THEOREM FOR FIXED POINTS, COMMON FIXED POINTS, COINCIDENCE POINTS BY MODIFIED ITERATIVE SEQUENCE

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Abstract

In this paper the technique of convergence for modified Mann and Ishikawa schemes are used to proving many results about the existence of fixed points, common fixed points and coincidence points.

1.Introduction

Let X denotes a Banach space, T a self-mapping of X. The **Mann iterative scheme**, [4], is defined by

$$x_0 \in C,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ ....(1.1)$$

where $0 \le \alpha_n \le 1.$ and $n > 0$ To guarantee convergence of the sequence $\{x_n\}$ in (1.1), other conditions are placed on α_n.

The **Ishikawa iteration scheme** is defined by

$$x_0 \in C,\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad .... (1.2)$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,$$

where $0 \le \alpha_n, \beta_n \le 1, n > 0$ with additional condition placed on $\alpha_n and\, \beta_n$ as needed. In defining this method, Ishikawa, [6] used the inequality $0 \le \alpha_n \le \beta_n \le 1$. As noted in, [1], replacing this inequality with $0 \le \alpha_n, \beta_n \le 1$.

Moreover, it allows one to immediately obtain a theorem for Mann iteration from the corresponding one for Ishikawa iteration by simply setting each $\beta_n = 0$, [3].

As noted in, [7], J. Schu introduced the following iterative sequences:

Let X be a normed space, C be a nonempty convex subset of X and T:C$\to$C be a given mapping. Then the **modified Mann iterative** sequence $\{x_n\}$ is defined by

$$x_1 \in C,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ ....(1.3)$$

Where $n \ge 1$ and $\{\alpha_n\}$ is appropriate sequence in [0, 1].

The **modified Ishikawa iterative** sequence $\{x_n\}$ is defined by

$$x_1 \in C,\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad .... (1.4)$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n,$$

where $x_0 \in C$, $n \ge 1$ and $\{\alpha_n\},\{\beta_n\}$ are appreciates sequence in [0,1].

Schu's work was to establish weak and strong convergence theorems of the modified Mann and Ishikawa sequence for asymptotically nonexpansive mappings in Hilbert space, [7]. In this paper, there are several theorems of the following type:

T is a self-mapping of a Banach space X, satisfying a contractive condition that may or not by strong enough to guarantee convergence of the ordinary iterates (Picard iterates) of T to a fixed point. It assumed that the modified Mann iterates and modified Ishikawa iterates of T converge, for certain$\{\alpha_n\},\{\beta_n\}$. It is then shown that they converge to a fixed point of T. In section two there is a generic theorem of this type for Mann iterates and some special cases. In section three a modified Mann iteration sequence for a pair of self- mappings is defined and theorem of common fixed point with its corollary are proved. Section four is developed to generalize the sequence in (1.3) for two self-mappings and then study their coincidence point. As well as section two, section five is concluded a generic theorem to show the convergence of modified Ishikawa iteration sequence in (1.4) to a fixed point. Also there are some special cases. Section six is developed to generalize the sequence in (1.4) for two self-mappings and the study their coincidences point. Section seven is developed to generalize the sequence in (1.4) for two self-mappings and then study their coincidence point.

2. General Principle for Modified Mann Iterations.

For the sequence $\{x_n\}$ in (1.3) we prove the following:

Theorem 2.1 :

Let T be a self-mapping of a closed convex subset C of a Banach space X and $\{x_n\}$ is defined in (1.3) with $\liminf_{n\to\infty} \alpha_n > 0$. Suppose that $\{x_n\}$ converges to a point $p \in C$. If there exists constants $\alpha, \lambda, \beta, \delta \geq 0, \delta < 1$ such that

$$\|T^n x_n - Tp\| \leq \alpha \|x_n - p\| + \lambda \|x_n - T^n x_n\| + \beta \|p - T^n x_n\| + \delta \max\{\|p - Tp\|, \|x_n - Tp\|\} \quad ....(2.1)$$

then p is a fixed point of T.

Proof:

By (1.3) implies that

$$x_{n+1} - x_n = (1-\alpha_n)x_n + \alpha_n T^n x_n - x_n$$
$$= \alpha_n (T^n x_n - x_n).$$

Thus $\lim_{n\to\infty} x_n = p$, which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \alpha_n \|T^n x_n - x_n\|$$
$$0 \qquad = \lim_{n\to\infty} \alpha_n \|T^n x_n - x_n\|$$

since $\liminf_{n\to\infty} \alpha_n > 0$, then $\lim_{n\to\infty} \|T^n x_n - x_n\|$. Which mean that $\lim_{n\to\infty} \|T^n x_n - p\| = 0, i.e. \lim_{n\to\infty} T^n x_n = p.$

Taking the limit of (2.1) as $n \to \infty$. Yields $\|p - Tp\| \leq \delta \|p - Tp\|$, which implies that p=Tp.

We can apply theorem (2.1) in the following:-

Corollary 2.1 :

Let X, T, C and $\{x_n\}$ as in Theorem (2.1) and $\{x_n\}$ converges to a point $p \in C$. If there exists constants $c, q \geq 0, q < 1$ such that, for all x, y in C

$$\|Tx - Ty\| \leq q \max\{c \|x - y\|, (\|x - Tx\| + \|y - Ty\|), (\|x - Ty\| + \|y - Tx\|)\} \quad ...(2.2)$$

then p is a fixed point of T.

proof:

Replacing Tx by $T^n x_n$, y by p, we have

$$\|T^n x_n - Tp\| \leq q \max\{c \|x_n - p\|, (\|x_n - T^n x_n\| + \|p - Tp\|), (\|x_n - Tp\| + \|p - T^n x_n\|)\} \leq$$
$$qc \|x_n - p\| + q \|x_n - T^n x_n\| +$$
$$q \|p - T^n x_n\| + q \max\{\|p - Tp\|, \|x_n - Tp\|\}$$

If we put $\alpha = qc, \beta = \lambda = q$, then another contractive condition (2.1) is satisfy and by theorem (2.1), p is fixed point of T.

