A FIXED POINT THEOREM IN GENERALIZED D-SEMI METRIC SPACES

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Abstract

The aim of this paper is to consider and study the concept of Δ-distance on a complete D-semi metric spaces and the fixed point theorem has been established on the $D-$semi metric spaces.

Keywords: Fixed point theorem, D-semi metric space, Δ-distance.

1. Introduction

Recently, Dhage [3] introduced the concept of D-metric. Afterwards, many authors Fred Galvin and S.D. Shore [4], N.S.Rao [5] and B.E.Rhoades [6] proved some fixed point theorems in D-metric spaces. Fred Galvin and S.D. Shore [4] introduced the completeness of semi metric space and Oleg Zubelevich [8] has been proved the fixed point theorem in semi metric space by using Schauder's fixed point theorem. In the present paper, we use an abstract notion of semi metric spaces which extends the notion of distance, and define a Δ-distance on a complete D-semi metric space, then by using the Δ-distance; we prove a fixed point theorem, which is the main result of this paper.

2. Preliminaries

We state the definitions of D-semi metric, Δ-distance which are necessary for the present work.

Definition 2.1 [1]

Let X is a non-empty set. A function $D:X\times X\times X\to[0,\infty)$ is called a D-semi metric if the following conditions are satisfied:

1. $D(x,y,z)\geq 0$, $\forall x,y,z\in X$ and $D(x,x,x)=0$.

2. $D(x,y,z)=D(p\{x,y,z\})$, where p is a permutation of x,y,z.

3. $D(x,y,z)\leq D(x,y,a)+D(x,a,z)+D(a,y,z)$ $\forall x,y,z,a\in X$.

Definition 2.2 [2]

A sequence $\{x_n\}$ in X is called a D-Cauchy sequence if for each $\varepsilon> 0$ there exists a positive integer n_0 such that, for all $m>n,p\geq n_0$, $D(x_m,x_n,x_p)<\varepsilon$.

Definition 2.3 [2]

A sequence $\{x_n\}$ in X is said to be D-convergent to a point $x\in X$ if for each $\varepsilon> 0$ there exists a positive integer n_0 such that, for all $m,n\geq n_0$, $D(x_m,x_n,x)<\varepsilon$.

Definition 2.4 [7]

The function $f:M\subseteq X\to(-\infty,\infty]$ is called lower semi-continuous iff for all $r\in R$ the set $\{x\in M:f(x)\leq r\}$ is relatively closed in M.

Definition 2.5 [8]

Let X is a semi metric space with semi metric D. Then a function $\Delta:X\times X\times X\to[0,\infty)$ is called a Δ-distance on X if the following conditions are satisfied:

1. $\Delta(x,y,z)\leq\Delta(x,y,a)+\Delta(x,a,z)+\Delta(a,y,z)$, $\forall\ x,y,z,a\in X$.

2. For any $x,y\in X,\Delta(x,y,.):X\to[0,\infty)$ is a lower semi-continuous.

3. for any $\varepsilon> 0$ there exists $\delta(\varepsilon)>0$ such that $\Delta(a,x,y)\leq\delta$, $\Delta(a,x,z)\leq\delta$ and $\Delta(a,y,z)\leq\delta$ imply $D(x,y,z)\leq\varepsilon$.

Definition 2.6 [8]

Let X be a non-empty set, then X is said to be Δ-bounded if there is a constant M > 0 such that $\Delta(x,y,z)\leq M$, $\forall\ x,y,z\in X$.

Definition 2.7 [5]

Let X be a semi-metric space with semi-metric D, then a self maps T of X is said to have a fixed point y if $Ty=y$.

3. The main results

It should be notice that the fixed point theorem on a complete D- semi metric has been developed by using lemma 3.1 and theorem 3.2.

Lemma 3.1

Let X is a semi metric space with the semi metric D and let Δ be a Δ-distance on X. Let $\{x_n\}, \{y_n\}$ be sequences in X and let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be sequences in $[0, \infty)$ converging to zero and assume that $x, y, z, a \in X$. Then:

1. If $\Delta(x_n, a_n, y_n) \leq \alpha_n$,
   $\Delta(x_n, a_n, z) \leq \beta_n$ and $\Delta(x_n, y_n, z) \leq \gamma_n$, for any $n \in N$, then $D(a_n, y_n, z) \to 0$.

2. If $\Delta(x_n, x_m, x_p) \leq \alpha_n$, for any p, m, $n \in N$ with m< n < p, then $\{x_n\}$ is a $D-$ Cauchy sequence.

Proof:

(1) Let $\varepsilon > 0$ be given. From the definition of Δ -distance, there exists $\delta > 0$ such that $\Delta(a, u, v) \leq \delta$, $\Delta(a, u, z) \leq \delta$ and $\Delta(a, v, z) \leq \delta$, imply $D(u, v, z) \leq \varepsilon$. Choose $n \in N$ such that $\Delta(a, u, v) \leq \delta$, $\Delta(a, u, z) \leq \delta$, and $\Delta(a, v, z) \leq \delta$ imply $D(u, v, z) \leq \varepsilon$. Choose $n \in N$ such that $\alpha_n \leq \delta, \beta_n \leq \delta$ and $\gamma_n \leq \delta$ for every $n \geq n_0$. Then for any $n \geq n_0$. we have $\Delta(x_n, a_n, y_n) \leq \alpha_n \leq \delta$, $\Delta(x_n, a_n, z) \leq \beta_n \leq \delta$, $\Delta(x_n, y_n, z) \leq \gamma_n \leq \delta$ and hence $D(a_n, y_n, z) \leq \varepsilon$. If we replace $\{\alpha_n\}$ with $\{y_n\}$, then $\{y_n\}$ converges to z.

