ON HOLLOW-WEAK LIFTING MODULES

Sahira M.Yaseen
Department of Mathematics, College of Science, University of Baghdad.

Abstract

Let R be any ring and let M be any right R-module. M is called hollow-weak lifting if every semisimple submodule N of M such that M/N is hollow has a cossential submodule that is a direct submmande of M. We prove that M is hollow-weak lifting iff every semisimple N of M such that M/N is hollow has strong supplemented in M.And show that M is hollow-weak lifting iff every semisimple submodule N of M such that M/N is hollow can be written as $N=K\oplus L$ with K is direct summand of M and L is small submodule of M.

1.Introduction and Preliminaries

Let R be a ring with identity and every R-module is unitaryright R-module. $A\leq M$ with mean A is submodule of M.Let M be an R-module and A submodule of M.A is called small submodule of M denoted by (A<<M) if for any $A\leq M$,M=A+X implies X=0. The module M is called hollow if every proper submodule is small in M. M is called semisimple if every submodule of M is direct summand od M.

A is said to be coessential submodule of B in M if B/A <<M/A.Let M be a module, for N, $L\leq M$,N is supplement of L in M if N is minimal with respect to M=N+L equivalenetely, M=N+L with $N\cap L$<<N. If M=N+L with $N\cap L$<<M, then N is called weak-supplement of L in M.M is called lifting or satifies (D_1) if for every submodule N of M, there exist a direct summand K of M such that K is coessential submodule of N in M.

M is called hollow-lifting if every submodule N of M with M/N hollow has a coessential submodule in M that is a direct summand of M [1].

In this paper we introduce the notion of hollow-weak lifting module and we give characterization of this kind of module and proved some basic properties of these modules.

2. Weak-lifting modules

Any module M is called lifting if for every submodule N of M, there exist a direct summand K of M such that $K\leq N$ and N/K<<M/K [2]. As a proper generalization of lifting modules, the weak lifting modules notion is introduced in [3] as the following.

Definition (2.1)[3]:

An R-module M is called weak-lifting module if for each semisimple submodule N of M, there exists a direct summand K of M such that $K\leq N$ and N/K<<M/K.

Equivalency, there exists a decomposition $M=M_1\oplus M_2$ such that $M_1\leq N$ and $M_2\cap N << M_2$ [4] .

It is clear that every lifting module is weak-lifting, thus hollow modules and semi simple are weak-lifting [5, Example 2.2] is an example of weak-lifting which is not lifting.

Proposition (2.2)[5]:

Any direct summand of weak-lifting modules M is also weak-lifting modules.

Example (2.5) in [5] show that a direct sum of two wek-lifting modules is not weak-lifting.

Z as Z-module is weak-lifting modules since the only semisimple submodule of Z is {0} and Z as Z-module is not lifting module.

3. Hollow weak-lifting.

M is called hollow-lifting if every submodule N of M with M/N hollow has a coessential submodule in M that is direct summand of M. We introduce the concept hollow-weak lifting.

Definition (3.1):

M is called hollow- weak lifting if every semisimple N of M with M/N hollow has a coessential submodule in M that is direct summand of M.

Equivalently, M is hollow-weak lifting module provided for each semisimple submodule N of M with M/N is hollow there exist a direct summand K of M such that $K\leq N$ and N/K<<M/K.

Remark (3.2):

Every hollow-lifting module is hollow-weak lifting.

It is clear that hollow and semisimple module are hollow-weak lifting.

The following remark is clear.

Remark (3.3):

Every weak lifting is hollow -weaklifting.

In the following proposition we show when the converse of Remark (3.3) is true.

Proposition (3.4):

Let H_1 and H_2 be hollow module, the following are equivalent for the module $M= H_1 \oplus H_2$.

(i) M is hollow-weak lifting.

(ii) M is weak-lifting.

Proof (i)→(ii):

Let N be a semisimple submodule of M.Consider the projections $\Pi_1:M\to H_1$ and $\Pi_2:M\to H_2$ if $\Pi_1(N)\neq H_1$ and $\Pi_2(N)\neq H_2$, then N<<M. Now if $\Pi_1(N)= H_1$, then $M=N+ H_2$ .Thus M/N is hollow hence there exists a direct summand K of M such that K≤N and N/K<<M/K thus M is weak lifting.

(ii) →(i) It is clear. Remark (3.3).

In the following example we show that in general a direct sum of tow hollow-weak lifting is not hollow-weak lifting.