Remark 2.1

In [2], Rhoades proved some results about convergence of Mann iteration sequence in (1.1) for the following cases:

Let X be a normed space

1-There exists a constant q, 0<q<1 such that, for all x, y in X

$$\|Tx - Ty\|^2 \leq q \max[\|x - Tx\| \cdot \|y - Ty\|, \|x - Ty\| \cdot \|y - Tx\|, \|x - Tx\| \cdot \|y - Tx\| \|x - Ty\| \cdot \|y - Ty\|\} \quad ..... (2.3)$$

2-There exists a constant q, 0<q<1 such that, for all x, y in X

$$\|Tx - Ty\| \leq q \max\left\{ \|x - y\| \cdot \frac{\|x - Tx\|(1 - \|x - Ty\|)}{1 + \|x - Tx\|}, \frac{\|Tx - y\|(1 - \|y - Ty\|)}{1 + \|Tx - y\|}, \frac{\|y - Ty\|(1 - \|Tx - y\|)}{1 + \|y - Tx\|} \right\} \quad .....(2.4)$$

3- For each x, y in X, at least one of the following conditions hold

(a) $\|x - Tx\| + \|y - Ty\| \leq \alpha \|x - y\|, \quad 1 \leq \alpha < 2,$

(b) $\|x - Tx\| + \|y - Ty\| \leq \beta \{\|x - Ty\| + \|y - Tx\| + \|x - y\|\}, \frac{1}{2} \leq \beta < \frac{2}{3},$

(c) $\|x - Tx\| + \|y - Ty\| + \|Tx - Ty\| \leq \lambda \{\|x - Ty\| + \|y - Tx\|\}, 1 \leq \lambda < \frac{2}{3},$

(d) $\|Tx-Ty\| \le \delta \max\left\{ \|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Ty\|+\|y-Tx\|}{2} \right\}, 0<\delta<1.$

Remark 2.2

It is easy to show that the cases (1) and (2) in remark (2.1) are special case of corollary (2.1) as follows:-

1- The condition (1) in remark (2.1) implies that

$$\|Tx-Ty\|^2 \le q \max\left\{ \max\{\|x-Tx\|^2, \|y-Ty\|^2\}, \max\left\{\|x-Ty\|^2, \|y-Tx\|^2\right\}, \max\left\{\|x-Tx\|^2, \|y-Tx\|^2\right\}, \max\left\{\|x-Ty\|^2, \|y-Ty\|^2\right\}\right\}.$$

which, in turn, implies that

$$\le q \max\left\{\|x-Tx\|^2, \|y-Ty\|^2, \|y-Tx\|^2, \|x-Ty\|^2\right\},$$

$$\|Tx-Ty\| \le \sqrt{q} \max\left\{\|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\right\}$$

2- For condition (2) in remark (2.1), consider

$$\frac{\|x-Tx\|\left(1-\|x-Ty\|\right)}{1+\|x-Tx\|}$$

for each pair x, y such that the numberator is negative, that term can be ignored, since it clearly cannot be the maximum. For each x, y such that the numerator is positive, the expression is dominated by $\|x-Tx\|$. Employing a similar argument to each each of the fractions in condition (2), it follows that condition (2), implies that

$$\|Tx-Ty\| \le q \max\{\|x-y\|, \|x-Tx\|, \|y-Tx\|, \|y-Ty\|\},$$

and condition (2) is a special case of corollary (2.1).

For condition (3) in remark (2.1), we prove the following:

Corollary 2.2

Let X, T, C and $\{x_n\}$ as in theorem (2.1), $\{x_n\}$ converges to a point $p \in C$. If there exists constants $\alpha, \beta, \lambda \ge 0, \delta < 1$ such that for each x, y in C one of the following conditions holds:

(a) $\|x-Tx\|+\|y-Ty\| \le \alpha\|x-y\|, 1 \le \alpha < 2,$

(b) $\|x-Tx\|+\|y-Ty\| \le \beta\{\|x-Ty\|+\|y-Tx\|+\|x-y\|\}, \frac{1}{2} \le \beta < \frac{2}{3},$

(c) $\|x-Tx\|+\|y-Ty\|+\|Tx-Ty\| \le \lambda\{\|x-Ty\|+\|y-Tx\|\}, 1 \le \lambda < \frac{2}{3},$

(d) $\|Tx-Ty\| \le \delta \max\left\{ \|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Ty\|+\|y-Tx\|}{2} \right\}, 0<\delta<1,$

then p is a fixed point of T.

Proof:

Set $Tx = T^n x_n$, y=p in each of (a)-d). Using the argument on page 754 in [2], it follows that

$$\|T^n x_n - Tp\| \le \max\left\{ \frac{\alpha}{2}\|x_n - p\| + \frac{\|x_n - Tp\| + \|T^n x_n - p\|}{2}, \frac{1+\beta}{2}\left(\|x_n - Tp\| + \|p - T^n x_n\|\right) + \frac{\beta}{2}\|x_n - p\|, \lambda\left(\|x_n - Tp\| + \|p - T^n x_n\|\right) - \|x_n - T^n x_n\| - \|p - Tp\|, \delta \max\left\{ \|x_n - p\|, \|x_n - T^n x_n\|, \|p - Tp\|, \frac{\|x_n - Tp\| + \|p - T^n x_n\|}{2} \right\} \right\} \le \max\left\{\frac{\alpha}{2}, \frac{\beta}{2}, \delta\right\}.$$

$$\|x_n - p\| + \delta\|T^n x_n - x_n\| + \max\left\{\frac{1}{2}, \frac{1+\beta}{2}, \lambda, \frac{\delta}{2}\right\}.$$

$$\|T^n x_n - p\| + \max\left\{\frac{1}{2}, \frac{1+\beta}{2}, \lambda, \delta\right\}$$

$$\max\left\{\|p - Tp\|, \|x_n - Tp\|\right\},$$

and (2.1) is satisfy.

