NUMERICAL METHODS FOR SOLVING THE FIRST ORDER LINEAR FREDHOLM-VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

H. H. Omran

Department of Mathematics, College of Education, Ibn Al-Haitham, University of Baghdad.

Abstract:

The aim of this work is to present numerical methods for solving the first order linear Fredholm-Volterra integro-differential equations of the second kind. These methods namely are the repeated Trapezoidal method and the repeated Simpson's 1/3 method. These techniques transform the integro-differential equations to a system of algebraic equations. Some numerical examples are presented to illustrate the efficiency and accuracy of these methods.

Introduction

Mathematical modeling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro - differential equations, stochastic equations .Many mathematical formulation of physical phenomena contain integro - differential equations, these equations arises in many fields like fluid dynamics, biological models and chemical kinetics.[1], [2].

There are several solution methods including quadrature, collocation, Galerkin, variational, for integro - differential equations have been studied in [3-8].

In this paper, use the repeated Trapezoidal quadrature formula and repeated Simpson's 1/3 quadrature formula to solve the first order linear Fredholm -Volterra integro-differential equations of the second kind

$$u'(x) = g(x) + \lambda\int_a^b L(x,y)u(y)dy + \mu\int_a^x K(x,y)u(y)dy \quad \text{.................................} (1.1)$$

with the initial condition u(a)= u_0

where $a \le x \le b$, λ and μ are scalar parameters, g(x), L(x, y) and K(x, y), are given continuous functions, u_0 ,a,b are known constants and u(x) is the unknown function to be determined.

The Repeated Trapezoidal Method

Consider the first order linear Fredholm-Volterra integro-differential equation of second kind given by equation (1.1). To solve this equation on the finite interval [a, b], we divide it into n smaller intervals of width h, where h=(b−a)/n. The i-th point of subdivision is denoted by x_i, such that $x_i = a + ih$, i = 0, 1,…,n. If we take $u'_{i-1}(x) \approx \dfrac{u_i(x) - u_{i-1}(x)}{h}$, i=1, 2, …, n.

Then, the approximated solution will be defined at the mesh point x_i is denoted by u_i and is given by.

$$u_i = u_{i-1} + h\left\{ g(x_{i-1}) + \lambda\int_a^b L(x_{i-1},y)u(y)dy + \mu\int_a^{x_{i-1}} K(x_{i-1},y)u(y)dy \right\}, \ i = 1,2,\ldots,n \quad \text{........................} (2.1)$$

with the initial condition u(a)= u_0 ,

If we approximate the integrals that appeared in equation (2.1) by the repeated Trapezoid formula which will yield the following system of equations:

$$u_1 = u_0 + hg_0 + \frac{\lambda h^2}{2}\left(L_{0,0}u_0 + 2\sum_{j=1}^{n-1} L_{0,j}u_j + L_{0,n}u_n \right),$$

$$u_i = u_{i-1} + hg_{i-1} + \frac{h^2}{2}\Bigg(\left(\lambda L_{i-1,0} + \mu K_{i-1,0}\right)u_0 + 2\sum_{j=1}^{i-2}\left(\lambda L_{i-1,j} + \mu K_{i-1,j}\right)u_j + \left(2\lambda L_{i-1,i-1} + \mu K_{i-1,i-1}\right)u_{i-1} + 2\lambda\sum_{j=i}^{n-1} L_{i-1,j}u_j + \lambda L_{i-1,n}u_n\Bigg),$$

$i = 2, 3, \ldots, n-1$

$$u_n = u_{n-1} + hg_{n-1} + \frac{h^2}{2}\Bigg(\left(\lambda L_{n-1,0} + \mu K_{n-1,0}\right)u_0 + 2\sum_{j=1}^{n-2}\left(\lambda L_{n-1,j} + \mu K_{n-1,j}\right)u_j + 2\left(\lambda L_{n-1,n-1} + \mu K_{n-1,n-1}\right)u_{n-1} + \lambda L_{n-1,n}u_n\Bigg) \quad \text{.................. .(2.2)}$$

where:

$K_{i,j} = K(x_i, x_j), i, j = 0, 1, \ldots, i,$

$L_{i,k} = L(x_i, x_k), i, k = 0, 1, \ldots, n,$

$g_i = g(x_i)$, i=0,1,…,n.

