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SOME CHARACTERIZATIONS OF WEAKLY* m-CONTINUOUS MULTIFUNCTIONS

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Abstract

In this paper we introduce some characterizations of weakly*-m-continuous multifunctions and some results about strongly-m-continuous multifunctions.

Introduction

The concept of minimal structure space was introduced in 1996 by H. Maki [1]. In 1968 Velicko [2] introduced the concept of θ-open set. This concept has been studied intensively by many authors Al-Asadi B. J. [3] defined the concept of θm_X -open and in 2008 Al-Asadi B. J. [4] introduced the concept of weakly*-m-continuous multifunction and in this paper we introduce some characterization of weakly*-m-continuous multifunction and some results about strongly m-continuous multifunction. We obtained some properties of weakly*-m-continuous multifunction about connectedness and compactness.

Preliminaries

Let (X,τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

Definition 1-1:

A subset A of a topological space (X,τ) is said to be

(1) regular closed (resp. regular open) if $cl(int(A))=A$ (resp. $int(cl(A))=A$) [5].

(2) Preopen [6] (resp.semi-open [7], α-open [5], β-open [8]) if $A \subset \text{int}(cl(A))$ (resp. $A \subset cl(\text{int}(A))$, $A \subset \text{int}(cl(\text{int}(A)))$, $A \subset cl(\text{int}(cl(A))))$.

(3)The family of all preopen (resp. semi-open, α-open, β-open) sets in X is denoted by $PO(X)$ (resp. $SO(X)$, $\alpha(X)$, $\beta(X)$).

(4) The complement of a preopen (resp. semi-open, α-open, β-open) set is said to be preclosed (resp. semi-closed, α-closed, β-closed).

(5) The intersection of all preclosed (resp. semi-closed, α-closed, β-closed) sets of X containing A is called the preclosure (resp. semi-closure, α-closure, β-closure) of A and is denoted by $pcl(A)$ (resp. $scl(A)$, $\alpha cl(A)$, $\beta cl(A)$).

(6) A subset A is called θ-open iff $A = \text{int}_\theta(A) = \bigcup\{u : cl(U) \subseteq A, U \in \tau\}$ [2].

(7) A subset A is called θ-closed iff
$$A = cl_\theta(A) = \bigcap\{F : A \subseteq cl(F), X \setminus F \in \tau\}$$

Definitions 1-2 [1] :

(1) A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a minimal structure (briefly, m-structure) on X if $\phi \in m_X$ and $X \in m_X$. Each member of m_X is said to be m_X-open and the complement of an m_X-open is said to be m_X-closed set. We denote by (X, m_X) the m-structure space.

(2) Let (X, m_X) be an m-structure space, for a subset A of X, the m_X-interior of A and the m_X-closure of A are defined as follows :

(a) $m_X - \text{int}(A) = \bigcup\{U : U \subseteq A, U \in m_X\}$

(b) $m_X - cl(A) = \bigcap\{F : A \subseteq F, X \setminus F \in m_X\}$

Not that $m_X - \text{int}(A)$ is not necessarily m_X-open, also $m_X - cl(A)$ is not necessarily m_X-closed see[1].

(3) An m-structure m_X on a nonempty set X is said to have the property (β) if the union of any family of subsets belonging to m_X belong to m_X.

Lemma 1-3 [1] :

Let (X, m_X) be an m-structure space, for a subset A of X the following hold:

(1) $m_X - cl(X \setminus A) = X \setminus m_X - \text{int}(A)$ and $m_X - \text{int}(X \setminus A) = X \setminus m_X - cl(A)$.

(2) If $X \setminus A \in m_X$, then $m_X - cl(A) = A$ and if $A \in m_X$, then $m_X - \text{int}(A) = A$.

(3) If $A \subseteq B$, then $m_X - cl(A) \subseteq m_X - cl(B)$ and $m_X - \text{int}(A) \subseteq m_X - \text{int}(B)$.

(4) $A \subseteq m_X - cl(A)$ and $m_X - \text{int}(A) \subseteq A$

(5) $m_X - cl(m_X - cl(A)) = m_X - cl(A)$ and $m_X - \text{int}(m_X - \text{int}(A)) = m_X - \text{int}(A)$.

Lemma 1-4 [1] :

Let (X, m_X) be an m-structure space, and A a subset of X. then $x \in m_X - cl(A)$ iff $U \cap A \neq \phi$, for every $U \in m_X$ containing x.

