0}, u_{n,1}, ..., u_{n,m}),$$
$$i = 1, 2, ..., n, m = 0, 1, ... \quad ....(4)$$

To determine Adomain polynomials, we consider the expansions:

$$u_{i,\lambda}(x_1,x_2,...,x_n)=\sum_{m=0}^{\infty}\lambda^m u_{i,m}(x_1,x_2,...,x_n), \quad ....(5)$$

$$N_{i,\lambda}(x_1,x_2,...,x_n)=\sum_{m=0}^{\infty}\lambda^m A_{i,m} \quad ....(6)$$

where λ is a parameter introduced for convenience. From equation (6) we obtain:

$$A_{i,m}=\frac{1}{m!}\left[\frac{d^m}{d\lambda^m}N_{i,\lambda}(u_1,u_2,...,u_n)\right]_{\lambda=0} \quad ..(7)$$

and from equations $(*)$,(5) and (7) we have:

$$A_{i,m}(u_{1,0},u_{1,1}...,u_{1,m},...,u_{n,0},u_{n,1},...,u_{n,m})=$$
$$\frac{1}{m!}\left[\frac{d^m}{d\lambda^m}\int_{a_1}^{b_1}\int_{a2}^{b_2}...\int_{an}^{bn}\sum_{j=1}^{n}k_{i,j}(x_1,x_2,...,x_n,y_1,y_2,...,y_n)\right.$$
$$\left.u_j(y_1,y_2,...,y_n)dy_n dy_{n-1}...dy_1\right]_{\lambda=0}$$
$$\left[\int_{a_1}^{b_1}\int_{a2}^{b_2}...\int_{an}^{bn}\sum_{m=0}^{\infty}k_{i,j}(x_1,x_2,...,x_n,y_1,y_2,...,y_n)\right.$$
$$\left.\sum_{m=0}^{n}\frac{1}{m!}\frac{d^m}{d\lambda^m}u_{i,m}(y_1,y_2,...,y_n)dy_n dy_{n-1}...dy_1\right]$$

$$................................(8)$$

So, the solution of the system given by equation(1) will be as follows:

$$u_{i,0}(x_1,x_2,...,x_n)=f_i(x_1,x_2,...,x_n),$$
$$u_{i,m+1}(x_1,x_2,...,x_n)=\int_{a_1}^{b_1}\int_{a2}^{b_2}...\int_{an}^{bn}\sum_{j=1}^{n}k_{i,j}(x_1,x_2,...,x_n,$$
$$y_1,y_2,...,y_n)u_{j,m}(y_1,y_2,...,y_n)dy_n dy_{n-1}...dy_1$$

3-Numerical Example

In this section we give two examples of systems of multi-dimensional linear Fredholm integral equations of the second kind with their approximated solutions via Adomain decomposition method.

Example (1):

Consider the tow-dimensional linear Fredholm integral equation of the second kind:

$$u(x,y)=xy^2-\frac{1}{8}x+\int_0^1\int_0^1 xmu(z,m)dzdm$$

This example is constructed such that the exact solution of it is *$u(x,y)=xy^2$*. Here we use Adomain decomposition method to find the solutions *u*. To do this we use the following Adomain scheme:

$$u_0(x,y)=xy^2-\frac{1}{8}x$$

and

$$u_{r+1}(x,y)=\int_0^1\int_0^1 xmu_r(z,m)dzdm, r=1,2,...$$

For the first iteration, we have:

$$u_1(x,y)=\int_0^1\int_0^1 xmu_0(z,m)dzdm$$
$$=\frac{3}{32}x.$$

Therefore the approximated solution of this example with two terms is:

$$Q_2(x,y)=u_0(z,m)+u_1(z,m)$$
$$=xy^2-\frac{1}{8}x+\frac{3}{32}x$$
$$=xy^2-\frac{1}{32}x.$$

For the second iteration, we have:

$$u_2(x,y)=\int_0^1\int_0^1 xmu_1(z,m)dzdm$$
$$=\frac{1}{32}x.$$

Therefore the approximated solution of this example with three terms is:

$$Q_3(x,y)=u_0(x,y)+u_1(x,y)+u_2(x,y)$$
$$=xy^2-\frac{1}{8}x+\frac{3}{32}x+\frac{1}{32}x$$
$$=xy^2$$

Not that $Q_3(x,y)=xy^2$ is the exact solution of this example.