By solving the system given by equation (2.2) which consists of n equations and n unknowns, the approximated solution of equation (1.1), is obtained.

The Repeated Simpson's 1/3 Method

Consider the first order linear Fredholm-Volterra integro-differential equation of second kind given by equation (1.1). Here we use Simpson's 1/3 method to find the solution of equation (1.1).To do this, we divide the finite interval [a, b] into 2n smaller intervals of width h, where h = (b − a)/2n, and we take.

$$u'_{i-1}(x) \approx \frac{u_i(x) - u_{i-1}(x)}{h}, \text{ i=1, 2, …,2n.}$$

The approximated solution at the odd nods x_{2i+1} is given by:

$$u(x_{2i+1}) = u(x_{2i}) + h\left\{g(x_{2i}) + \lambda\int_a^b L(x_{2i}, y)u(y)dy + \mu\int_a^{x_{2i}} K(x_{2i}, y)u(y)dy\right\}, \; i = 0, 1, \ldots, n-1 \quad \text{............................ (3.1)}$$

and in the even nods x_{2i} is given by

$$u(x_{2i}) = u(x_{2i-1}) + h\left\{g(x_{2i-1}) + \lambda\int_a^b L(x_{2i-1}, y)u(y) + \mu\int_a^{x_{2i-1}} K(x_{2i-1}, y)u(y)dy\right\},$$

$i = 1, 2, \ldots, n$ ....(3.2)

with the initial condition u(a)= u_0,

By using the repeated Simpson's 1/3 formula to approximate the integrals that appeared in equations (3.1) - (3.2) one can get the following system of equations:

$$u_1 = u_0 + hg_0 + \frac{\lambda h^2}{3}\left(L_{0,0}u_0 + 4\sum_{j=1}^{n} L_{0,2j-1}u_{2j-1} + 2\sum_{j=1}^{n-1} L_{0,2j}u_{2j} + L_{0,2n}u_{2n}\right),$$

$$u_{2i+1} = u_{2i} + hg_{2i} + \frac{h^2}{3}\Bigg(\left(\lambda L_{2i,0} + \mu K_{2i,0}\right)u_0 + 4\sum_{j=1}^{i}\left(\lambda L_{2i,2j-1} + \mu K_{2i,2j-1}\right)u_{2j-1} + 2\sum_{j=1}^{i-1}\left(\lambda L_{2i,2j} + \mu K_{2i,2j}\right)u_{2j} + \left(2\lambda L_{2i,2i} + \mu K_{2i,2i}\right)u_{2i} + 4\lambda\sum_{j=i+1}^{n} L_{2i,2j-1}u_{2j-1} + 2\lambda\sum_{j=i+1}^{n-1} L_{2i,2j}u_{2j} + \lambda L_{2i,2n}u_{2n}\Bigg)$$

$i = 1, 2, \ldots, n-1$

$$u_{2i} = u_{2i-1} + hg_{2i-1} + \frac{h^2}{3}\Bigg(\left(\lambda L_{2i-1,0} + \mu K_{2i-1,0}\right)u_0 + 4\sum_{j=1}^{i}\left(\lambda L_{2i-1,2j-3} + \mu K_{2i-1,2j-3}\right)u_{2j-3} + 2\sum_{j=1}^{i-1}\left(\lambda L_{2i-1,2j-2} + \mu K_{2i-1,2j-2}\right)u_{2j-2} + \left(2\lambda L_{2i-1,2i-2} + \frac{5}{2}\mu K_{2i-1,2i-2}\right)u_{2i-2} + \left(4\lambda L_{2i-1,2i-1} + \frac{3}{2}\mu K_{2i-1,2i-1}\right)u_{2i-1} + 4\lambda\sum_{j=i+1}^{n} L_{2i-1,2j-1}u_{2j-1} + 2\lambda\sum_{j=i}^{n-1} L_{2i-1,2j}u_{2j} + \lambda L_{2i-1,2n}u_{2n}\Bigg), \quad i = 0, 1, \ldots, n-2$$

$$u_{2n} = u_{2n-1} + hg_{2n-1} + \frac{h^2}{3}\Big(\big(\lambda L_{2n-1,0} + \mu K_{2n-1,0}\big)u_0 + 4\sum_{j=1}^{n-2}\big(\lambda L_{2n-1,2j-1} + \mu K_{2n-1,2j-1}\big)u_{2j-1} + 2\sum_{j=1}^{n-1}\big(\lambda L_{2n-1,2j} + \mu K_{2n-1,2j}\big)u_{2j} + \big(4\lambda L_{2n-1,2n-1} + \mu K_{2n-1,2n-1}\big)u_{2n-1} + \lambda L_{2n-1,2n}u_{2n}\Big) \quad (3.3)$$

By solving this system which consists of n equations and n unknowns, the approximated solution of equation (1.1) is obtained.