Lemma 1-5 [1] :

For an m-structure m_X on a non-empty set X, the following are equivalent :

(1) m_X has property (β).

(2) If $m_X - \text{int}(V) = V$, then $V \in m_X$.

(3) If $m_X - cl(F) = F$, then F is m_X -closed.

Lemma 1-6 [1] :

Let (X, m_X) be an m-structure space with property (β). For a subset A of X, the following properties hold :

(1) $A \in m_X$ iff $m_X - \text{int}(A) = A$.

(2) A is m_X -closed iff $m_X - cl(A) = A$.

(3) $m_X - \text{int}(A) \in m_X$ and $m_X - cl(A)$ is m_X -closed.

Definition 1-7 [3]:

Let (X, m_X) be an m-structure space, for a subset A of X**:**

(1)The θm_X -interior of A is defined by

$\theta m_X - \text{int}(A) = \cup \{U : m_X - cl(A) \subseteq A, U \in m_X\}$

(2) A is called θm_X -open iff $\theta m_X - \text{int}(A) = A$ and the complement of A is called θm_X -closed.

(3) A point x of X is said to be a θm_X -cluster of a subset A if $m_X - cl(U) \cap A \neq \phi$ for every m_X -open set containing x.

(4) The set of all θm_X -cluster points of A is said to be θm_X -closure of A and denoted by $\theta m_X - cl(A)$.

Remark 1-8 [1] :

If an m-structure space m_X on a non-empty subset X satisfy (β), then we have every θm_X -open is m_X -open.

Remark 1-9 [3] :

Let (X, m_X) be an m-structure space. For subsets A and B of X, the following hold :

(1) $\theta m_X - cl(X \setminus A) = X \setminus \theta m_X - \text{int}(A)$ and $\theta m_X - \text{int}(X \setminus A) = X \setminus \theta m_X - cl(A)$

(2) $\theta m_X - \text{int}(A) \subseteq m_X - \text{int}(A) \subseteq A$ and $A \subseteq m_X - cl(A) \subseteq \theta m_X - cl(A)$

(3) If $A \subseteq B$, then $\theta m_X - cl(A) \subseteq \theta m_X - cl(B)$ and $\theta m_X - \text{int}(A) \subseteq \theta m_X - \text{int}(B)$

(4) A is θm_X -closed iff $\theta m_X - cl(A) = A$

Remark 1-10 [3] :

Let (X, m_X) be an m-structure space and A,B are subsets of X, then :

(1) $\theta m_X - cl(A \cup B) = \theta m_X - cl(A) \cup \theta m_X - cl(B)$

(2) $\theta m_X - cl(A \cap B) \subseteq \theta m_X - cl(A) \cap \theta m_X - cl(B)$

(3) $\theta m_X - \text{int}(A \cup B) \supseteq \theta m_X - \text{int}(A) \cup \theta m_X - \text{int}(B)$

(4) $\theta m_X - \text{int}(A \cap B) \subseteq \theta m_X - \text{int}(A) \cap \theta m_X - \text{int}(B)$

w^* -m-continuous multifunction

Recall the definition of multifunction. A multifunction $F : X \to Y$ from a topological space (X, τ) into a topological space (Y, σ) is a point to set correspondence such that $F(x) \neq \phi$ for all $x \in X$.

Definition 2-1 [9]:

Let $F:(X,\tau)\to(Y,\sigma)$ be a multifunction from a topological space (X,τ) into a topological space (Y,σ).

(1) The upper and lower inverse of a set B of the space Y are denoted by $F^{+}(B)$ and $F^{-}(B)$, respectively and defined as $F^{+}(B)=\{x\in X : F(x)\subseteq B\}$, $F^{-}(B)=\{x\in X : F(x)\cap B\neq\phi\}$.

(2) Let P(Y) be the collection of all non-empty subsets of *Y*, we define $V^{+}=\{B\in P(Y):B\subseteq V\}$ and $V^{-}=\{B\in P(Y):B\cap V\neq\phi\}$.