Corollary 2.3

Let X, T, C and $\{x_n\}$ as in Theorem (2.1), $\{x_n\}$ converges to a point $p \in C$. If there exists constants $\alpha, q > 0$ and $\alpha < \frac{1}{2}, q < 1$ such that, for each x, y in C one of the following conditions holds:

1- $\|Tx - Ty\| \le \max\{\|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2}, \frac{\|x - Ty\| + \|y - Tx\|}{2}\}$ ....(2.5)

2- $\|Tx - Ty\| \le \alpha(\|x - Tx\| + \|y - Ty\|) + (1-2\alpha)\max\left\{\|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2}, \frac{\|x - Ty\| + \|y - Tx\|}{2}\right\}$ ....(2.6)

3- $\|Tx - Ty\| \le q\max\left\{\|x - y\|, \frac{\|y - Ty\|[1+\|x - Tx\|]}{1+\|x - y\|}, \frac{\|x - y\|(1+\|x - Tx\| + \|y - Ty\|)}{2(1+\|x - y\|)}\right\}$ ....(2.7)

then p is a fixed point of T.

Proof:

For (1) set $Tx = T^n x_n$, y=p in (2.5), given

$$\|T^n x_n - Tp\| \le \max\left\{\|x_n - p\|, \frac{\|x_n - T^n x_n\| + \|p - Tp\|}{2}, \frac{\|x_n - Tp\| + \|p - T^n x_n\|}{2}\right\}$$

$$\le \|x_n - p\|, \frac{1}{2}\|x_n - T^n x_n\| + \frac{1}{2}\|p - T^n x_n\| + \frac{1}{2}\max\{\|p - Tp\|, \|x_n - Tp\|\},$$

and (2.1) is satisfied.

In 2,

$$\|Tx - Ty\| \le \alpha(\|x - Tx\| + \|y - Ty\|) + (1-2\alpha)\max\left\{\|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2}, \frac{\|x - Ty\| + \|y - Tx\|}{2}\right\} \le$$

$$\max\{(1-2\alpha)\|x - Tx\|, \max\{2\alpha, (1-2\alpha)\frac{\|x - Tx\| + \|y - Ty\|}{2}, (1-2\alpha)\frac{\|x - Ty\| + \|y - Tx\|}{2}\}$$

$$\le \max\left\{\|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2}, \frac{\|x - Ty\| + \|y - Tx\|}{2}\right\},$$

which is a special case of condition (2.5).

For 3, set $Tx = T^n x_n$, y=p in (2.7), given

$$\|T^n x_n - Tp\| \le q\max\left\{\|x_n - p\|, \frac{\|p - Tp\|(1+\|x_n - T^n x_n\|)}{1+\|x_n - p\|}, \frac{\|x_n - p\|(1+\|x_n - T^n x_n\|) + \|p - Tp\|}{2(1+\|x_n - p\|)}\right\}$$

$$\le \left(q + \frac{1}{2}\right)\|x_n - p\| + \left(q\|p - Tp\| + \frac{1}{2}\right)\|x_n - T^n x_n\| + \max\left\{q, \frac{1}{2}\right\}\|p - Tp\|$$

and (2.1) is satisfied.

3. Common Fixed Point For Modified Mann Iteration of A Pair Mapping

We shall define a general Mann iteration for a pair of mappings S and T as follows:

$$x_0 \in C,$$
$$x_{2n+1} = (1-\alpha_{2n})x_{2n} + \alpha_{2n} S^n x_{2n}, \qquad \text{......... (3.1)}$$
$$x_{2n+2} = (1-\alpha_{2n+1})x_{2n+1} + \alpha_{2n+1} T^n x_{2n+1}$$

where $0 \le \alpha_{2n}, \alpha_{2n+1} \le 1, n \ge 1$.

Theorem 3.1

Let T and S be a self- maps of a closed convex subset C of a Banach space X and $\{x_n\}$ is defined in (3.1) with $\liminf_{n\to\infty} \alpha_{2n} > 0$. Suppose that $\{x_n\}$ converges to a point $p \in C$. If there exists constants $\alpha, \lambda, \beta, \delta, \alpha', \lambda', \beta', \delta' \ge 0, \delta, \delta' < 1$ such that

$$\|S^n x_{2n} - Tp\| \le \alpha\|x_{2n} - p\| + \beta\|x_{2n} - S^n x_{2n}\| + \lambda\|p - S^n x_{2n}\| + \delta \max\{\|p - Tp\|, \|x_{2n} - Tp\|\} \quad \text{...........(3.2)}$$

and

$$\|Sp - T^n x_{2n+1}\| \le \alpha\ \|x_{2n+1} - p\| + \beta\ \|x_{2n+1} - T^n x_{2n+1}\| + \lambda\ \|p - T^n x_{2n+1}\| + \delta\ \max\{\|p - Sp\|, \|x_{2n+1} - Sp\|\} \quad \text{.......(3.3)}$$

then p is a common fixed point of S and T.

Proof:

By (3.1) implies that

$x_{2n+1} - x_{2n} = \alpha_{2n}(S^n x_{2n} - x_{2n})$.

Thus $\lim_{n\to\infty} x_n = p$, which implies that

$$\lim_{n\to\infty}\|x_{2n+1} - x_{2n}\| = \lim_{n\to\infty}\alpha_{2n}\ \|S^n x_{2n} - x_{2n}\|$$

$$0 = \lim_{n\to\infty}\alpha_{2n}\ \|S^n x_{2n} - x_{2n}\|$$

since $\liminf_{n\to\infty}\alpha_{2n} > 0$, then $\lim_{n\to\infty}\|S^n x_{2n} - x_{2n}\| = 0$. Which mean that $\lim_{n\to\infty}\|S^n x_{2n} - p\| = 0, i.e. \lim_{n\to\infty} S^n x_{2n} = p.$

Taking the limite of (3.2) as $n \to \infty$. Yields $\|p - Tp\| \le \delta\ \|p - Tp\|$, which implies that p=Tp.

