CONVERGENCE ACCELERATION OF HERMITE-FEJÉR POLYNOMIALS BASED ON LEGENDRE NODES IN THE INTERVAL [-1,1]

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Abstract

The rate of uniform convergence for Hermite-Fejer polynomials to any continuous function *f(x)* in each closed sub-interval of *(-1,1)* has been given by Schonhage in 1971 by means of estimating the rate of convergence. The present paper deals with the acceleration of convergence and the rate of convergence by improving the estimate given by Schonhage, throughout two parallel ways, firstly, by use of the averaged moduli of smoothness or τ-moduli that gives much better estimation than that of the moduli of continuity or ω-moduli. Secondly, by make use of the necessary and sufficient conditions that we borrow from Szego in 1959 together with the well-known Fejer's identity (3.8) and the properties of τ-moduli in addition to some known results that have been given by Murray Spiegel in 1981 pp299-345.

Keywords: Hermite-Fejer's polynomials, Legendre nodes, acceleration of convergence, moduli of smoothness.

Introduction

Let *f(x)* be a continuous function defined on the closed interval *[-1,1]*. Recall Legendre polynomials of the first kind (see **[6]** pp. 343-345) which are solutions to the Legendre differential equation:

$$\left(1-x^2\right)y''+2y'+m(m+1)y=0 \quad \text{..........} (1.1)$$

where *m* is a real number. This equation can be solved by means of Frobinous series method (see **[7]**) as follows: Since *x=0* is an ordinary point of the equation, then the solution will be in the form

$$y=\sum_{j=-\infty}^{\infty} a_j x^j \quad \text{.....................................} (1.2)$$

From now on, we shall omit the limits of the summation which are from -∞, to ∞. The singular points of this series are *x=±1*, that means it should be converge at least in the open interval *(-1,1)*

From (1.2) we have

$$y=\sum a_j x^j,\ y'=\sum j a_j x^{j-1},\ y''=\sum j(j-1)a_j x^{j-2} \quad \text{...} (1.3)$$

Substituting (1.3) into (1.1) we get

$$\sum j(j-1)a_j x^{j-2}-\sum j(j-1)a_j x^j-\sum 2ja_j x^j+\sum m(m+1)a_j x^j=0$$

i.e.

$$\sum\left[(j-2)(j+1)a_{j+2}-j(j-1)a_j-2ja_j+m(m+1)a_j\right]x^j=0$$

or

$$\sum\left[(j+2)(j+1)a_{j+2}+\left[m(m+1)-j(j+1)a_j\right]\right]x^j=0$$

Since $x^j \neq 0$, then

$$(j+2)(j+1)a_{j+2}+\left[m(m+1)-j(j+1)\right]=0 \quad \text{..............................} (1.4)$$

Putting *j=-2* in (1.4) shows that a_0 is arbitrary. Putting *j=-1* in (1.4) shows that a_1 is arbitrary. from (1.4) the general solution is given as: **[7]**

$$a_{j+2}=-\frac{\left[m(m+1)-j(j+1)\right]}{(j+2)(j+1)}a_j$$

Putting *j=0,1,2,3,…* in secession, we find

$$a_2=-\frac{\left[m(m+1)\right]}{2!}a_0,$$

$$a_3=-\frac{\left[m(m+1)-1\cdot 2\right]}{3!}a_1,$$

$$a_4=-\frac{\left[m(m+1)-2\cdot 3\right]}{4\cdot 3}$$

$$a_2=\frac{m(m+1)\left[m(m+1)-2\cdot 3\right]}{4!}a_0,$$

$$a_5=-\frac{\left[m(m+1)-3\cdot 4\right]}{5\cdot 4}$$

$$a_3=\frac{\left[m(m+1)-1\cdot 2\right]\left[m(m+1)-3\cdot 4\right]}{5!}a_1,\ etc.$$

$$\therefore y = a_0[1 - \frac{m(m+1)}{2!}x^2 + \frac{m(m+1)[m(m+1)-2\cdot 3}{4!}x^4 - \mathbf{L}] + $$
$$+ a_1[x - \frac{[m(m+1)-1\cdot 2]}{3!}x^2 + \frac{[m(m+1)-1\cdot 2][m(m+1)-3\cdot 4}{5!}x^5 - \mathbf{L}]$$