Definition 2-2 [4]:

A multifunction $F:(X,m_X)\to(Y,\sigma)$ where X is non-empty set with an m-structure m_X into a topological space (Y,σ) is said to be weakly*-m-continuous, (briefly w*-m-continuous) (resp. strong-m- continuous, briefly s-m-continuous) iff for each $x\in X$ and for each open sets V_1, V_2 of Y such that $F(x)\in V_1^{+}\cap V_2^{-}$, there exists $U\in m_X$ containing x such that $F(u)\in[cl(V_1)]^{+}\cap[cl(V_2)]^{-}$ (resp. $F(u)\in V_1^{+}\cap V_2^{-}$) for all $u\in m_X-cl(U)$.

Definitions 2-3:

A subset A of a topological space (X,τ) is said to be

1) α-regular [10] if for each $a\in A$ and each open set U containing a, there exists an open set G of X such that $a\in G\subset cl(G)\subset U$,
2) α-paracompact [11] if every *X*-open cover of A has an *X*-open refinement which covers A and is locally finite for each point of *X*.

For a multifunction $F:X\to(Y,\sigma)$, by $cl(F):X\to(Y,\sigma)$ we denote a multifunction defined as follows : $cl(F)(x)=cl(F(x))$ for each $x\in X$. Similarly, we denote $scl(F):X\to(Y,\sigma)$, $pcl(F):X\to(Y,\sigma)$, $\alpha cl(F):X\to(Y,\sigma)$ and $\beta cl(F):X\to(Y,\sigma)$ [9].

Lemma 2-4 [9]:

If $F:(X,m_X)\to(Y,\sigma)$ is a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x\in X$, then

1) $G^{+}(V)=F^{+}(V)$ for each open set *V* of *Y*,
2) $G^{-}(K)=F^{-}(K)$ for each closed set *K* of *Y*,

where G denotes $cl(F), pcl(F), scl(F), \alpha cl(F)$, or $\beta cl(F)$.

Lemma 2-5 [9]:

For a multifunction $F:(X,m_X)\to(Y,\sigma)$, the following properties hold :

(1) $G^{-}(V)=F^{-}(V)$ for each open set *V* of *Y*,
(2) $G^{+}(K)=F^{+}(K)$ for each closed set *K* of *Y*,

where G denotes $cl(F), pcl(F), scl(F), \alpha cl(F)$, or $\beta cl(F)$.

Theorem 2-6 [4]**:**

For a multifunction $F:(X,m_X)\to(Y,\sigma)$, the following are equivalent :

1) *F is w*-m-* continuous.
2) $F^{+}(G_1)\cap F^{-}(G_2)\subseteq \theta m_X-\text{int}(F^{+}(cl(G_1))\cap F^{-}(cl(G_2)))$ for every open set G_1, G_2 of *Y*.
3) $\theta m_X-cl(F^{-}(\text{int}(K_1))\cup F^{+}(\text{int}(K_2)) \subseteq F^{-}(K_1)\cup F^{+}(K_2)$ for every closed sets K_1, K_2 of *Y*.
4) $\theta m_X-cl(F^{-}(\text{int}(cl(B_1)))\cup F^{+}(\text{int}(cl(B_2)))) \subseteq F^{-}(cl(B_1))\cup F^{+}(cl(B_2))$ for every subsets B_1, B_2 of *Y*.
5) $F^{+}(\text{int}(B_1))\cap F^{-}(\text{int}(B_2)) \subseteq \theta m_X-\text{int}(F^{+}(cl(B_1))\cap F^{-}(cl(B_2)))$ for every subsets B_1, B_2 of *Y*.
6) $\theta m_X-cl(F^{-}(G_1)\cup F^{+}(G_2)) \subseteq F^{-}(cl(G_1))\cup F^{+}(cl(G_2))$ for every open set G_1, G_2 of *Y*.

Lemma 2-7 :

If $F:(X,m_X)\to(Y,\sigma)$ is a multi-function such that $F(x)$ is α-regular and α-paracompact for each $x\in X$, then

1) $G^{+}(V_1)\cap G^{-}(V_2)=F^{+}(V_1)\cap F^{-}(V_2)$ for each open sets V_1, V_2 of *Y*.
2) $G^{+}(K_1)\cap G^{-}(K_2)=F^{+}(K_1)\cap F^{-}(K_2)$ for each closed sets K_1, K_2 of *Y*,

where G denotes
$cl(F), pcl(F), scl(F), \alpha cl(F), \beta cl(F)$.

Proof :

The proof follows from lemma 2-4 and 2-5.