Example (3.5):

Let P be any prime integer. Consider the z-module $M=Z/PZ\oplus Z/P^3Z$ .It is well known that Z/PZ and Z/P^3Z are hollow local module then they are Hollow-weak lifting. But M is not weak-lifting (Ex(2.5) [5]) thus by proposition (3.4) is not weak-lifting.

Let R be a prime ring and M an R-module. Let U and V be two submodule of M. We will say that V is strong supplement of U in M if V is supplement of U in M and V∩U is direct summand of U [6].

Proposition (3.6):

Let N be a semisimple submodule of R-module M, then the following are equivalent.

(i) N has strong supplement in M.

(ii) N has a cossential submodule that is direct summand of M.

Proof (i)→(ii):

Let V be a strong supplement of N in M and let W≤M such that N+X=M and (N∩V)+W+X=M . Since N∩V<<V , we have W+X=M. Hence, X=M, thus N/W<<M/W.

(ii) →(i)

Let A be a cossential submodule of N that is direct summand of M. Let B be submodule of M with $M=A\oplus B$, thus $N= A\oplus (B\cap N)$ and N+B=M. If (N∩B)+X=B, then A+(N∩B)+X=M. Hence N+X=M and N/A+(X+A)/A=M/A. Since N/A<<M/A, we have X+A=M but X≤B, then X=B. Then N∩B is small in B therefore B is supplement of N in M.

Corollary (3.7):

Let M be any module, then the following are equivalent.

(i)M is hollow-weak lifting.

(ii) Every submodule N of M such that M/N is hollow has a strong supplement in M.

Proposition (3.8):

Let M be an R-module, the following are equivalent

(i)M is hollow-weak lifting.

(ii) Every semisimple submodule N of M such that M/N is hollow can be written as $N=K\oplus L$ with K is a direct summand of M and L is small submodule of M.

Proof (i) →(ii) :

Let N be semisimple submodule of M such that M/N is hollow. Since M is hollwo-weak lifting .There exists a direct summand K of M such that K≤N and N/K<<M/K .Let F be a submodule of M with $M=K\oplus F$. So $N=K\oplus(F\cap N)$.If X≤F with (F∩N)+X=F, then N+X=M ,since N/K<<M/K, we have K+X=M.Hence X=F and F∩N<<F. Thus let L=F∩N, L<<M.

(ii)→(i)

Let N be semisimple submodule of M such that M/N is hollow .Then N can be written as $N=K\oplus L$ with K is direct summand of M and L is small in M .Let X be submodule of M such that K≤X and N/K+X/K=M/K .Thus N+X=M.So, K+L+X=M and K+X=M.But K≤X. Then X=M and N/K<<M/K.Then M is hollow-weak-lifting.

Remark (3.9):

It is clear that every module having no hollow factor of semisimple module is hollow-weak lifting module.

Proposition (3.10):

Let M be an indec- omposable module, the following are equivalent.

(i) M is hollow-weak lifting.

(ii) Every semisimple submodule is small or else M has no hollow factor of semisimple submodule of M.

Proof (i) →(ii) :

Suppose that M has a hollow factor module. Then there exists a proper semisimple N of M such that M/N is hollow .Since M is hollow–weak lifting, there is K a direct summand of M such that N/K is small M/K. But M is indecomposable, then K=0 and K<<M.

(ii) →(i). It is clear.

References

[1] K. Oshiro, "Lifting modules", "extending modules and their applications to generalized unserial rings", Hokkaido Math. J., Vol.13, No.3, 1984, pp.339-346.

[2] S .M. Mohammed and B. J. Muller," Continuous and discrete modules", London Math. Soc. LNS 147, Cambridge Univ-press, Cambridge 1990.

[3] D. Keskin and N. Orhan CCSR, "Modules and weak lifting ", East- weast Journal of Math.Vol.5,No.1 2003, pp. 89-96.

[4] N.Orhan, D.Keskin, "Generalization of weak lifting modules", Soochow Journal of Math. Vol.32, No.1. January 2006, pp.71-76.

[5] D. Keskin and R. Tribak, "On lifting modules and weak lifting modules", Kynn pook Math. J. Vol. 45, 2005.

[6] N. Orham, D.Keskin and R.Tribak, "On Hollow–lifting modules", Taiwanense Journal of Math. Vol. 11, No.2, June 2007, pp. 545-568.

الخلاصة

لتكن R حلقة و M مقاس معرف علــى R . يقــال ان المقاس M مجوف ملتو - ضعيف، اذا كان كل مقاس جزئي شبه بسيط من M بحيث ان M/N مقاس مجــوف يملــك مقاس جزئي كبير مضاد يكون مجموع مباشر الى M.