.... (1.5)

Since m is a real number (not an integer), (see **[7]** p.344) both of the two series in (1.5) converge when $-1<x<1$, but they diverge for $x=\pm 1$. If m is a positive integer or zero, one of these series becomes a polynomial, while the other series converges for $-1<x<1$ but diverges for $x=\pm 1$. To find the polynomial solutions, for $m=0,1,2,3,\ldots$, we obtain $1,\ x,\ 1-3x^2,\ x-5/3x^3,\ldots$ which are polynomials of degree $0,1,2,3,\ldots$ respectively.

Multiplying each of these polynomials by a constant so chosen that the resulting polynomial has the value 1 when $x=1$. The resulting polynomials are called Legendre polynomials and are denoted by $P_n(x)$. Where

$$p_0(x) = 1,$$
$$p_1(x) = x,$$
$$p_2(x) = \frac{1}{2}(3x^2 - 1),$$
$$p_3(x) = \frac{1}{2}(5x^3 - 3x),$$
$$p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$$
$$p_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

Remarks:

Legendre polynomials satisfy the following properties (see **[7]**):

1. $P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n+1}(x)$.
2. $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$.
3. $\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$
4. Legendre polynomials are the only solution of Legendre equation **[4]** which are bounded in $-1 \le x \le 1$, since the series giving all other solutions diverge for $x=\pm 1$
5. If $P_n(x)=0$ for $n=-1,-2,\ldots$, then we get $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$, $\int P_n(x)dx = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1} + C$
6. $P'_{n+1}(x) - P'_n(x) = (n+1)P_n(x)$
7. $xP'_n(x) - P'_{n-1}(x) = nP_n(x)$
8. $P_{2n+1}(0) = 0$
9. $P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}$
10. $P_n(-x) = (-1)^n P_n(x),\quad P_n(1) = 1,\quad P_n(-1) = (-1)^n$

Fig.(1) below (see**[6]**) represents Legendre polynomials of the first kind $p_0(x),\ p_1(x),\ \ldots,\ p_5(x)$ defined on the closed interval $[-1,1]$.

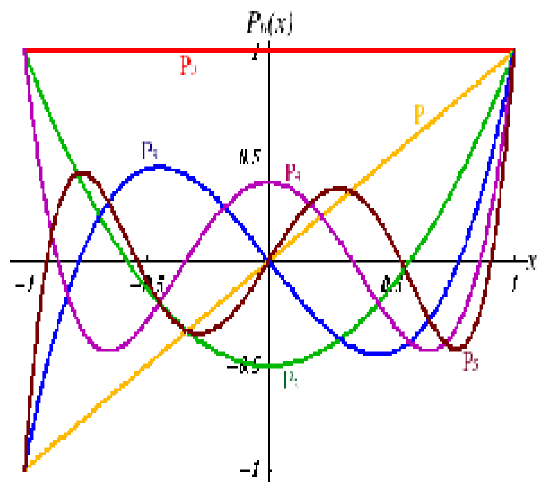

Fig.(1) : Legndre polynomials $p_i(x)$, $i=0,1,\ldots,5$ of the first kind [6].

2. Hermite-Fejer's polynomials and the estimate of the rate of convergence [9]

Let $1>x_{1,n}>x_{2,n}>\ldots>x_{n,n}>-1$ be the roots of the Legendre polynomial $p_n(x)$ of degree n. The general form of Hermite-Fejer interpolation polynomials of degree $\le 2n-1$(see **[9]**) is

$$H_n(f,x) = \sum_{k=1}^{n} f(x_{k,n}) \frac{1-2xx_{k,n}+x_{k,n}^2}{1-x_{k,n}^2} m_{k,n}^2(x)$$