Numerical examples

In this section, we give two numerical examples to illustrate the repeated Trapezoid method and the repeated Simpson's 1/3 method. The computations associated with the example were performed using Matlab 7.

Example 1

Consider the first order linear Fredholm-Volterra integro-differential of the second kind:

$$u'(x) = f(x) - \left(\int_0^x x^2 y u(y)dy + \int_0^1 x(x-y)u(y)dy\right)$$

with the initial condition u(0)=1, $0 \le x \le 1$, where

$$f(x) = -\sin x - x^2\cos x - x^3\sin x + x^2 - x^2\sin(1) + x\cos(1) + x\sin(1) - x$$

and the exact solution is u(x) = cos(x).

Table (1) and (2) show that the absolute errors at some mesh points obtained by using the repeated Trapezoid method and the repeated Simpson's 1/3 method respectively for h =0.1, 0.025, 0.01. Therefore, Table (1) and (2) show that the repeated Simpson's 1/3 method gave accurate results than the repeated Trapezoid method. Fig.(1) and (2) : Plot the exact and numerical solutions for example 1 by using repeat Trapezoidal method and Simpson's method respectively.

Table (1)
The absolute errors at some mesh points of example 1 obtained by using the repeated Trapezoidal method.

Points	h=0.1	h=0.025	h=0.01
x=0.1	4.99583×10^{-3}	1.23347×10^{-3}	4.92742×10^{-4}
x=0.2	9.74286×10^{-3}	2.42274×10^{-3}	9.69158×10^{-4}
x=0.3	1.42462×10^{-2}	3.56771×10^{-3}	1.42909×10^{-3}
x=0.4	1.85161×10^{-2}	4.66954×10^{-3}	1.87288×10^{-3}
x=0.5	2.25712×10^{-2}	5.73176×10^{-3}	2.30186×10^{-3}
x=0.6	2.64430×10^{-2}	6.76177×10^{-3}	2.71897×10^{-3}
x=0.7	3.01819×10^{-2}	7.77279×10^{-3}	3.12960×10^{-3}
x=0.8	3.38651×10^{-2}	8.78614×10^{-3}	3.54248×10^{-3}
x=0.9	3.76048×10^{-2}	9.83394×10^{-3}	3.97088×10^{-3}
x=1	4.18856×10^{-2}	1.09670×10^{-2}	4.43427×10^{-3}

Table (2)
The absolute errors at some mesh points of example 1 obtained by using the repeated Simpson's 1/3method.

Points	h=0.1	h=0.025	h=0.01
x=0.1	4.99583×10^{-3}	1.23163×10^{-3}	4.92400×10^{-4}
x=0.2	9.70297×10^{-3}	2.41496×10^{-3}	9.67886×10^{-4}
x=0.3	1.42533×10^{-2}	3.55152×10^{-3}	1.42649×10^{-3}
x=0.4	1.81611×10^{-2}	4.64357×10^{-3}	1.86873×10^{-3}
x=0.5	2.22962×10^{-2}	5.69578×10^{-3}	2.29612×10^{-3}
x=0.6	2.57598×10^{-2}	6.71664×10^{-3}	2.71179×10^{-3}
x=0.7	2.95997×10^{-2}	7.72036×10^{-3}	3.12126×10^{-3}
x=0.8	3.29792×10^{-2}	8.72905×10^{-3}	3.53340×10^{-3}
x=0.9	3.68341×10^{-2}	9.77547×10^{-3}	3.96158×10^{-3}
x=1	3.95266×10^{-2}	1.08002×10^{-2}	4.40738×10^{-3}