Theorem2-8 :

Let $F:(X, m_X)\to(Y,\sigma)$ be a multi-function such that $F(x)$ is α-regular and α-paracompact for each $x\in X$. Then the following properties are equivalent:

1) F is w^*-m- continuous.
2) $cl(F)$ is w^*-m- continuous.
3) $scl(F)$ is w^*-m- continuous.
4) $pcl(F)$ is w^*-m- continuous.
5) $\alpha\, cl(F)$ is w^*-m- continuous.
6) $\beta\, cl(F)$ is w^*-m- continuous.

Proof :

We put $G = cl(F)$, $scl(F)$, $pcl(F)$, $\alpha\, cl(F)$ or $\beta\, cl(F)$ in the sequel.

Necessity Suppose that F is w^*-m-continuous. Then it follows from theorem 2-6 and Lemma 2-7 that for every open sets V_1, V_2 of Y,

$$G^+(V_1)\cap G^-(V_2)=F^+(V_1)\cap F^-(V_2)$$
$$\subseteq \theta m_X-\text{int}(F^+(cl(V_1))\cap F^-(cl(V_2)))$$
$$=\theta m_X-\text{int}(G^+(cl(V_1))\cap G^-(cl(V_2)))$$

By Theorem 2-6, G is w^*-m- continuous.

Sufficiency. Suppose that G is w^*-m-continuous. Then it follows from theorem 2-6 and Lemma 2-7 that for every open sets V_1, V_2 of Y,

$$F^+(V_1)\cap F^-(V_2)=G^+(V_1)\cap G^-(V_2)$$
$$\subseteq \theta m_X-\text{int}(G^+(cl(V_1))\cap G^-(cl(V_2)))$$
$$=\theta m_X-\text{int}(F^+(cl(V_1))\cap F^-(cl(V_2)))$$

It follows from theorem 2-6 that F is w^*-m- continuous.

Theorem 2-9 :

For a multifunction $F:(X, m_X)\to(Y,\sigma)$ the following properties are equivalent :

(1) F is w^*-m- continuous.

(2) $\theta m_X-cl(F^-(\text{int}(cl(V_1)))\cup F^+(\text{int}(cl(V_2))))$
$\subseteq F^-(cl(V_1))\cup F^+(cl(V_2))$
for every preopen sets V_1, V_2 of Y.

(3) $\theta m_X-cl(F^-(V_1)\cup F^+(V_2))$
$\subseteq F^-(cl(V_1))\cup F^+(cl(V_2))$ for every preopen sets V_1, V_2 of Y.

(4) $F^+(V_1)\cap F^-(V_2)$
$\subseteq \theta m_X-\text{int}(F^+(cl(V_1))\cap F^-(cl(V_2)))$ for every preopen sets V_1, V_2 of Y.

Proof :

(1)$\Rightarrow$(2) Let V_1, V_2 be any preopen sets of Y. Since $\text{int}(cl(V_1))$ and $\text{int}(cl(V_2))$ are open, by Theorem 2-6 we have

$$\theta m_X-cl(F^-(\text{int}(cl(V_1)))\cup F^+(\text{int}(cl(V_2))))$$
$$\subseteq F^-(cl(\text{int}(cl(V_1))))\cup F^+(cl(\text{int}(cl(V_2))))$$
$$\subseteq F^-(cl(V_1))\cup F^+(cl(V_2)).$$

(2)$\Rightarrow$(3) Let V_1, V_2 be any preopen sets of Y. Then by (2) we have

$$\theta m_X-cl(F^-(V_1)\cup F^+(V_2))$$
$$\subseteq \theta m_X-cl(F^-(\text{int}(cl(V_1)))\cup$$
$$F^+(\text{int}(cl(V_2))))\subset F^-(cl(V_1))\cup F^+(cl(V_2))$$

(3)$\Rightarrow$(4) Let V_1, V_2 be any preopen sets of Y. Then by (3) and Remark 1-9 we have :

$$X\setminus\theta m_X-\text{int}(F^+(cl(V_1))\cap F^-(cl(V_2)))$$
$$=\theta m_X-cl((X\setminus F^+(cl(V_1))\cap F^-(cl(V_2)))$$
$$=\theta m_X-cl(X\setminus(F^+(cl(V_1))\cup$$
$$(X\setminus F^-(cl(V_2)))$$
$$=\theta m_X-cl(F^-(Y\setminus cl(V_1))\cup F^+(Y\setminus cl(V_2)))$$
$$\subseteq F^-(cl((Y\setminus cl(V_1)))\cup F^+(cl(Y\setminus cl(V_2)))$$