By (3.1), implies that

$$x_{2n+2} - x_{2n+1} = (1-\alpha_{2n+1})x_{2n+1} + \alpha_{2n+1}T^n x_{2n+1} - x_{2n+1}$$

$$\Rightarrow \quad 0 = \alpha_{2n+1}(T^n x_{2n+1} - x_{2n+1}).$$

Thus $\lim_{n\to\infty} x_n = p$, which implies that

$$\lim_{n\to\infty}\|x_{2n+2} - x_{2n+1}\| = \lim_{n\to\infty}\alpha_{2n+1}\ \|T^n x_{2n+1} - x_{2n+1}\|$$

$$0 = \lim_{n\to\infty}\alpha_{2n+1}\ \|T^n x_{2n+1} - x_{2n+1}\|$$

since $\liminf_{n\to\infty}\alpha_{2n+1} > 0$, then $\lim_{n\to\infty}\|T^n x_{2n+1} - x_{2n+1}\|$. Which mean that $\lim_{n\to\infty}\|T^n x_{2n+1} - p\| = 0, i.e. \lim_{n\to\infty} T^n x_{2n+1} = p.$

Taking the limit of (3.3) as $n \to \infty$. Yields $\|p - Sp\| \le \delta\ \|p - Sp\|$, which implies that p=Sp. Thus, p is a common fixed point of S and T.

4. Coincidence Points by Generalizing Iteration Sequence in (1.3).

Let F, G be two self-mappings of X. The following iteration scheme, which is a **generalization of Mann iteration** process, is defined as follows:-

$$x_0 \in X,\ Gx_{2n+1} = (1-\alpha_n)Gx_n + \alpha_n F^n x_n, \quad \text{........(4.1)}$$

where $0 < \alpha_n \le 1$, $n \ge 1$ and $\liminf_{n\to\infty}\alpha_n > 0$.

Definition 4.1 [5]

Two maps F and G said to be compatible if, for each sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n$, it is the case that $\lim_{n\to\infty}\|FGx_n - GFx_n\| = 0.$

Theorem 4.1

Let F,G be a self-mappings of a Banach space X, for any $n \ge 1, F^n, G$ are compatible, G is continuous and $\{Gx_n\}$ is defined in (4.1) converges to a point $p \in X$. If there exists constants $\alpha, \lambda, \beta, \delta \ge 0, \delta < 1$ such that

$$\|F^n Gx_n - Fp\| \le \alpha\|G^2 x_n - Gp\| + \beta\|G^n x_n - F^n Gx_n\| + \lambda\|Gp - F^n Gx_n\| + \delta \max\{\|Gp - Fp\|, \|G^2 x_n - Fp\|\} \quad \text{.....(4.2)}$$

then p is a coincidence point of F and G.

Proof:

By (4.1) implies that

$$Gx_{n+1} - Gx_n = (1-\alpha_n)Gx_n + \alpha_n F^n x_n - Gx_n = \alpha_n(F^n x_n - Gx_n).$$

Thus $\lim_{n\to\infty} Gx_n = p$, which implies that

$$\lim_{n\to\infty}\|Gx_{n+1} - Gx_n\| = \lim_{n\to\infty}\alpha_n\ \|F^n x_n - Gx_n\|$$

$$0 = \lim_{n\to\infty}\alpha_n\ \|F^n x_n - Gx_n\|$$

since $\liminf_{n\to\infty}\alpha_n > 0$, then $\lim_{n\to\infty}\|F^n x_n - Gx_n\| = 0$

Which mean that

$\lim_{n\to\infty}\|F^n x_n - p\| = 0, i.e. \lim_{n\to\infty} F^n x_n = p.$

Since F^n and G are compatible, $\lim_{n\to\infty}\|F^n Gx_n - GF^n x_n\| = 0.$ Since G is continuous, $\lim_{n\to\infty} GF^n x_n = Gp$, and hence $\lim_{n\to\infty} F^n Gx_n = Gp.$

Using the triangular inequality,

$$\|GF^n x_n - Fp\| \le \|GF^n x_n - F^n Gx_n\| + \|F^n Gx_n - Fp\| \quad \text{........ (4.3)}$$

Taking the limit of (4.3) as $n \to \infty$. Yields $\|Gp - Fp\| \le \delta \|Gp - Fp\|$, which implies that Gp=Fp.

Lemma 4.1

Let F,G be a self-mappings of a Banach space X, for any $n \ge 1, F^n, G$ are compatible, then F^n and G commute at coincidence points.

Proof

Let x coincidence point of F^n and G, then $F^n x = Gx = y$.

By definition of compatibility,

$$\Rightarrow \lim_{n\to\infty} \|F^n Gx - GF^n x\| = 0.$$

$$\Rightarrow \lim_{n\to\infty} \|F^n y - Gy\| = 0$$

$$\Rightarrow \lim_{n\to\infty} F^n y - \lim_{n\to\infty} Gy = 0.$$

$$\Rightarrow \lim_{n\to\infty} F^n y = \lim_{n\to\infty} Gy$$

$\Rightarrow \lim_{n\to\infty} F^n Gx = \lim_{n\to\infty} GF^n x$; thus, F^n and G commute at x.

Corollary 4.1

Let F,G be a self-mappings of a Banach space X, for any $n \ge 1, F^n, G$ are compatible, G is continuous and for any x, y in X,

$$\|Fx - Fy\| \le a(x,y)\|Gx - Gy\| + b(x,y) \left(\|Gx - Fx\| + \|Gy - Fy\|\right) + c(x,y) \left(\|Gx - Fy\| + \|Gy - Fx\|\right) \quad \text{..... (4.4)}$$

where a, b, c, : $X \times X \to [o,1]$ such that $\sup_{x,y\in X} (a + 2b + 2c)(x,y) = 1.$ Suppose that $\{Gx_n\}$ is defined in (4.1) converges to a point $p \in X$ then p is coincidence point of F and G. If

$$\inf_{x,y\in X} b(x,y) > 0 \quad \text{............................... (4.5)}$$

then Fp is a unique common fixed point of F and G.

proof:

To show that p is a coincidence point, it suffices to show that (4.4) implies (4.2). substituting into (4.4) with $x = Gx_n, Fx = F^n Gx_n, y = p$, we have

$$\|F^n Gx_n - Fp\| \le a\|G^2 x_n - Gp\| + b\left(\|G^2 x_n - F^n Gx_n\| + \|Gp - Fp\|\right) + c\left(\|G^2 x_n - Fp\| + \|Gp - F^n Gx_n\|\right) \le$$

$$a\|G^2 x_n - Gp\| + b\|G^2 x_n - F^n Gx_n\| + c\|Gp - F^n Gx_n\| + \gamma \max\left\{\|Gp - Fp\|, \|G^2 x_n - Fp\|\right\},$$

where a, b and c are evaluated at (Gx_n, p), and $\gamma = \sup_{x,y\in X} (b+c)(x,y)$. If $\gamma = 0$, then

(4.2) is clearly satisfied.

