Journal of Al-Nahrain University Vol.12 (3), September, 2009, pp.155-160 Science

# VARIATIONAL FORMULATION OF ONE-DIMENSIONAL PARABOLIC MOVING BOUNDARY PROBLEMS WITH SINGULARITY

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## Abstract

In this paper, the problem of solving one-dimensional parabolic moving boundary value problem (heat equation) which involves a singularity in its initial and boundary conditions is solved using the varitional approach which is by using the direct Ritz method.

## Introduction

The term moving-boundary problems (M.B.P) are commonly used when the boundary is associated with time dependent problems, and the boundary is unknown in advance and has to be determined as a part of the problem which is a function of time and space. A moving-boundary problem will be taken with time-dependent problems governed by a parabolic partial differential equation with a prescribed initial condition and boundary condition. Examples of such type of problems, are shock waves in gas dynamics, cracks in solid mechanics, melting of the polar ice cap, ... , etc, [4].

The term "Stefan problem" is generally used for heat transfer problems with phase-changes such as from the liquid to the solid. Stefan problems have some characteristics that are typical of them, but certain problems arising in fields such as mathematical physics and engineering also exhibit characteristics similar to them. The term "classical" distinguishes the formulation of these problems from their weak formulation, in which the solution need not possess classical derivatives. Under suitable assumptions, a weak solution could be as good as a classical solution. In hyperbolic Stefan problems, the characteristic features of Stefan problems are present but unlike in Stefan problems, discontinuous solutions are allowed because of the hyperbolic nature of the heat equation. The numerical solutions of inverse Stefan problems, and the analysis of direct Stefan problems are so integrated that it is difficult to discuss one without referring to the other. So no strict line of demarcation can be identified between the classical Stefan problem and other similar problems. On the other hand, including every related problem in the domain of classical Stefan problem would require several volumes for their description. A suitable compromise has to be made. The basic concepts, modeling, and analysis of the classical Stefan problems have been extensively investigated and there seems to be a need to report the results at one place.[10]

A moving boundary value problem is linear initial-boundary value problem with a moving boundary whose position has to be determined as part of the solution. Parabolic moving boundary problems describe many phenomenas of interest that arising in physical and biological sciences, engineering, metallurgy, soil mechanics, decision and control theory, etc. (see [4]).

Consider the following class of moving boundary value problems of the parabolic type

$$\left.\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + f(x,t) \quad \text{on} \quad t>0,\ 0<x<s(t)\\
&\alpha(t)u(0,t)+\beta(t)\frac{\partial u}{\partial x}(0,t)=\gamma(t),\\
&\alpha^2+\beta^2 \neq 0, \quad g \neq 0\\
&u(s(t),t) = p\frac{ds}{dt}(t)\\
&\frac{\partial u}{\partial x}(s(t),t) = q\frac{ds}{dt}(t)\\
&s(0)=a, \quad u(x.0)=\begin{cases} r(x) & \text{for } 0 \le x \le a\\ 0 & \text{for } a<x\end{cases}
\end{aligned}\right\} 1$$

Where f and r are given functions which are constant of their arguments, p and q are given constants and s(t) is the unknown moving boundary.

As a consequence obtaining analytical solutions for problems belonging to the class (1) is a difficult task (see [4]).

Several numerical methods have been proposed for the solution of moving boundary problems. Let us list here those of more frequent use: finite deference method, finite element method, isotherm migration, method of lines, enthalpy, method (alternating phase) and variational inequalities method. For the fundamental aspects of those methods, as well as for an extended bibliography, We refer the interested reader to [4].

As far as the performance of different methods is concerned the introductory remark in a survey paper by Fox [5] is pertinent: ``Problems of the same general nature can differ enough in detail to make a good method for one problem less satisfactory and even mediocre for another almost similar problem". This point of view justifies the development of so many deferent numerical methods. At a more general level the numerical approaches for the solution of moving boundary problems belong to three main classes, namely front-tracking, fixing-domain and fixed-domain. In a front-tracking approach the position of the moving boundary is computed explicitly by the numerical algorithm. The method of lines is an example of the front-tracking strategy. On the other hand in a fixed-domain or in a fixing-domain approach the moving boundary can be recovered a posteriori from the solution properties.

For a fixing-domain approach a variable transformation is used in order to reduce the problem to a computational domain. The isotherm migration method belongs to the fixing-domain class. A weak formulation of the problem is usually used for the fixed-domain approach. The enthalpy method is within the fixed-domain class. Our approach belongs to the front-tracking class.