................................(2.1)

these polynomials satisfy

$$H_n(f,x_{k,n}) = f(x_{k,n}),\quad H'_n(f,x_{k,n}) = 0,$$

$$m_{k,n}(x)=\frac{P_n(x)}{(x-x_{k,n})P_n'(x_{k,n})}.\ \text{where}$$

L. Fejer has proved (see **[8]**) that the sequence $H_n(f,x)$ converges uniformly to the continuous function $f(x)$ for $|x|<1$ in each closed subinterval of $(-1,1)$. Also he proved that the following limit at the end points of the interval $[-1,1]$ is satisfied

$$\lim_{n\to\infty} H_n(f,\pm1)=\frac{1}{2}\int_{-1}^{1} f(x)dx \quad \text{.....................(2.2)}$$

It has been given in **[1]** that the condition

$$f(\pm1)=\frac{1}{2}\int_{-1}^{1} f(x)dx \quad \text{..................................(2.3)}$$

is necessary and sufficient for

$$\lim_{n\to\infty}\left\| f(x)-H_n(f,x)\right\|_{C[-1,1]}=0, \quad |x|\le 1 \quad \text{..... (2.4)}$$

The rate of convergence has been estimated in **[9]** for a continuous function $f(x)$ in $[-1,1]$ as:

$$|f(x)-H_n(f,x)|=$$

$$\max\left[\left|f(1)-\frac{1}{2}\int_{-1}^{1}f(x)dx\right|,\ \left|f(-1)-\frac{1}{2}\int_{-1}^{1}f(x)dx\right|\right]\times \quad \text{........ (2.5)}$$

$$\times O\left(\frac{1}{n\sqrt{1-x^2}}\right)+O\left(\omega_f\left(\frac{1}{\sqrt{\frac{\log n}{n}}}\right)\right)$$

where $\omega_f(.)=\omega(f,\delta)$ denotes the modulus of continuity of $f(x)$ **[2,3]**.

The graphical representation (see **[11]**) of Legendre polynomials of the second kind is shown in Fig.(2) below

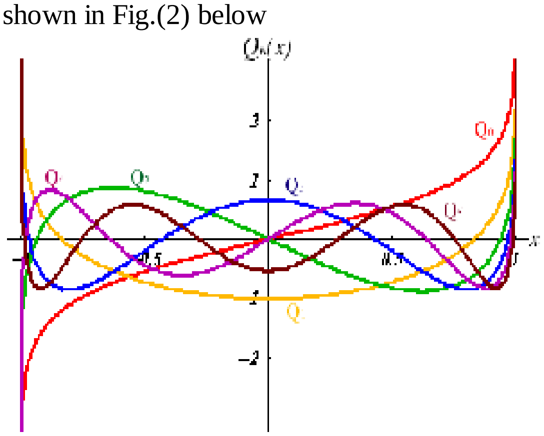

Fig.(2): [11] Legendre polynomials of the second kind Q_n, $n=0,1,2,\ldots,5$ (solutions).

Definition: [7] Hermite Equation

$$y''-2xy'+2my=0$$

is a special case of the Sturm-Liouville Boundary Value Problem. It arises in the treatment of the harmonic oscillator in quantum mechanics. This equation has solutions called Hermite polynomials when $n=0,1,2,\ldots$ and are denoted by $H_n(x)$ and have many important properties analogous to those of Bessel functions and Legendre polynomials **[7]** such as:

1. $H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x)$
2. $H_n(x)=(-1)^n e^{x^2}\dfrac{d^n}{dx^n}\left(e^{-x^2}\right)2.$
3. $e^{2tx-t^2}=\displaystyle\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}$

Below is the first six Hermite polynomials:

$$H_0(x)=1,\ H_1(x)=2x,\ H_2(x)=4x^2-2,$$
$$H_3(x)=8x^3-12x,\ H_4(x)=16x^4-48x^2+12,$$
$$H_5(x)=32x^5-160x^3+120x$$

Note: In Fig.(3) below Hermite polynomials are scaled down by a factor of n^2 in order to be fit on the same plot i.e.