If $\gamma > 0$, then $\gamma < 2\gamma \le 1$, and again (4.2) is satisfied.

From lemma (4.1), F^n and G commute at coincidence points. Substituting x=Gp, y=p into (4.4) gives

$$\|FGp - Fp\| \le a\|G^2 p - Gp\| + b\left(\left(\|G^2 p - FGp\|\right) + \|Gp - Fp\|\right) + c\left(\|Gp - Fp\| + \|Gp - FGp\|\right),$$

$$\le (a+c)\|FGp - Fp\|$$

$$\le \gamma\|FGp - Fp\|$$

where a, b and c are evaluated at (Gp, p). Condition (4.5) implies that $\gamma < 1$. Hence, Fp=FGp=GFp and Fp is a fixed point of G. $F^2 p$ =FGp= GFp= Fp, and Fp is a fixed point of F.

To prove uniqueness, suppose that u is also a common fixed point of F and G. Substituting into (4.4), with x=u, y=Fp, we have

$$\|u - Fp\| = \|Fu - F^2 p\| \le a\|Gu - GFp\| + b\left(\|Gu - GFu\| + \|GFp - F^2 p\|\right) + c\left(\|Gu - F^2 p\| + \|GFp - Fu\|\right),$$

$$=(a+2c)\ \|u - Fp\| .$$

By condition (4.5) implies that u=Fp.

5. Fixed point by modified Ishikawa Iteration sequence.

To find a fixed point for a self-mapping T using modified Ishikawa iteration scheme in (1.4), we prove the following:

Theorem 5.1

Let T be a self- mapping of a closed convex subset C of a Banach space X and $\{x_n\}$ is defined in (1.4) with $\liminf_{n\to\infty} \alpha_n > 0$. Suppose that $\{x_n\}$ converges to a point $p \in C$. If there exists constants $\alpha, \lambda, \beta, \delta, \alpha, \beta \geq 0 and \delta, \beta < 1$ such that

$$\|T^n x_n - T^n y_n\| \leq \alpha \|x_n - T^n y_n\| + \beta \|x_n - T^n x_n\| \quad \text{........ (5.1)}$$

and

$$\|T^n x_n - Tp\| \leq \alpha \|x_n - p\| + \lambda \|x_n - T^n x_n\| + \beta \|p - T^n x_n\| + \delta \max\{\|p - Tp\|, \|x_n - Tp\|\} \quad \text{.........(5.2)}$$

then p is a fixed point of T.

Proof:

By (1.4) implies that

$$x_{n+1} - x_n = (1-\alpha_n)x_n + \alpha_n T^n y_n - x_n = \alpha_n (T^n y_n - x_n).$$

Thus $\lim_{n\to\infty} x_n = p$, which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \alpha_n \|T^n y_n - x_n\|$$

$$0 = \lim_{n\to\infty} \alpha_n \|T^n y_n - x_n\|$$

since $\liminf_{n\to\infty} \alpha_n > 0$, then $\lim_{n\to\infty} \|T^n y_n - x_n\|$. Which mean that $\lim_{n\to\infty} \|T^n y_n - p\| = 0, i.e. \lim_{n\to\infty} T^n y_n = p$. Put $\lim_{n\to\infty} T^n y_n = p$ in (5.1), we have $\|T^n x_n - p\| \leq \beta \ \|T^n x_n - p\|$. since $\beta < 1$, then $\lim_{n\to\infty} \|T^n x_n - p\| = 0, i.e. \lim_{n\to\infty} T^n x_n = p$.

Taking the limit of (5.2) as $n \to \infty$. Yields $\|p - Tp\| \leq \delta \|p - Tp\|$, which implies that p=Tp.

Corollary 5.1

Let C be a closed convex subset of a Banach space X, T a self- mapping satisfying the condition: there exists constants $c, q \geq 0, q < 1$ such that, for all x, y in C

$$\|Tx - Ty\| \leq q \max\{c\|x-y\|, (\|x - Tx\| + \|y - Ty\|), (\|x - Ty\| + \|y - Tx\|)\} \quad \text{..... (5.3)}$$

If the Ishikawa scheme in (1.4), with $\lim_{n\to\infty} \beta_n = 0$, converges to a point p, then p is a fixed point of T.

Proof:

It is sufficient to show that (5.1) and (5.2) are satisfied.

Replacing Tx by $T^n x_n$, y by y_n in (5.3), we have

$$\|T^n x_n - T^n y_n\| \leq q \max\{c\|x_n - y_n\|, (\|x_n - T^n x_n\| + \|y_n - T^n y_n\|), (\|x_n - T^n y_n\| + \|y_n - T^n x_n\|)\}$$

then by (1.4)

$$y_n = (1-\beta_n)x_n + \beta_n T^n x_n,$$

We have

$$\|x_n - y_n\| = \beta_n \|x_n - T^n x_n\|,$$

$$\|y_n - T^n y_n\| = \|(1-\beta_n)x_n + \beta_n T^n x_n - T^n y_n\| \leq (1-\beta_n)\|x_n - T^n y_n\| + \beta_n \|T^n x_n - T^n y_n\|$$

and

$$\|y_n - T^n x_n\| = \|(1-\beta_n)x_n + \beta_n T^n x_n - T^n x_n\|, = \|x_n - T^n x_n - \beta_n (x_n - T^n x_n)\|, = (1-\beta_n)\|x_n - T^n x_n\|.$$