The moving boundary conditions in (1) are called explicit when p $\neq 0$ or q $\neq 0$ (implicit otherwise). In the case of explicit moving boundary conditions it is possible to apply a finite difference formula to find a first approximation of the moving boundary position at the next time step. Of course that is not possible when implicit moving boundary conditions are prescribed. Moreover, existence and uniqueness of solution is easier to prove for problems with explicit boundary conditions than for problems with implicit ones (see [11]). Problems with implicit moving boundary conditions arise in diffusion of oxygen and lactic acid in tissues [6], in the theory of diffusion flames [2], and in statistical decision theory [3].

The governing differential equation and the moving boundary conditions are also nonlinear and may depend on the free boundary and its derivative. In this paper we will transform our problem which is especial case o eq.(1) to the varitional formulation then with the cooperation of the direct Ritz method we shall get the desired approximate solution .

## Mathematical Formulation of the Classical Stefan problem

Among the class of free boundary value problems for partial differential equations,the one-dimensional parabolic problems have been examined in some detail. Perhaps the best understood problem of this kind is the formulation for the melting of a slab of ice in the contact with a viscous fluid. If one assumes that the ice is held at 0°C throughout, and that the heat transfer in the fluid occurs by conduction only, then the temperature distribution is described by the usual heat equation

$$u_{xx} - cu_t = 0 \qquad (2)$$

subject to the initial and boundary conditions

$$u(0,t) = \alpha(0,t) \quad , \; u(s(t),t) = 0 \quad ,$$

$$u_x(s(t),t) = -\lambda \frac{ds}{dt} \quad , \; s\,(0)=0 \qquad (3)$$

Here u denotes the temperature in the fluid between a wall at x and 0 held at temperature α(t) and the unknown and moving boundary s(t) between the fluid and ice. The flux condition $u_x(s(t),t) = -\lambda \frac{ds}{dt}$.is obtained from an energy balance and indicates that the heat flowing toward the ice is used to melt it rather than raise its temperature. The condition s(0) = 0 means that initially no fluid is present. The constants c and λ are determined from the conductivity, heat capacity, and latent heat of water. The original formulation of this paper considered by Sackett has a singularity in the boundary data at the initial time. To overcome this difficulty Sackett used a sophisticated similarity transformation whereas Meyer applied a simple subtracting of the singularity from the boundary data. In my problem the governing differential equation is singular at the initial time and the moving boundary conditions are implicit [12].

The following formulation is a special case of problem (1) which will be considered in this paper:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{.........................................} (4)$$

with the boundary

$$u(0,t) = 1 \quad \text{.........................................} (5)$$

$$u(s(t),t) = 0 \quad \text{......................................} (6)$$

$$\frac{\partial u}{\partial x}(s(t),t) = -\frac{1}{s}\frac{ds}{dt}(t) \quad \text{............................} (7)$$

$$s(0)=\in \quad \text{..............................................} (8)$$

$$u(x,0)=0 \text{ for } 0 < x \text{ and } 0< t \quad \text{...............} (9)$$

Where $\in$ is a very small number which is chosen in order to avoid the singularity occurred in this problem

**Variational Formulation of the Heat Diffusion Problem**

The variational formulation corresponding to diffusion type problem will be found, the formulation is considered for homogenous and non- homogenous initial conditions.

***1-Heat Diffusion Problem with Homogeneous Initial Conditions:***

In this case, the initial condition equals to zero, therefore we take the bilinear form (u, v) to be the of the following type[13]:

$$(u,v)=\int_R \int_0^T u(x,t)\overline{v}(x,t)\, dtdR \quad \text{......} (10)$$

Where:

$$\overline{v}(x,t) = \int_{s=0}^{S=T-t} k(t,s)v(x,s)\, ds$$

and satisfying the symmetry and non-degeneracy for (u, v) and symmetry for the linear operator $L=\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}$.(for complete details about these notions see [1].[7],[9],[13]

Now, we turn to the problem of evaluating the variational formulation corresponding to the heat diffusion problem with homogeneous initial condition.