$$H_0(x)=1$$
$$H_1(x)=2x$$
$$H_2(x)/4=x^2-\frac{1}{2}$$
$$H_3(x)/9=\frac{8}{9}x^3-\frac{4}{3}x$$
$$H_4(x)/16=x^4-3x^2+\frac{3}{4}$$
$$H_5(x)/25=\frac{32}{25}x^5-32x^3+24x$$

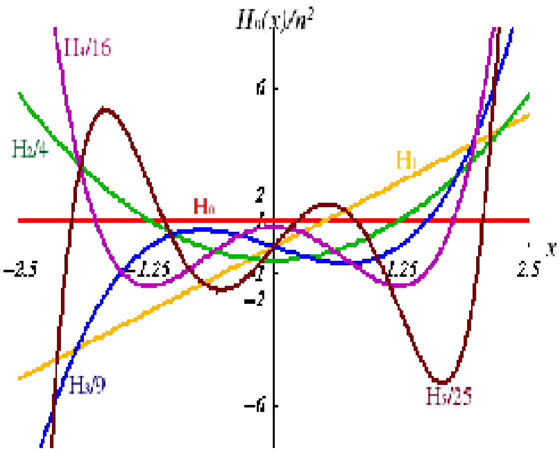

Fig.(3) : graphical representation of Hermite polynomials $H_n(x)/n^2$, n=0,1,2,3, 4,5 (see [10]).

If equation (2.1) holds, Schonhage has proved in **[1]** the following estimate as a corollary:

$$\left|f(x)-H_n(f,x)\right| = O\left(\omega_f\left(\frac{\log n}{n}\right)\right), \qquad |x|\le 1 \qquad \text{.............................. (2.6)}$$

3. Preliminaries and moduli of smoothness

In order to improve error estimates (2.5) and (2.6) by use of the well known averaged moduli of smoothness **[2]** whenever possible. In addition to these moduli,the given estimates in section 4 (see **[5]**) will definitely give a better acceleration to the rate of convergence and minimize the error of the estimation of Hermite-Fejer polynomials.

Some notations and definitions [2]:

The modulus of continuity is used to measure the continuity of a function $f\in C_{[a,b]}$ and is defined as

$$\omega(f;\delta)=\omega(\delta)$$
$$=\sup\left\{\left|f(x)-f(x')\right| : |x-x'|\le\delta, x, x'\in[a,b]\right\} \qquad \text{............................ (3.1)}$$

The k^{th} difference with step h at a point x for every continuous function f is

$$\Delta_h^k f(x)=\sum(-1)^m\binom{k}{m}f(x+mh),$$

$$\binom{k}{m}=\frac{k!}{m!(k-m)!} \quad \text{where}$$

is the binomial coefficient

The modulus of smoothness of order k of a continuous function f is

$$\omega_k(f;\delta)=\sup\left\{\left|\Delta_h^k f(x)\right| : |h|\le\delta, x, x+kh\in[a,b]\right\}$$
$$\text{.............................. (3.2)}$$

and has the following property:

$$\omega_k(f;\lambda\delta)\le(\lambda+1)^k\omega_k(f;\delta), \lambda,\delta>0 \text{ .... (3.3)}$$

The integral L_p-modulus (or p-modulus) of order k of the function f is

$$\omega_k(f;\delta)_{L_p} = \sup_{0\le h\le\delta}\left\{\int_a^{b-kh}\left|\Delta_h^k f(x)\right|^p\right\}^{1/p}$$
$$\text{.............................. (3.4)}$$

The local modulus of smoothness of the continuous function f of order k at a point $x\in[a,b]$ is defined as

$$\omega_k(f,x;\delta)=\sup\left\{\left|\Delta_h^k f(t)\right| : t, t+kh\in\left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\cap[a,b]\right\}$$
$$\text{.............................. (3.5)}$$

so

$$\omega_k(f;\delta)=\left\|\omega_k(f,\cdot;\delta)\right\|_{C_{[a,b]}} \text{ .... (3.6)}$$