[since every open set is preopen set]

$$=F^-(Y\setminus int(cl(V_1)))\cup F^+(Y\setminus \text{int}(cl(V_2)))$$
$$=(X\setminus F^+(\text{int}(cl(V_1))))\cup(X\setminus F^-(\text{int}(cl(V_2))))$$
$$=X\setminus(F^+(\text{int}(cl(V_1)))\cap F^-(\text{int}(cl(V_2))))$$

$\subseteq X\setminus(F^+(V_1)\cap F^-(V_2))$ [since V_1, V_2 are preopen sets]

Therefore, we obtain $F^+(V_1)\cap F^-(V_2)$
$\subset \theta m_X-\text{int}(F^+(cl(V_1))\cap F^-(cl(V_2)))$

(4)$\Rightarrow$(1) Let V_1, V_2 be any open sets of Y. Since every open is preopen, then V_1, V_2 are preopen and by (4)

$F^+(V_1)\cap F^-(V_2)$
$\subset \theta m_X-\text{int}(F^+(cl(V_1))\cap F^-(cl(V_2)))$. By Theorem 2-6, F is w^*-m- continuous.

Theorem 2-10 :

If $F:(X, m_X)\to(Y,\sigma)$ is a multifunction such that F is s-m- continuous, then the following properties are satisfying :

1) $F^{+}(V_1) \cap F^{-}(V_2)$
$=\theta m_X - \text{int}(F^{+}(V_1) \cap F^{-}(V_2))$ for every open sets V_1, V_2 of Y.

2) $F^{+}(K_1) \cup F^{-}(K_2) = \theta m_X - cl(F^{+}(K_1) \cup F^{-}(K_2))$ for every closed sets K_1, K_2 of Y.

3) $\theta m_X - cl(F^{+}(B_1) \cup F^{-}(B_2)) \subseteq F^{+}(cl(B_1)) \cup F^{-}(cl(B_2))$ for every closed sets B_1, B_2 of Y.

Proof :

(1) Let V_1, V_2 be any two open subsets of Y and let $x \in F^{+}(V_1) \cap F^{-}(V_2)$ then there exists $U \in m_X$ containing x such that $F(u) \in V_1^{+} \cap V_2^{-}$ for all $u \in m_X - cl(U)$. Thus $F(u) \subseteq V_1$ and $F(u) \cap V_2 \neq \phi$ for all $u \in m_X - cl(U)$. This implies that $u \in F^{+}(V_1)$ and $u \in F^{-}(V_2)$ for all $u \in m_X - cl(U)$, then $u \in F^{+}(V_1) \cap F^{-}(V_2)$ for all $u \in m_X - cl(U)$.
Hence $m_X - cl(U) \subseteq F^{+}(V_1) \cap F^{-}(V_2)$, then $x \in \theta m_X - \text{int}(F^{+}(V_1) \cap F^{-}(V_2))$.

(2) Let K_1, K_2 be any two closed subsets of Y. Since $Y \setminus K_1, Y \setminus K_2$ are open sets in Y, then by (1) and Remark 1-9 we get

$$F^{+}(K_1) \cup F^{-}(K_2)$$
$$= (X \setminus F^{-}(Y \setminus K_1)) \cup (X \setminus F^{+}(Y \setminus K_2))$$
$$= X \setminus (F^{-}(Y \setminus K_1) \cap F^{+}(Y \setminus K_2))$$
$$= X \setminus \theta m_X - \text{int}(F^{-}(Y \setminus K_1) \cap F^{+}(Y \setminus K_2))$$
$$= X \setminus \theta m_X - \text{int}((X \setminus F^{+}(K_1)) \cap (X \setminus F^{-}(K_2)))$$
$$= X \setminus \theta m_X - \text{int}(X \setminus (F^{+}(K_1) \cup F^{-}(K_2)))$$
$$= \theta m_X - cl(F^{+}(K_1) \cup F^{-}(K_2))$$

(3) Let B_1, B_2 be any two closed subsets of Y, the by (2) we get:

$$= \theta m_X - cl(F^{+}(B_1) \cup F^{-}(B_2))$$
$$= F^{+}(B_1) \cup F^{-}(B_2)$$
$$\subseteq F^{+}(cl(B_1)) \cup F^{-}(cl(B_2))$$

Theorem 2-11 :

If $F:(X, m_X) \to (Y, \sigma)$ is a multi-function such that F is s-m- continuous, then the following properties are satisfying :

(1) $F^{-}(cl_\theta(B_1)) \cup F^{+}(cl_\theta(B_2))$
$= \theta m_X - cl(F^{-}(cl_\theta(B_1)) \cup F^{+}(cl_\theta(B_2)))$ for every subsets B_1, B_2 of Y.