Using the definition, we have the following general form of variational formulation:

$$J(u) = \frac{1}{2} <Lu, u> - <f, u>$$

Since L is symmetric and by definition <u, v> = (u, Lv), we have:

$$J(u) = \frac{1}{2}(Lu, Lu) - (f, Lu) \text{ , where f is}$$

equal zero in this problem

$$J(u) = \frac{1}{2}(Lu, Lu)$$

$$= \frac{1}{2}\int_R \int_0^T Lu(x,t)\,\overline{Lu}(x,t)\, dt\, dR$$

$$= \frac{1}{2}\int_R \int_0^T Lu(x,t)\left\{\int_{s=0}^{s=T-t} k(t,s)\frac{\partial}{\partial s}u(x,s)\, ds\right\} dt dR$$

$$= \frac{1}{2}\int_R \int_0^T \int_0^{T-t} \{u_{xx}(x,t)$$

$$-u_t(x,t)\}k(t,s)\frac{\partial}{\partial s}u(x,s)dsdtdR$$

$$= \frac{1}{2}\int_R \int_0^T \int_0^{T-t} k(t,s)\left\{u_{xx}(x,t)\frac{\partial}{\partial s}u(x,s)\right.$$

$$\left. - u_t(x,t)\frac{\partial}{\partial s}u(x,s)\right\}dsdtdR \quad \text{.....} (11)$$

Now, by using the divergence theorem, we have:

$$J(u)= \frac{1}{2}\int_R \int_0^T \int_0^{T-t} k(t,s)\left\{-\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}\frac{\partial}{\partial s}u(x,s)\right.$$

$$\left. - u_t(x,t)\frac{\partial}{\partial s}u(x,s)\right\}ds\, dt\, dR \quad \text{....} (12)$$

Now, taking k(t, s) ≡ 1 and taking the derivation and integration with respect to s, eq. (12) will be of the following form:

$$J(u)=-\frac{1}{2}\int_R \int_0^T \left\{\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}[u(x,T-t)-u(x,0)]+\right.$$

$$\left. u_t(x,t)\Big[u(x,T-t)-u(x,0)\Big]\right\}dtdR$$

And using the homogenous initial condition, we get:

$$J(u)=-\frac{1}{2}\int_R\int_0^T\left\{\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}u(x,T-t)+u_t(x,t)u(x,T-t)\right\}dtdR \text{ ........... (13)}$$

This represents the variational formulation for the diffusion problem with homogenous initial condition.

***2-Heat Diffusion Problem with Non-Homogenous Initial Conditions:***

In this case we consider the bilinear form (u, v) of the form:

$$(u,v)=\int_R\int_0^T u(x,t)\bar{v}(x,t)\,dtdR$$

Where,

$$\bar{v}(x,t)=\int_{s=0}^{s=T-t}k(t,s)v(x,s)\,\mathbf{m}\,v(x,0)\,ds$$

With non- homogenous initial conditions.

One can notice that adding v(x, 0) is a necessary condition in order to satisfy, the initial condition equals zero and for its utility in proving the non- degeneracy of the bilinear form (u, v) and the symmetry of the operator L. Now, we turn to the problem of evaluating the variational formulation corresponding to the heat diffusion problem with non homogeneous initial condition. Using the definition $J(u)=\frac{1}{2}<Lu, u> - <f, u>$, we have the following general form of variational formulation:

$$J(u)=\frac{1}{2}<Lu,u>-<f,u>$$

And by definition <u, v> = (u, Lv), we have:

$$J(u)=\frac{1}{2}(Lu,Lu)-(f,Lu)$$

$$=\frac{1}{2}\int_R\int_0^T Lu(x,t)\overline{Lu}(x,t)\,dt\,dR$$

$$=\frac{1}{2}\int_R\int_0^T Lu(x,t)\left\{\int_{s=0}^{s=T-t}k(t,s)u(x,s)+u(x,0)\,ds\right\}dtdR$$

$$J(u)=\frac{1}{2}\int_R\int_0^T\int_0^{T-t}\{u_{xx}(x,t)-u_t(x,t)\}k(t,s)\frac{\partial}{\partial s}u(x,s)\,dsdtdR+u(x,0)$$

$$=\frac{1}{2}\int_R\int_0^T\int_0^{T-t}k(t,s)\left\{u_{xx}(x,t)\left[\frac{\partial}{\partial s}u(x,s)+u(x,0)\right]-u_t(x,t)\left[\frac{\partial}{\partial s}u(x,s)+u(x,0)\right]\right\}dsdtdR \text{ ..... (14)}$$