Thus

$$\|T^n x_n - T^n y_n\| \leq q \max\{c\beta_n \|x_n - T^n x_n\|, \|x_n - T^n x_n\| + (1-\beta_n)\|x_n - T^n y_n\| + \beta_n \|x_n - T^n y_n\|, \|x_n - T^n y_n\| + (1-\beta_n)\|x_n - T^n x_n\|\}$$

$$= \max\{qc\beta_n \|x_n - T^n x_n\|, q\|x_n - T^n x_n\| + q(1-\beta_n)\|x_n - T^n y_n\| + q\beta_n \|T^n x_n - T^n y_n\|, q\|x_n - T^n y_n\| + q(1-\beta_n)\|x_n - T^n x_n\|\} \leq$$

$$\max\{qc\beta_n \|x_n - T^n x_n\|, q\|x_n - T^n x_n\| + q(1-\beta_n)\|x_n - T^n y_n\|, q\|x_n - T^n y_n\| + q(1-\beta_n)\|x_n - T^n x_n\|\} + q\beta_n \|T^n x_n - T^n y_n\|$$

so that

$$(1-q\beta_n)\|T^n x_n - T^n y_n\| \le \max\{qc\|x_n - T^n x_n\|,$$
$$q\|x_n - T^n x_n\| + q(1-\beta_n)\|x_n - T^n y_n\|,$$
$$q\|x_n - T^n y_n\| + q(1-\beta_n)\|x_n - T^n x_n\|\}$$

then

$$\|T^n x_n - T^n y_n\| \le \max\{qc\beta_n\|x_n - T^n x_n\|,$$
$$\frac{q}{q(1-\beta_n)}\|x_n - T^n x_n\| +$$
$$\frac{q(1-\beta_n)}{1-q\beta_n}\|x_n - T^n y_n\|, q\|x_n - T^n y_n\| +$$
$$q(1-\beta_n)\|x_n - T^n x_n\|\}$$

Fix N, N is positive integer number since $\beta_n \to 0$, so that for n>N implies $c\beta_n < 1$ and $\beta_n < \frac{1-q}{q}$. Then for n>N,

$$\beta = \max\left\{c\beta_n, \frac{q}{1-q\beta_n}, (1-q\beta_n)\right\},$$
$$\|T^n x_n - T^n y_n\| \le q\|x_n - T^n y_n\|$$
$$+\beta\|x_n - T^n x_n\|$$

and (5.1) satisfied. Therefore $\lim_{n\to\infty} T^n x_n = p$.

Again form (5.3)

$$\|Tp - T^n x_n\| \le q\max\{c\|p - x_n\|, \|p - Tp\| +$$
$$\|x_n - T^n x_n\|, \|p - T^n x_n\| + \|x_n - Tp\|\} \le$$
$$q\|p - x_n\| + q\|x_n - T^n x_n\| + q\|p - T^n x_n\| +$$
$$q\max\{\|p - Tp\|, \|x_n - Tp\|\}$$

and (5.2) is satisfied.

Corollary 5.2

Let C be a closed convex subset of a Banach space X, T self-mapping of C satisfying at least one of the following condition of x, y in X

(a) $\|x - Tx\| + \|y - Ty\| \le \alpha\|x - y\|$, $1 \le \alpha < 2$,

(b) $\|x - Tx\| + \|y - Ty\| \le \beta\{\|x - Ty\| + \|y - Tx\| + \|x - y\|\}, \frac{1}{2} \le \beta < \frac{2}{3}$,

(c) $\|x - Tx\| + \|y - Ty\| + \|Tx - Ty\| \le \lambda(\|x - Ty\| + \|y - Tx\|), 1 \le \lambda < \frac{2}{3}$,

(d) $\|Tx - Ty\| \le \delta\max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \frac{\|x - Ty\| + \|y - Tx\|}{2}\}, 0 < \delta < 1.$

If modified Ishikawa iteration scheme (1.4) converges to a point p, then p is a fixed point of T.

Proof:

It is sufficient to show that T satisfy condition (5.1) and (5.2) from conditions (a)-d). Replace x by x_n, Tx by $T^n x_n$, y=p then;

(1) $\|x_n - T^n x_n\| + \|p - Tp\| \le \alpha\|x_n - p\|$, $1 \le \alpha < 2$,

(2) $\|x_n - T^n x_n\| + \|p - Tp\| \le \beta\{\|x_n - Tp\| + \|p - T^n x_n\| + \|x_n - p\|\}, \frac{1}{2} \le \beta < \frac{2}{3}$,

(3) $\|x_n - T^n x_n\| + \|p - Tp\| + \|T^n x_n - Tp\| \le \lambda\{\|x_n - Tp\| + \|p - T^n x_n\|\}, 1 \le \lambda < \frac{2}{3}$,

(4) $\|T^n x_n - Tp\| \le \delta\max\{\|x_n - p\|, \|x_n - T^n x_n\|, \|p - Tp\|, \frac{\|x_n - Tp\| + \|p - T^n x_n\|}{2}\}$, $0 \le \delta < 1$.

Using triangles inequality

$$\|T^n x_n - Tp\| \le \|T^n x_n - x_n\| + \|x_n - Tp\|$$

so that

$$\|T^n x_n - Tp\| - \|x_n - Tp\| \le \|T^n x_n - x_n\| \quad ...(5.4)$$

and

$$\|T^n x_n - Tp\| \le \|T^n x_n - p\| + \|p - Tp\|$$

so that

$$\|T^n x_n - Tp\| - \|T^n x_n - p\| \le \|p - Tp\| \quad ....(5.5)$$

If x_n, p satisfying (1) and using (5.4) and (5.5), then since

$$\|x_n - T^n x_n\| + \|p - Tp\| \le \alpha\|x_n - p\|$$
$$\Rightarrow \|T^n x_n - Tp\| - \|x_n - Tp\| + \|T^n x_n - Tp\| -$$
$$\|T^n x_n - p\| \le \alpha\|x_n - p\|,$$

$$\Rightarrow 2\|T^n x_n - Tp\| - \|x_n - Tp\| - \|T^n x_n - p\| \le \alpha\|x_n - p\|$$

$$\Rightarrow \|T^n x_n - Tp\| \le \frac{\alpha}{2}\|x_n - p\| + \frac{1}{2}\left(\|x_n - Tp\| + \|T^n x_n - p\|\right)$$

If x_n, p satisfying (2) and using (5.4) and (5.5), then

$$\|x_n - T^n x_n\| + \|p - Tp\| \le \beta\left(\|x_n - Tp\| + \|p - T^n x_n\| + \|x_n - p\|\right),$$

$$\Rightarrow \|T^n x_n - Tp\| - \|x_n - Tp\| - \|T^n x_n - p\| + \|T^n x_n - Tp\| \le \beta\left(\|x_n - Tp\| + \|p - T^n x_n\| + \|x_n - p\|\right),$$

then

$$\|T^n x_n - Tp\| \le \frac{\beta+1}{2}\left(\|x_n - Tp\| + \|p - T^n x_n\|\right) + \frac{\beta}{2}\|x_n - p\|$$