Example (2):

Consider the system of the two-dimensional linear Fredholm integral equations:

$$u_1(x,y)=\frac{7}{12}xy+\int_0^1\int_0^1 xy(u_1(z,m)+u_2(z,m))dzdm$$
$$u_2(x,y)=\frac{17}{24}x^2y+\int_0^1\int_0^1 x^2yz(u_1(z,m)+u_2(z,m))dzdm$$

This example is constructed such that the exact solution of it is

$$u_1(x,y)=xy \text{ and } u_2(x,y)=x^2y.$$

Here we use Adomain decomposition method to find the solutions u_1, u_2 of this example. To do this we use the following Adomain scheme:

$$u_{1,0}(x,y) = \frac{7}{12}xy \approx 0.5833xy$$

$$u_{2,0}(x,y) = \frac{17}{24}x^2y \approx 0.7083x^2y$$

and

$$u_{1,r+1}(x,y) = \int_0^1\int_0^1 xy(u_{1,r}(z,m) + u_{2,r}(z,m))dzdm, r = 1,2,...$$

$$u_{2,r+1}(x,y) = \int_0^1\int_0^1 x^2yz(u_{1,r}(z,m) + u_{2,r}(z,m))dzdm, r = 1,2,...$$

For the first iteration, we have:

$$u_{1,1}(x,y) = \int_0^1\int_0^1 xy(u_{1,0}(z,m) + u_{2,0}(z,m))dzdm = \frac{9}{72}xy \approx 0.2639xy.$$

$$u_{2,1}(x,y) = \int_0^1\int_0^1 x^2yz(u_{1,0}(z,m) + u_{2,0}(z,m))dzdm = \frac{107}{576}x^2y \approx 0.1858x^2y.$$

Therefore the approximated solutions of this example with two terms are:

$$Q_{1,2}(x,y) = u_{1,0}(x,y) + u_{1,1}(x,y) = \frac{7}{12}xy + \frac{19}{72}xy \approx 0.5833xy + 0.2639xy \approx 0.8472xy$$

$$Q_{2,2}(x,y) = u_{2,0}(x,y) + u_{2,1}(x,y) = \frac{17}{24}x^2y + \frac{107}{576}x^2y \approx 0.7083x^2y + 0.1858x^2y \approx 0.8941x^2y$$

For the second iteration, we have:

$$u_{1,2}(x,y) = \int_0^1\int_0^1 xy(u_{1,1}(z,m) + u_{2,1}(z,m))dzdm = \frac{335}{3456}xy \approx 0.0969xy.$$

$$u_{2,2}(x,y) = \int_0^1\int_0^1 x^2yz(u_{1,1}(z,m) + u_{2,1}(z,m))dzdm = \frac{929}{13824}x^2y \approx 0.0672x^2y.$$

Therefore the approximated Solutions of this example with three terms are:

$$Q_{1,3}(x,y) = u_{1,0}(x,y) + u_{1,1}(x,y) + u_{1,2}(x,y) = \frac{7}{12}xy + \frac{19}{72}xy + \frac{335}{3456}xy = \frac{3263}{3456}xy \approx 0.9442xy.$$

$$Q_{2,3}(x,y) = u_{2,0}(x,y) + u_{2,1}(x,y) + u_{2,2}(x,m) = \frac{17}{24}x^2y + \frac{107}{576}x^2y + \frac{929}{13824}x^2y = \frac{13289}{13824}xy^2 \approx 0.9613x^2y.$$

In the same way, the components $Q_{1,k}$ (x,y) and $Q_{2,k}$ (x,y) can be calculated for k=4, 5,… The solutions with ten terms are given as:

$$Q_{1,10}(x,y) = \sum_{i=0}^{9} u_{1,i}(x,y) = \frac{47544222539155}{47552535724032}xy \approx 0.9998xy.$$

$$Q_{2,10}(x,y) = \sum_{i=0}^{9} u_{2,i}(x,y) = \frac{63395395174439}{63403380965376}x^2y \approx 0.9999x^2y.$$

4- Conclusion

As seen before Admian decomposition method have been successfully employedto obtain the approximated solutions of systems of the multi-dimensional linear Fredholm integral equations of the second kind. More accurate results can be obtained by increasing the number of iteration . On the other hand, finding the approximated solutions of systems of the multi-dimensional nonlinear Fredholm integral equations of the second kind by using Adomian decomposition method is a good subject for further research.

5-References

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