(2) Let $\varepsilon > 0$ be given. As in the poof of (1), choose $\delta > 0$ and then $n_0 \in N$. Then , for any p>n>m>n, $\Delta(x_{n_0}, x_n, x_m) \leq \alpha_{n_0} \leq \delta$, $\Delta(x_{n_0}, x_n, x_p) \leq \beta_{n_0} \leq \delta$, $\Delta(x_{n_0}, x_m, x_p) \leq \gamma_{n_0} \leq \delta$ and hence $D(x_n, x_n, x_p) \leq \varepsilon$. This implies that $\{x_n\}$ is a $D-$ Cauchy sequence.

Theorem 3.2

Let X be a Δ-bounded complete semimetric space with semi metric D, and Δ

is a Δ-distance on X. Then a self map T from X into itself has a unique fixed point if :

(1) $\Delta(Tx, T^2 x, Tw) \leq r \Delta(x, Tx, w)$; for all x , w $\in$ X and r$\in$ (0,1) .

(2) $\Delta(x, y, z) > 0$, for all x, , y and z $\in$ X.

Proof:

By using condition (2), we get:
$$\inf \{\Delta(x, Tx, y) + \Delta(x, Tx, T^2 x)$$
$$+ \Delta(x, T^2 x, y) ; x \in X\} > 0;$$

For all y $\in$ X with Ty $\neq$ y. Let u $\in$ X and define a sequence $\{u_n\}$ in X, by $u_n = T^n u$; For all $n \in N$. Then for all t, $n \in N$ we have $\Delta(u_n, u_{n+1}, u_{n+t}) \leq r \Delta(u_{n-1}, u_n, u_{n+t-1})$ $\leq ... \leq r^n \Delta(u, u_1, u_t)$. Thus, for any p > m > n for which m = n + k and p = m + t; $t, k \in N$ .we have
$$\Delta(u_n, u_m, u_p) = \Delta(u_n, u_{n+k}, u_{n+k+t})$$
$$\leq \Delta(u_n, u_{n+k}, u_{n+1}) + \Delta(u_n, u_{n+1}, u_{n+k+t})$$
$$+ \Delta(u_{n+1}, u_{n+k}, u_{n+k+t}) \leq \Delta(u_n, u_{n+1}, u_{n+k})$$
$$+ \Delta(u_n, u_{n+1}, u_{n+k+t}) + \Delta(u_{n+1}, u_{n+k}, u_{n+k+t})$$
$$\leq r^n + r^n + \Delta(u_{n+1}, u_{n+k}, u_{n+k+t})$$

By repeated application of tetrahedral inequality we obtain
$$\Delta(u_n, u_m, u_p) \leq \Delta(u_n, u_{n+1}, u_{n+k}) + \Delta(u_n, u_{n+1}, u_{n+k+t})$$
$$+ \Delta(u_{n+1}, u_{n+2}, u_{n+k}) + \Delta(u_{n+1}, u_{n+2}, u_{n+k+t}) + ...$$
$$\leq 2(r^n + r^{n+1} + r^{n+2} + ...) M$$
$$\leq 2M \frac{r^n}{1-r} \to 0 \text{ as } n \to \infty.$$

By part (2) of lemma 3.1, $\{u_n\}$ is a $D-$ Cauchy sequence. Since X is complete, $\{u_n\}$ converges to a point z $\in$ X. Let $n \in \aleph$ be fixed. Then by lower semi-continuity of Δ, we have
$$\Delta(u_n, u_m, z) \leq \liminf_{p \to \infty} \Delta(u_n, u_m, u_p)$$
$$\leq 2M \frac{r^n}{1-r}.\text{Assume that z } \neq \text{ Tz .Then we}$$
have:
$$0 < \inf \{\Delta(x, Tx, z) + \Delta(x, Tx, T^2 x) + \Delta(x, T^2 x, z)\}$$
$$\leq \inf \{\Delta(u_n, u_{n+1}, z) + \Delta(u_n, u_{n+1}, u_{n+2})$$
$$+ \Delta(u_n, u_{n+2}, z) : n \in N\}$$

$$\leq \inf\left\{2M\frac{r^n}{1-r}+r^n M+2M\frac{r^{n+1}}{1-r} : n\in N\right\}=0$$

This is a contradiction. Therefore, we have
$z = Tz$.

Now if x and y are two fixed points such that
$x \neq y$, then

$$\Delta(x, y, y) = \Delta(Tx, Ty, Ty) \leq r\Delta(x, y, y)$$

$$\leq r\,\Delta(Tx, Ty, Ty)$$

$$\leq r^2\,\Delta(x, y, y) \leq .... \leq r^n\,\Delta(x, y, y)$$

This is not possible since r < 1.

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الخلاصة

الهدف من هذا البحث هو لدراسة دالة المسافة - Δ في الفضاءات الشبه مترية – D الكاملة ومن ثم دراسة مبرهنة النقطة الصامدة في الفضاءات الشبه مترية - D .