If x_n, p satisfying (3) and using (5.4) and (5.5),since

$$\|x_n - T^n x_n\| + \|p - Tp\| + \|T^n x_n - Tp\| \le \lambda\left(\|x_n - Tp\| + \|p - T^n x_n\|\right).$$

$$\Rightarrow \|T^n x_n - Tp\| - \|x_n - Tp\| + \|T^n x_n - Tp\| - \|T^n x_n - p\| + \|T^n x_n - Tp\| \le \lambda\left(\|x_n - Tp\| + \|p - T^n x_n\|\right),$$

then

$$\|T^n x_n - Tp\| \le \frac{1+\lambda}{3}\left(\|x_n - Tp\| + \|T^n x_n - p\|\right)$$

If x_n, p satisfying (4), then

$$\|T^n x_n - Tp\| \le \delta \max\left\{ \|x_n - p\|, \|x_n - T^n x_n\|, \frac{\|x_n - Tp\| + \|p - T^n x_n\|}{2}\right\}$$

In all cases, we have

$$\|T^n x_n - Tp\| \le \max\left\{\frac{\alpha}{2}, \frac{\beta}{2}, \delta\right\}\|x_n - p\| + \max\left\{\frac{1}{2}, \frac{1+\beta}{2}, \frac{1+\lambda}{3}, \frac{\delta}{2}\right\}\|p - T^n x_n\| + \delta\|x_n - T^n x_n\| + \max\left\{\frac{1}{2}, \frac{1+\beta}{2}, \frac{1+\lambda}{3}, \delta\right\} \max\left\{\|x_n - Tp\|, \|p - Tp\|\right\} \quad \text{.........(5.6)}$$

then (5.2) satisfied. Now: substituting p by y_n in (5.6)

$$\|T^n x_n - T^n y_n\| \le \max\left\{\frac{\alpha}{2}, \frac{\beta}{2}, \delta\right\}\|x_n - y_n\| + \max\left\{\frac{1}{2}, \frac{1+\beta}{2}, \frac{1+\lambda}{3}, \frac{\delta}{2}\right\}\|y_n - T^n x_n\|$$

$$+\delta\|x_n - T^n x_n\| + \max\left\{\frac{1}{2}, \frac{1+\beta}{2}, \frac{1+\lambda}{3}, \delta\right\} \max\left\{\|x_n - T^n y_n\|, \|y_n - T^n y_n\|\right\}$$

Let A= $\max\left\{\frac{\alpha}{2}, \frac{\beta}{2}, \delta\right\}$, B=$\max\left\{\frac{1}{2}, \frac{1+\beta}{2}, \frac{1+\lambda}{3}, \frac{\delta}{2}\right\}$ and C=$\max\left\{\frac{1}{2}, \frac{1+\beta}{2}, \frac{1+\lambda}{3}, \delta\right\}$, then

$$\|T^n x_n - T^n y_n\| \le A\|x_n - y_n\| + B\|y_n - T^n x_n\| + \delta\|x_n - T^n x_n\| + C\max\left\{\|x_n - T^n y_n\|, \|y_n - T^n y_n\|\right\} \le A\beta_n\|x_n - T^n x_n\| +$$

$$\delta\|x_n - T^n x_n\| + B\|y_n - T^n x_n\| + C\|x_n - T^n y_n\| + C\|y_n - T^n y_n\| \le$$

$$A\beta_n\|x_n - T^n x_n\| + \delta\|x_n - T^n x_n\| + B\|y_n - T^n x_n\| + C(1-\beta_n)\|x_n - T^n x_n\| +$$

$$C\left\|y_n - T^n x_n\right\| + C\left\|T^n x_n - T^n y_n\right\|$$

$$\left\|T^n x_x - T^n y_n\right\| - C\left\|T^n x_x - T^n y_n\right\| \le$$

$$\left(A\beta_n + \delta + C(1-\beta_n) + C\right)\left\|x_n - T^n x_n\right\| + B\left\|y_n - T^n y_n\right\|,$$

thus,

$$(1-C)\left\|T^n x_x - T^n y_n\right\| \le \left(A\beta_n + \delta + 2C - \beta_n C\right)$$

$$\left\|x_n - T^n x_n\right\| + B\left\|y_n - T^n x_n\right\| \left\|T^n x_x - T^n y_n\right\| \le \frac{\left(A\beta_n + \delta + 2C - \beta_n C\right)}{1-C}\left\|x_n - T^n x_n\right\| + \frac{B}{1-C}\left\|y_n - T^n x_n\right\|.$$

Let $\alpha = \dfrac{\left(A\beta_n + \delta + 2C - \beta_n C\right)}{1-C}$ *and* $\beta = \dfrac{B}{1-C}$.

then (5.1) satisfied. Thus,by Theorem (5.1), p is a fixed point of T.

6. Common Fixed Point for Modified Ishikawa Iteration of a Pair Mapping

We shall define a general Ishikawa iteration for a pair of mappings S and T as follows:

$$x_0 \in C,\; y_n = (1-\beta_n)x_n + \beta_n T^n x_n \qquad \text{............ (6.1)}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n$$

where $0 \le \alpha_n, \beta_n \le 1, n \ge 1$.