(2) $F^{-}(K_1) \cup F^{+}(K_2)$
$= \theta m_X - cl(F^{-}(K_1) \cup F^{+}(K_2))$ for every θ-closed sets K_1,K_2 of Y.

(3) $F^{-}(V_1) \cap F^{+}(V_2)$
$= \theta m_X - \text{int}(F^{-}(V_1) \cap F^{+}(V_2))$ for every θ-open sets V_1, V_2 of Y.

Proof :

(1) Let B_1, B_2 be any subsets of Y, then $cl_\theta(B_1)$ and $cl_\theta(B_2)$ are closed in Y. By Theorem 2-10(2), we get :

$$F^{-}(cl_\theta(B_1)) \cup F^{+}(cl_\theta(B_2))$$
$$= \theta m_X - cl(F^{-}(cl_\theta(B_1)) \cup F^{+}(cl_\theta(B_2)))$$

(2) Let K_1, K_2 be θ-closed sets of Y, then $cl_\theta(K_1) = K_1$ and $cl_\theta(K_2) = K_2$.
Therefore by (1) we get :

$$F^{-}(K_1) \cup F^{+}(K_2)$$
$$= \theta m_X - cl(F^{-}(K_1) \cup F^{+}(K_2))$$

(3) let V_1, V_2 be θ-open sets of Y, then $Y \setminus V_1$ and $Y \setminus V_2$ are θ-closed and by (2), we get :

$$F^{-}(Y \setminus V_1) \cup F^{+}(Y \setminus V_2)$$
$$= \theta m_X - cl(F^{-}(Y \setminus V_1) \cup F^{+}(Y \setminus V_2)),$$

hence $X \setminus (F^{+}(V_1) \cap F^{-}(V_2)) =$

$$(X \setminus F^{+}(V_1)) \cup (X \setminus F^{-}(V_2))$$
$$= \theta m_X - cl((X \setminus F^{+}(V_1)) \cup (X \setminus F^{-}(V_2)))$$
$$= \theta m_X - cl(X \setminus (F^{+}(V_1) \cap F^{-}(V_2)))$$
$$= X \setminus \theta m_X - \text{int}(F^{+}(V_1) \cap F^{-}(V_2))$$

Therefore we have :

$$F^{-}(V_1) \cap F^{+}(V_2)$$
$$= \theta m_X - \text{int}(F^{-}(V_1) \cap F^{+}(V_2)).$$

Remark 2-12:

For Theorem 2-11, we have the following :
If F is s-m- continuous, then

(1) $F^{-}(cl_\theta(B_1)) \cup F^{+}(cl_\theta(B_2))$ is θ-closed set of X, for every subsets B_1, B_2 of Y.

(2) $F^{-}(K_1) \cup F^{+}(K_2)$ is θ-closed set of X, for every θ-closed sets K_1, K_2 of Y.

(3) $F^{-}(V_1) \cap F^{+}(V_2)$ is θ-open set of *X*, for every θ-open sets V_1, V_2 of *Y*.

Definition 2-13 [12]:

A nonempty set *X* with a minimal structure m_X is said to be m-connected if X can not be written as the union of two nonempty disjoint m_X-open sets.

Definition 2-14 [13]:

A topological space (X,τ) is said to be semi-connected (resp. preconnected, α-connected, β- connected) if X can not be written as the union of two nonempty disjoint semi-open (resp. preopen, α-open, β-open) sets.

Theorem 2-15 :

Let (X,m_X) be a nonempty set with a minimal structure m_X satisfying property (β) and (Y,σ) a topological space. If $F:(X,m_X)\to(Y,\sigma)$ is w*-m-continuous surjective multifunction such that *F*(*x*) is connected for each $x\in X$ and (X,m_X) is m-connected, then (Y,σ) is connected.

Proof :

Suppose that (Y,σ) is not connected. Then there exist non-empty open sets $U, V\in\sigma$ such that $U\cap V=\phi$ and $U\cup V=Y$.