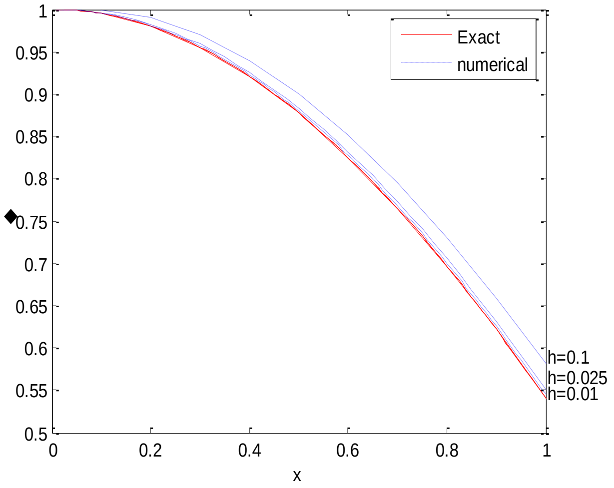

Fig. (1): The numerical and exact solutions for example 1 by using repeated Trapezoidal method.

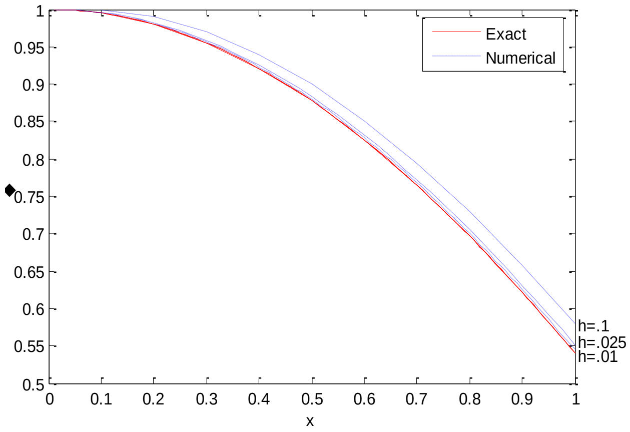

Fig.(2): The numerical and exact solutions for example 1 by using the repeated Simpson's 1/3method.

Example 2

Consider the first order linear Fredholm-Volterra integro-differential of the second kind:

$$u(x) = -\frac{2}{3}e^{-x} - \frac{2}{3}x + e^{-2}x + e^{-2} + \int_0^x (x-y)u(y)dy + \int_0^2 (xy+y)u(y)dy, \quad 0 \le x \le 2$$

with the initial condition u(0)=1/3, for which the exact solution is: $u(x) = \frac{1}{3}e^{-x}$.

Table (3) and (4) show that the absolute errors at some mesh points obtained by using the repeated trapezoid method and the repeated Simpson's 1/3 method respectively for h =0.2, 0.05, 0.025. Therefore, Table (3) and (4) show that the repeated Simpson's 1/3 method gave accurate results than the repeated Trapezoid method.

Fig. (3) and (4): Plot the exact and numerical solutions for example 2 by using the repeated Trapezoidal method and the repeated Simpson's formula respectively.

Table (3)
The absolute errors at some mesh points of example 2 obtained by using the repeated Trapezoidal method.

Points	h=0.2	h=0.05	h=0.025
x=0.2	4.31370×10^{-3}	7.84211×10^{-4}	3.06958×10^{-4}
x=0.4	5.52089×10^{-3}	1.13585×10^{-3}	4.49388×10^{-4}
x=0.6	5.30779×10^{-3}	1.13976×10^{-3}	4.52554×10^{-4}
x=0.8	3.96325×10^{-3}	8.50526×10^{-4}	3.37084×10^{-4}
x=1	1.71296×10^{-3}	3.12860×10^{-4}	1.20008×10^{-4}
x=1.2	1.27599×10^{-3}	4.38837×10^{-4}	1.85535×10^{-4}
x=1.4	4.89173×10^{-3}	1.38095×10^{-3}	5.70574×10^{-4}
x=1.6	9.07553×10^{-3}	2.50116×10^{-3}	1.03058×10^{-3}
x=1.8	1.38208×10^{-2}	3.79902×10^{-3}	1.56579×10^{-3}
x=2	1.79054×10^{-2}	5.27061×10^{-3}	2.18054×10^{-3}

Table (4)
The absolute errors at some mesh points of example 2 obtained by using the repeated Simpson's 1/3method.