Evidently, the averaged modulus of smoothness of order k (τ-modulus) of the function $f\in M_{[a,b]}$ (M is the set of bounded and measurable functions on $[a,b]$) is defined as

$$\tau_k(f;\delta)=\left\|\omega_k(f,\cdot;\delta)\right\|_{L_p}=\left[\int_a^b(\omega_k(f,x;\delta))^p dx\right]^{1/p}$$
$$\text{.............................. (3.7)}$$

and has the following properties

$$\tau_k(f,\lambda\delta)_p\le(2(\lambda+1))^{k+1}\tau_k(f;\delta)_p, \lambda,\delta>0$$
$$\text{.............................. (3.8)}$$
$$\tau_k(f;\delta')\le\tau_k(f;\delta''), \delta'\le\delta'' \text{ .... (3.9)}$$

4. Acceleration of Convergence

For the sake of simplicity, from now on we shall write x_k for $x_{k,n}$.

Recall the following estimates (see **[5]**):

$$P_n'(x_k)\approx n^2\sqrt{\frac{1}{k^3}}, k=1,2,\ldots,\left[\frac{n}{2}\right], \text{ .... (4.1)}$$

is the greatest integer less than $\left[\frac{n}{2}\right]$ where

$\frac{n}{2}$ or equal to

$$p_n(x_k) \approx n^2\sqrt{n+1-k} \quad ,k=\left[\frac{n}{2}\right]+1,\ldots,n \qquad (4.2)$$

$$1-x_k^2 > \frac{\left(k-\frac{1}{2}\right)^2}{\left(n+\frac{1}{2}\right)^2}, k=1,2,\ldots,\left[\frac{n}{2}\right] \qquad (4.3)$$

$$1-x_k^2 > \left(n-k+\frac{1}{2}\right)^2\sqrt{n+\frac{1}{2}}, k=\left[\frac{n}{2}\right]+1,\ldots,n \qquad (4.4)$$

$$\frac{\left(k-\frac{1}{2}\right)\pi}{n+\frac{1}{2}} < \theta_k < \frac{k\pi}{n+\frac{1}{2}}, k=1,2,\ldots,n \qquad (4.5)$$

$$P_n(x)\sqrt[4]{1-x^2} \le \frac{1}{\sqrt{n}\sqrt{2/\pi}}, \qquad (4.6)$$

where $x_k=\cos\theta_k$; $x=\cos\theta$

These estimates will play a basic role in the proof of the following lemma.

Lemma:

Let x_j be the root of $P_n(x)$ which is nearest to x, then

$$A_k(x)=\frac{P_n^2(x)(1-x^2)}{P_n'^2(x_k)(1-x_k^2)(x-x_k)^2}=O\left(\frac{1}{i^2}\right)$$

$$when\ k\neq j, k=j\pm 1$$

Proof:

To prove that

$$A_k(x)=O\left(\frac{1}{i^2}\right)\ when\ k\neq j, k=j\pm 1$$

From (4.6) we have

$$(*)[P_n(x)\sqrt[4]{1-x^2}]^2=\sqrt{(1-x^2)}p_n^2(x)$$

For $x=\cos\theta$ and $x_k=\cos\theta_k$ (4.6) we find from calculus (see for example, Thomas "calculus and analytic geometry" 4th eddition) that when $x_k=\cos\theta_k$ then

$$x_k=\cos\theta_k$$

$$x_k^2=\cos^2\theta_k$$

$$1-x_k^2=1-\cos^2\theta_k$$

$$(1-x_k^2)=\sin^2\theta_k$$

$$\sqrt{1-x_k^2}=\sin\theta_k$$

Therefore eq. (*) and the ineq. (**) yield

$$A_k(x)\le\frac{P_n^2(x)\sqrt{1-x^2}}{P_n'^2(x_k)(1-x_k^2)}\cdot\frac{1}{2\sin^2\frac{\theta-\theta_k}{2}\sin\frac{\theta+\theta_k}{2}}$$