Similarly as in homogenous case and by using the divergence theorem, we obtain that:

$$J(u)=\frac{1}{2}\int_R\int_0^T\int_0^{T-t}k(t,s)\left\{-\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}\left[\frac{\partial}{\partial s}u(x,s)+u(x,0)\right]-u_t(x,t)\left[\frac{\partial}{\partial s}u(x,s)+u(x,0)\right]\right\}dsdtdR \text{ ........ (15)}$$

taking k(t, s) ≡ 1 and differentiating and integrating with respect to s, eq.(15), takes the form:

$$J(u)=\frac{1}{2}\int_R\int_0^T\left\{-\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}u(x,T-t)-u_t(x,t)u(x,T-t)\right\}dt\,dR \text{ ............... (16)}$$

The final version of eq. (16), takes the form:

$$J(u)=-\frac{1}{2}\int_R\int_0^T\left\{\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}u(x,T-t))+u_t(x,t)u(x,T-t)\right\}dt\,dR \text{ ................ (17)}$$

**Numerical Solutions of the Poblem**

Now the direct variational approach, (to namely direct Ritz method) will be used to find the critical points, corresponding to the functional (13,17), derived above which represent the solution of the original problem . This will be made with cooperation of computer programs,

As a numerical application, consider the Heat diffusion problems. governed by

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \text{ ........................................ (18)}$$

With the initial and boundary condition

$$u(0,t)=1 \text{ ..................................... (19)}$$

$$u(s(t),t)=0 \text{ ................................... (20)}$$

$$\frac{\partial u}{\partial x}(s(t),t)=-\frac{1}{s}\frac{ds}{dt}(t) \text{ ........................... (21)}$$

$$s(0)=\in \text{ ........................................... (22)}$$

$$u(x,0)=0 \text{ for } 0<x \text{ and } 0<t \text{ ................ (23)}$$

Where ∈ a very small number is tends to 0 in order to a void the singularity.

From the chemical and physical interpretation of the problem, and for numerical solution of the above problem propose the moving boundary s(t) of this problem requires the following conditions to be satisfied :-

1- When t increases, s(t) increases.
2- When t increases, s(t) decreases.
3- When t = 0, s(t) = $s_0$, where $s_0$ is the initial moving boundary where s(0)=$\in$ .

The following definition of s(t) could be taken, which satisfies the above three conditions,

$s(t) = bt+\in$

Now, instead of solving the problem analytically, we can find the critical points of the functional:

$$J(u)=-\frac{1}{2}\int_R\int_0^T\left\{\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}u(x,T-t)+ u_t(x,t)u(x,T-t)\right\}dtdR$$

for the homogeneous condition, and the functional:

$$J(u)=-\frac{1}{2}\int_R\int_0^T\left\{\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}u(x,T-t))+ u_t(x,t)u(x,T-t)\right\}dt\ dR$$

for the Non-homogeneous condition.

In order to use the direct Ritz methods, we approximate the solution $\phi$ (x, t) as the follows:

$\phi(x, t)=\psi(x, t) + W(x,t)$.

Where $\psi(r,t)$ is any function which satisfies the non-homogeneous boundary conditions, and W(x,t) any function which satisfies the homogeneous boundary conditions.

One of the choices for W(x, t) which fits our needs is the following function:

$W(x, t) = (x - s)^2\ t\ (a_1+a_2x+a_3t)$

Where $a_1$, $a_2$, $a_3$,b are constant

In addition, for the non-homogenous boundary condition, and by using mathematical inspection and induction, we can take $\psi(x, t)$ to be as

$$\psi(x,t)=\frac{-\dot{s}(x^2-xs)}{s^2}+1$$

Where $\dot{s}=\dfrac{ds}{dt}$

Which satisfies the non- homogenous boundary conditions.

The numerical result obtained upon carrying the computer program written in MathCAD software we are given by:

$a_1$=0.143, $a_2$ = -1.1, $a_3$=0.775 and b=0.3

With functional minimum equals to zero. Hence the approximate solution is given by:

$$\phi(x,t)=\frac{\left[-.3\cdot x^2+.6\cdot x\cdot\left(.3\cdot t+1.\cdot 10^{-3}\right)\right]}{\left(.3\cdot t+1.\cdot 10^{-3}\right)^2}+1$$
$$+(x-s)^2\ t\ (0.143-1.1x+0.775t)$$

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**الخلاصة**

في هذا البحث ، تمت دراسة حلول معادلة الحـــرارة وهي معادلة تفاضلية جزئيه حدودية ذات ســـطح متحـــرك والتي تضمنت نقطة انفراديـــة فـــي الشـــروط الحدوديـــة باستخدام الطرق التغايرية. وذلك بتطبيق الاساليب المباشرة للحلول العددية.