Since *F*(*x*) is connected for each $x\in X$, either $F(x)\subset U$ or $F(x)\subset V$. If $x\in F^{+}(U\cup V)$, then $F(x)\subset U\cup V$ and hence $x\in F^{+}(U)\cup F^{+}(V)$. Moreover, since *F* is surjective, there exist *x* and *y* in *X* such that $F(x)\subset U$ and $F(y)\subset V$, hence $x\in F^{+}(U)$ and $y\in F^{+}(V)$. Therefore, we obtain the following :

1) $F^{+}(U)\cup F^{+}(V)=F^{+}(U\cup V)=X$

2) $F^{+}(U)\cap F^{+}(V)=F^{+}(U\cap V)=\phi$

3) $F^{+}(U)\neq\phi$ and $F^{+}(V)\neq\phi$

Now, we show that $F^{+}(U)$ and $F^{+}(V)$ are m_X-open in X.

Let $x\in F^{+}(U)$, then $F(x)\subset U$, hence $F(x)\cap U\neq\phi$. Thus $x\in F^{-}(U)$ and $x\in F^{+}(U)\cap F^{-}(U)$. Since *F* is w*-m-continuous, then there exists $W\in m_X$ containing x such that $F(w)\in[cl(U)]^{+}\cap[cl(U)]^{-}$ for all $w\in m_X-cl(W)$. That is $w\in F^{+}(cl(U))\cap F^{-}(cl(U))$ for all $w\in m_X-cl(W)$, hence

$m_X-cl(W)\subset F^{+}(cl(U)\cap F^{-}(cl(U))$. Then we get $m_X-cl(W)\subset F^{+}(cl(U))=F^{+}(U)$ since U is clopen. Therefore $x\in\theta m_X-\text{int}(F^{+}(U))$, that is $F^{+}(U)$ is θm_X-open. Since *X* satisfying (β) then $F^{+}(U)$ is m_X-open.

Similarly, we obtain $F^{+}(V)$ is m_X-open in *X*. Consequently, this shows that (X,m_X) is not m-connected. This completes the proof.

Corollary 2-16 :

Let (X,τ) and (Y,σ) be topological spaces and $F:(X,\tau)\to(Y,\sigma)$ be a surjective multifunction such that *F(x)* is connected for each $x\in X$. If (X,τ) is connected (resp. semi-connected, preconnected, α-connected, β-connected) and *F* is w*-m-continuous (resp. semi-continuous, percontinuous, α-continuous, β-continuous), then (Y,σ) is connected.

Proof:

Let $m_X=\tau$ (resp. *so*(*X*), *po*(*X*), $\alpha(X)$, $\beta(X)$) and $F:(X,m_X)\to(Y,\sigma)$ be w*-m-continuous surjective multifunction such that *F(x)* is connected for each $x\in X$. Then by Theorem 2-15 we obtain the result.

Definition 2-17 [9] :

A nonempty set *X* with a minimal structure m_X is said to be m-compact if every cover of X by m_X-open sets has a finite subcover.

A subset K of a nonempty set *X* with a m-structure m_X is said to be m-compact if every cover of K by m_X-open sets has a finite subcover.

Definition 2-18 [9] :

A topological space (X,τ) is said to be quasi H-closed if for every open cover $\{U_\alpha:\alpha\in\Delta\}$ of *X*, there exists a finite subset Δ_0 of Δ such that $X=\cup\{cl(U_\alpha):\alpha\in\Delta_0\}$.

Theorem 2-19 :

Let $F:(X,m_X)\to(Y,\sigma)$ is w*-m-continuous surjective multifunction such that *F(x)* is m-compact for each $x\in X$. If (X,m_X) is m-compact, then (Y,σ) is quasi H-closed.

Proof :

Let $\{V_\alpha:\alpha\in\Delta\}$ be any open cover of *Y*. For each $x\in X$, F(x) is compact and there exists a finite subset $\Delta(x)$ of Δ such that $F(x)\subset\bigcup\{V_\alpha:\alpha\in\Delta(x)\}$. Now, set $V(x)=\bigcup\{V_\alpha:\alpha\in\Delta(x)\}$, then we have $F(x)\subset V(x)$.