$$\le\frac{P_n^2(x)\sqrt{1-x^2}}{P_n'^2(x_k)(1-x_k^2)}\frac{1}{2\sin\frac{\theta-\theta_k}{2}\sqrt{1-x_k^2}}$$

$$\le\frac{P_n^2(x)\sqrt{1-x^2}}{\sin^2\frac{\theta-\theta_k}{2}P_n'^2(x_k)\sqrt{(1-x_k^2)^3}}=O\left(\frac{1}{i^2}\right)$$

when $k\neq j\pm i$ and x_j is the zero of $P_n(x)$ which is the nearest to x, by hypothesis, then

$$A_k(x)=O\left(\frac{1}{i^2}\right) \qquad \blacksquare$$

Theorem 1:

Let $f(x)$ be a continuous function defined on the interval $[-1,1]$, then

$$|f(x)-H_n(f,x)|=$$

$$=\max\left[\left|f(\pm1)-\frac{1}{2}\int_{-1}^{1}f(x)dx\right|\right]P_n^2(x)+$$

$$O\left[P_n^2(x)\tau\left(f;\frac{1}{n}\right)\right]+\sum_{i=1}^{n}\frac{1}{i^2}\tau\left(f;\frac{i|P_n(x)|\sqrt[3]{(1-x^2)^4}}{\sqrt{n}}\right),$$

where $x\in[-1,1]$.

Hermite-Fejer's polynomial (2.1) yields

$$H_n(f,x)=\sum_{k=1}^{n}f(x_k)\frac{P_n^2(x)}{P_n'^2(x_k)(1-x_k^2)}+\sum_{k=1}^{n}f(x_k)\frac{1-x^2}{1-x_k}m_k^2(x) \qquad (4.7)$$

and Fejer's identity **[4]** is

$$\sum_{k=1}^{n}\frac{1}{P_n'^2(x_k)(1-x_k^2)}=1 \qquad (4.8)$$

Using the last identity with equation (4.7) to obtain the following identity

$$P_n(x)+\sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}l_k^2(x)=1 \qquad (4.9)$$

This identity and equation (3.7) give

$$f(x)-H_n(f,x)=$$

$$P_n^2(x)\left[f(x)-\sum_{k=1}^{n}\frac{f(x_k)}{P_n'^2(x_k)(1-x_k^2)}\right]+$$

$$[f(x)-f(x_k)]A_k(x)=T_1+T_2 \qquad (4.10)$$

Let us estimate T_1 firstly as $0\le x\le 1$. Then

$$|T_1| \le P_n^2(x)\left[\begin{array}{l}|f(x)-f(1)|+\left|f(1)-\frac{1}{2}\int_{-1}^{1}f(x)dx\right|+\\ \left|\frac{1}{2}\int_{-1}^{1}f(x)dx-\sum_{k=1}^{n}\frac{f(x_k)}{P_n'^2(x_k)(1-x_k^2)}\right|\end{array}\right]$$

.... (4.11)

Recall estimation (2.6) and the fact that $|P_n(x)|\le 1$ (see **[5]**) and τ-property (3.9), and the last inquality (4.11) to get

$$S_1 \equiv P_n^2(x)|f(x)-f(1)| \le P_n^2(x)\tau_k(f;(1-x))$$

$$\le P_n^2(x)\tau_k(f;1-x^2)$$

$$\le \tau_k\left(f;\left[\left[\frac{\left|P_n(x)\sqrt[4]{(1-x^2)^3}\right|}{\sqrt{n}}\right]\right]\left[1+\frac{\sqrt{n}\sqrt[4]{(1-x^2)^3}}{|P_n(x)|}\right]\right)$$

$$P_n^2(x) \le \frac{5}{2}\tau_k\left(\frac{\left|P_n(x)\sqrt[4]{(1-x^2)^3}\right|}{\sqrt{n}}\right) \quad ....(4.12)$$