Theorem 6.1

Let T and S be a self- mappings of a closed convex subset C of a Banach space X and $\{x_n\}$ is defined in (6.1) with $\liminf_{n\to\infty} \alpha_{2n} > 0$.Suppose that $\{x_n\}$ converges to a point $p \in C$. If there exists constants $\alpha, \lambda, \beta, \delta, \alpha\ , \lambda\ , \beta\ , \delta\ , \alpha\ , \beta\ \ge 0, \delta, \delta\ , \beta\ < 1$ such that

$$\left\|S^n y_n - Tp\right\| \le \alpha\left\|x_{n+1} - p\right\| + \beta\left\|x_{n+1} - S^n y_n\right\| \qquad \text{...... (6.2)}$$

$$+\lambda\left\|p - S^n y_n\right\| + \delta \max\{\left\|p - Tp\right\|, \left\|x_{n+1} - Tp\right\|\},$$

$$\left\|T^n x_{n+1} - S^n y_n\right\| \le \alpha\ \left\|x_{n+1} - S^n y_n\right\| + \beta\ \left\|x_{n+1} - T^n x_{n+1}\right\| \text{...... (6.3)}$$

and

$$\left\|Sp - T^n x_{n+1}\right\| \le \alpha\ \left\|x_{n+1} - p\right\| + \beta\ \left\|x_{n+1} - T^n x_{n+1}\right\|$$

$$+\lambda\ \left\|p - T^n x_{n+1}\right\| + \delta\ \max\left\{\left\|p - Sp\right\|, \left\|x_{n+1} - Sp\right\|\right\} \text{...(6.4)}$$

then p is a common fixed point of S and T.

Proof:

By (6.1) implies that

$$x_{n+1} - x_n = (1-\alpha_n)x_n + \alpha_n S^n y_n - x_n = \alpha_n (S^n y_n - x_n).$$

Thus $\lim_{n\to\infty} x_n = p$, which implies that

$$\lim_{n\to\infty}\left\|x_{n+1} - x_n\right\| = \lim_{n\to\infty}\alpha_n\left\|S^n y_n - x_n\right\|$$

$$0 = \lim_{n\to\infty}\alpha_n\left\|S^n y_n - x_n\right\|$$

since $\liminf_{n\to\infty} \alpha_n > 0$, then $\lim_{n\to\infty}\left\|S^n y_n - x_n\right\| = 0$. Which mean that $\lim_{n\to\infty}\left\|S^n y_n - p\right\| = 0, i.e. \lim_{n\to\infty} S^n y_n = p.$

Taking the limit of (6.2) as $n \to \infty$. Yields $\left\|p - Tp\right\| \le \delta\left\|p - Tp\right\|$, which implies that p=Tp. Put $\lim_{n\to\infty} S^n y_n = p$ in (6.3),implies that $\left\|T^n x_{n+1} - p\right\| \le \beta\ \left\|T^n x_{n+1} - p\right\|$, so that $\lim_{n\to\infty} T^n x_{n+1} = p.$

Taking the limit of (6.4) as $n \to \infty$. Yields $\left\|p - Sp\right\| \le \delta\ \left\|p - Sp\right\|$, which implies that p=Sp. Thus, p is a common fixed point of S and T.

7. Coincidence Point by Generalizing Iteration Sequence in (1.4).

Let F, G be two self-mapping S of X. The following iteration scheme, which is a generalization of Ishikawa iteration process, is defined as follows:-

$x_0 \in X$,

$$Gx_{n+1} = (1-\alpha_n)Gx_n + \alpha_n F^n y_n, \text{ ....(7.1)}$$

where $0 < \alpha_n \le 1$, $n \ge 1$ and $\liminf_{n\to\infty} \alpha_n > 0$.

Theorem 7.1

Let F, G be self-mapping S of a Banach space X, for any $n \ge 1, F^n, G$ are compatible, G is continuous and $\{Gx_n\}$ is defined in (7.1) converges to a point $p \in X$. If there exists constants $\alpha, \lambda, \beta, \delta, \alpha \ , \beta \ \geq 0, \beta \ , \delta < 1$ such that

$$\|F^n x_n - F^n y_n\| \leq \alpha \ \|x_n - F^n y_n\| + \beta \ \|x_n - F^n x_n\| \quad \text{.......... (7.2)}$$

and

$$\|F^n Gx_n - Fp\| \leq \alpha \|G^2 x_n - Gp\| + \beta \|G^n x_n - F^n Gx_n\| + \lambda \|Gp - F^n Gx_n\| + \delta \max\{\|Gp - Fp\|, \|G^2 x_n - Fp\|\} \quad \text{.......... (7.3)}$$

then p is a coincidence fixed point of F and G.

Proof:

By (7.1) implies that

$$Gx_{n+1} - Gx_n = (1-\alpha_n)Gx_n + \alpha_n F^n y_n - Gx_n = \alpha_n (F^n y_n - Gx_n).$$

Thus $\lim_{n\to\infty} Gx_n = p$, which implies that

$$\lim_{n\to\infty} \|Gx_{n+1} - Gx_n\| = \lim_{n\to\infty} \alpha_n \ \|F^n y_n - Gx_n\|$$

$$0 \qquad = \lim_{n\to\infty} \alpha_n \ \|F^n y_n - Gx_n\|$$

Since $\lim_{n\to\infty} \inf \alpha_n > 0$, then $\lim_{n\to\infty} \|F^n y_n - Gx_n\|$. Which mean that $\lim_{n\to\infty} \|F^n y_n - p\| = 0$, $i.e. \lim_{n\to\infty} F^n y_n = p$. Put $\lim_{n\to\infty} F^n y_n = p$ in (7.2), implies that $\|F^n x_n - p\| \leq \beta \ \|F^n x_n - p\|$, so that $\lim_{n\to\infty} F^n x_n = p$.

Since F^n and G are compatible, $\lim_{n\to\infty} \|F^n Gx_n - GF^n x_n\| = 0$. Since G is continuous, $\lim_{n\to\infty} GF^n x_n = Gp$, and hence $\lim_{n\to\infty} F^n Gx_n = Gp$.

Using the triangular inequality,

$$\|GF^n x_n - Fp\| \leq \|GF^n x_n - F^n Gx_n\| + \|F^n Gx_n - Fp\| \quad \text{.......... (7.4)}$$

Taking the limit of (7.4) as $n \to \infty$, and using (7.3), gives $\|Gp - Fp\| \leq \delta \|Gp - Fp\|$, which implies that Gp=Fp.

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الخلاصة

خلال هذا البحث استخدم اسلوب التقارب لتكرارات مان المطـــورة (modified Mann schemes) وتكـــرارات اشـــيكاوا المطـــورة (modified Ishikawa schemes) لبرهنة عدة نتائج حول وجود النقـــاط الـــصامده والنقـــاط الصامدة المشتركة ونقاط التطابق.