Since F is w*-m-continuous, there exists $U(x)\in m_X$ containing x such that $F(u)\in[cl(V(x))]^+\cap[cl(V(x))]^-$ for all $u\in m_X-cl(U(x))$ this implies that

$m_X-cl(U(x))\subset F^+(cl(V(x)))\cap$
$F^-(cl(V(x)))$

$\Rightarrow m_X-cl(U(x))\subset F^+(cl(V(x)))$

$\Rightarrow F(m_X-cl(U(x)))\subset cl(V(x))$

$\Rightarrow F(U(x))\subset F(m_X-cl(U(x)))\subset cl(V(x))$

$\Rightarrow F(U(x))\subset cl(V(x))$

The family $\{U(x):x\in X\}$ is a cover of X by m_X -open sets.

Since X is m-compact, there exists a finite number of points, say, $x_1,x_2,\ldots,x_n$ in *X* such that $X=\bigcup\{U(x_i):x_i\in X,i=1,\ldots,n\}$. Hence we obtain

$Y=F(X)=\bigcup\{F(U(x_i)):i=1,\ldots,n\}$

$\subset\bigcup\{cl(V(x_i)):i=1,\ldots,n\}$

$\subset\bigcup\{cl(V_\alpha):\alpha\in\Delta(x_i),i=1,\ldots,n\}$.

This show that (Y,σ) is quasi H-closed.

Definition 2-20 [9] :

Let (X,m_X) be a m-structure space and $A\subseteq X$. The set $\theta m_X-fr(A)$ $=\theta m_X-cl(A)\setminus\theta m_X-\mathrm{int}(A)$ is said to be θm_X -frontier of *A*.

Theorem 2-21 [9] :

Let (X,m_X) be a m-structure space and $A\subseteq X$, then

$\theta m_X-fr(A)$
$=\theta m_X-cl(A)\cap\theta m_X-cl(X\setminus A)$

Theorem 2-22 :

Let *X* be a non empty set with a m-structure m_X and (Y,σ) a topological space. The set of all points *x* of *X* of which a multifunction $F:(X,m_X)\to(Y,\sigma)$ is not w*-m-continuous is identical with the union of the θm_X -frontier of the inverse images of the closures of open sets which containing and meeting *F*(*x*).

Proof :

Let *x* be a point of *X* at which *F* is not w*-m- continuous. Then there exists an open sets V_1,V_2 in *Y* such that $F(x)\in V_1^+\cap V_2^-$ and such that $F(u)\notin[cl(V_1)]^+\cap[cl(V_2)]^-$ for every $U\in m_X$ containing x and for all $u\in m_X-cl(U)$.

Hence $u\notin F^+(cl(V_1))\cap F^-(cl(V_2))$ for all $u\in m_X-cl(U)$, then

$m_X-cl(U)\not\subset F^+(cl(V_1))\cap F^-(cl(V_2))$ thus

$m_X-cl(U)\cap(X\setminus(F^+(cl(V_1))\cap$
$F^-(cl(V_2))))\neq\phi$. By definition 1-7, we have

$x\in\theta m_X-cl(X\setminus(F^+(cl(V_1))\cap$
$F^-(cl(V_2))))$ Since $x\in F^+(V_1)\cap F^-(V_2)$

$\subseteq F^+(cl(V_1))\cap F^-(cl(V_2))$

$\subseteq\theta m_X-cl(F^+(cl(V_1))\cap F^-(cl(V_2)))$, we have by Theorem 2-21

$x\in\theta m_X-fr(F^+(cl(V_1))\cap F^-(cl(V_2)))$.

Conversely, if *F* is w*-m- continuous at *x*, then for any open sets V_1,V_2 in Y such that $F(x)\in V_1^+\cap V_2^-$, there exists $U\in m_X$ containing x such that $F(u)\in[cl(V_1)]^+\cap[cl(V_2)]^-$ for all $u\in m_X-cl(U)$.

Hence

$m_X-cl(U)\subset F^+(cl(V_1))\cap F^-(cl(V_2))$.

Therefore, we obtain $x\in m_X-cl(u)$

$\subset\theta m_X-\mathrm{int}(F^+(cl(V_1))\cap F^-(cl(V_2)))$. This contradicts that

$x\in\theta m_X-fr(F^+(cl(V_1))\cap F^-(cl(V_2)))$.

الخلاصة

في هذا البحث قدمنا بعض المكافئات للدوال المتعـــددة القيم الضعيفة* المستمرة من النمط m وقمنا بإيجاد بعـــض النتائج بما يخص الدوال المتعددة القيم القوية من النمط m .

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