Let $E_n(f)$ denotes the deviation from $f(x)$ of the polynomial $q_n(x)$ of degree less than or equal n of best approximation on the interval $[-1,1]$, then by Jackson's theorem [3] we deduce

$$E_n(f) \le c\tau\left(f;\frac{1}{n}\right), \qquad c>0 \quad .... (4.13)$$

The Gauss-Jacobi quadrature formula is exact for polynomials of degree $\le 2n-1$. Thus

$$\frac{1}{2}\int_{-1}^{1}q_n(x)dx = \sum_{k=1}^{n}\frac{q_n(x_k)}{P_n'^2(x_k)(1-x_k^2)} \quad .... (4.14)$$

In virtue of Fejer's identity (4.8) and eqs.(4.11) and (4.12) we find (4.12)

where $c_1>0$

Equations (4.11), (4.12) and (4.15) imply the estimate

$$|T_1| = O\left[\tau\left(f;\frac{|P_n(x)|\sqrt[4]{(1-x^2)^3}}{\sqrt{n}}\right)+P_n^2(x)\tau\left(f;\frac{1}{n}\right)\right]+$$

$$P_n^2(x)\left|f(1)-\frac{1}{2}\int_{-1}^{1}f(x)dx\right|$$

.... (4.16)

Similarly, when $-1\le x\le 0$ we find the estimate

$$|T_1| = O\left[\tau_k\left(f;\frac{|P_n(x)|\sqrt[4]{(1-x^2)^3}}{\sqrt{n}}\right)+P_n^2(x)\tau_k\left(f;\frac{1}{n}\right)\right]+$$

$$P_n^2(x)\left|f(-1)-\frac{1}{2}\int_{-1}^{1}f(x)dx\right| \quad ....(4.17)$$

Therefore, when $-1\le x\le 1$, the two equations (4.16) and (4.17) imply the estimate

$$|T_1| = O\left[\tau\left(f;\frac{|P_n(x)|\sqrt[4]{(1-x^2)^3}}{\sqrt{n}}\right)+P_n^2(x)\tau\left(f;\frac{1}{n}\right)\right]+$$

$$P_n^2(x)\left|f(1)-\frac{1}{2}\int_{-1}^{1}f(x)dx\right| \quad ....(4.18)$$

Now, to estimate T_2, we have

$$|T_2| \le \sum_{k=1}^{[n/2]}|f(x)-f(x_k)|A_k(x)+$$

$$\sum_{[n/2]+1}^{n}|f(x)-f(x_k)|A_k(x)$$

The previous lemma and the estimates (4.1), (4.3), (4.6) and (3.8) when $k=1$ all of them simultaneously imply the following estimate for the first summation on the right side of the last inequality above:

$$\sum_{k=1}^{[n/2]}|f(x)-f(x_k)|A_k(x) \le$$

$$\sum_{\substack{k\ne j\\ k=j\pm 1}}\tau_k\left[\frac{|P_n(x)||j-k|\sqrt[4]{(1-x^2)^3}}{\sqrt{n}}\right]$$

$$\left[1+\frac{|x-x_j|\sqrt{n}}{|P_n(x)|\sqrt[4]{(1-x^2)^3}}\right]A_j(x)$$

$$= O\left(\sum_{i=1}^{n}\frac{1}{i^2}\tau_k\left(f;\frac{|P_n(x)|i\sqrt[4]{(1-x^2)^3}}{\sqrt{n}}\right)\right)$$

Similarly, we get the estimation of the second summation. Whence,

$$(4.19)\ |T_2| = O\left(\sum_{i=1}^{n}\frac{1}{i^2}\tau_k\left(\frac{|P_n(x)|i\sqrt[4]{(1-x^2)^3}}{\sqrt{n}}\right)\right)$$

■ The estimates given in (3.10), (4.18) and (4.19) end the proof of theorem *1*.

Corollary:

Let *f(x)* be a continuous function in the interval *[-1,1]* such that (2.3) is satisfied, then

$$\left|f(x)-H_n(f;x)\right| =$$
$$=O\left[P_n^2(x)\tau_k\left(f;\frac{1}{n}\right)+\sum_{i=1}^{n}\frac{1}{i^2}\tau_k\left(\frac{\left|P_n(x)\right| i\sqrt[4]{\left(1-x^2\right)^3}}{\sqrt{n}}\right)\right]$$

for $-1\le x\le 1$. Moreover, if $\tau_k(f;t)=t^{\alpha}$ for $0<\alpha<1$, then using the estimate (14) and theorem 1 to obtain

$$\left|f(x)-H_n(f,x)\right| =$$
$$\left[\max\left|\left(f(\pm 1)-\frac{1}{2}\int_{-1}^{1}f(x)\,dx\right)\right|+\frac{1}{n^2}\right]$$
$$P_n^2(x)+O\left(\frac{\log n}{n}\sqrt{1-x^2}\right)$$

where $x\in[-1,1]$.

Definition:

(see **[3]**) A function f defined on $A=[a,b]$, satisfies a Lipschitz condition with constant M and exponent α, or belongs to the class $Lip_M\alpha$, $M\ge 0$, $0<\alpha\le 1$ if

$$\left|f(x')-f(x)\right|\le M\left|x'-x\right|^{\alpha},\ x,x'\in A$$

The following theorem 2 proves that the estimation included in this paper is precise for $f(x)\in Lip1$, $-1<x<1$ **[3]**.

Theorem 2:

There exists a function $f(x)\in Lip1$ and a constant c such that

$$\left|H_n(f,0)-f(0)\right|\ge c\,\frac{\log n}{n},$$

where n is even integer.

Proof:

Let $f(x)=|x|, x=\cos\theta$, $\theta=\pi/2$ and n be even, we have **[5]**,

$$\left|P_n(0)\right|=\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot(n-2)n}>\frac{1}{3\sqrt{n}} \quad \ldots\ldots (4.21)$$

Estimations (4.1), (4.3), (4.5), and (4.20) give for (4.21) and for $\theta=\pi/2$, the following

$$H_n(f,0)-f(0)>\frac{1}{9n^3}\sum_{k=1}^{n/2}\frac{k}{\cos\theta_k}=\frac{1}{9n^2}\sum_{k=1}^{n/2}\frac{k}{\cos\theta_k-\cos\theta}>$$
$$>\frac{1}{18n^2}\sum_{k=1}^{n/2}\frac{k}{\sin\frac{\pi}{2}\sin\frac{\theta-\theta_k}{2}}>$$
$$>\frac{2(2n+1)}{9n^3}\sum_{k=1}^{n/2}\frac{k}{2n-4k+3}\ge c\frac{\log n}{n},$$

and the proof is done. ■

References

[1] Schonhage, "Zur Konvergenz der Stufen Polynome ube den Nullstellen der Legendre Polynomen". Proceedings of the Conference on Abstract Spaces and Approximation, held in Oberwolfach, Birkhauser Verla 1971,.

[2] Bl. Sendov. V. A. Popov, "The Averaged Moduli of Smoothness". A. Wiley Interscience Publication. John Wiley & Sons. Sofia 1988.

[3] G.G. Lorentz, "Approximation of functions",Chlsea Pub. Co. New york, N.Y. 1986.

[4] S. J.Smith.,"On the Fundamental Polynomials for Hermite - Fejer Interpolation of Legendre Type on the Chebyshev Nodes". Division of Maths, LaTrobe Univ., P.O. Box 199, Bending Victorie 3552, Australia 1998.

[5] G. Szego, "Oryhogonal Polynomials". Amer. Math soc., CollCollaq. Publ. 23(1959).

[6] Wolfram MathWord. On Line.

[7] R. Murray Spiegel,"Applied Differential Equations". Printice Hall. Inc., Englewood Cliffs. N. J. 07632 third edition 1981.

[8] L. Fejer, "Uber Interpolation". Gottinger Nachrichten (1916), 66-91.

[9] J. Szabados, "On the Convergence of Hermite-Fejer Interpolation Based on the Roots of Legendre Polynomials". Acta Sci. Math. Szeged, 1973.

[10] http://www.info@efunda.com.

[11] file://E:\Legendre Polynomials.htm.