Points	h=0.2	h=0.05	h=0.025
x=0.2	4.46978×10^{-3}	7.88346×10^{-4}	3.08586×10^{-4}
x=0.4	5.45059×10^{-3}	1.14498×10^{-3}	4.51834×10^{-4}
x=0.6	6.15298×10^{-3}	1.14961×10^{-3}	4.55127×10^{-4}
x=0.8	3.69175×10^{-3}	8.61163×10^{-4}	3.39781×10^{-4}
x=1	2.55684×10^{-3}	3.27431×10^{-4}	1.23315×10^{-4}
x=1.2	1.54949×10^{-3}	4.15047×10^{-4}	1.80783×10^{-4}
x=1.4	3.87938×10^{-3}	1.34119×10^{-3}	5.63296×10^{-4}
x=1.6	8.93698×10^{-3}	2.43763×10^{-3}	1.01952×10^{-3}
x=1.8	1.22055×10^{-2}	3.70316×10^{-3}	1.54953×10^{-3}
x=2	1.50389×10^{-2}	5.11028×10^{-3}	2.15621×10^{-3}

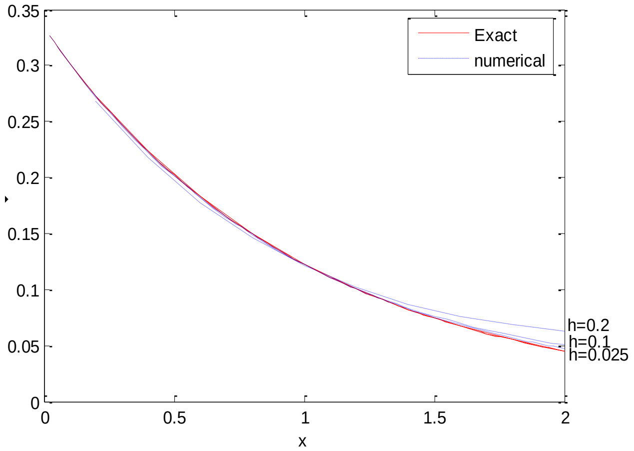

Fig.(2): The numerical and exact solutions for example 2 by using repeated Trapezoidal method.

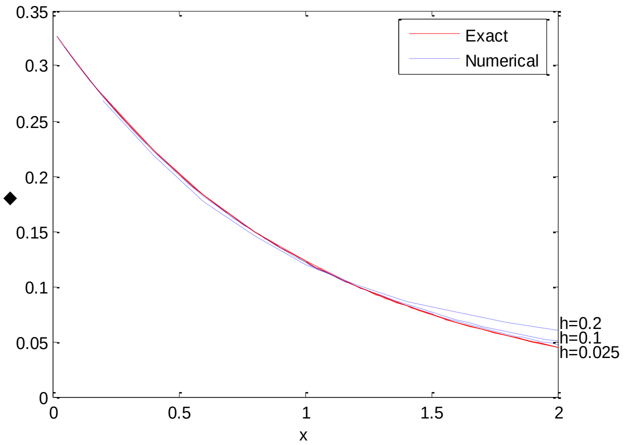

Fig. (4): The numerical and exact solutions for example 2 by using repeated Simpson's 1/3method.

Conclusions and Recommendations

The integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximated method. For this reason, the presented methods have been proposed for approximated solutions to the first order linear Fredholm-Volterra integro-differential equations of the second kind. From numerical examples it can be seen that the proposed numerical methods are efficient and accurate to estimate the solution of these equations, Also, we show that when the values of h decreases, the absolute errors decrease to smaller values. We will use these methods to study systems of linear Fredholm – Volterra integro – differential equations of the second kinds in our future work.

References

[1] K. V. Lakshm and G. M. R. M. Rao," Theory of Integro-Differential Equations ", Gordon and Breach Publishers, 1995.

[2] B. Batiha, M.S.M. Noorani and I. Hashim, "Numerical Solutions Of The Nonlinear Integro-Differential Equations", Int. J. Compt. Math., Vol.1, No.1,2008, pp 34-42.

[3] L. M. Delves and J. Walsh, "Numerical Solution of Integral Equations", Clarendon Press, Oxford, 1974.

[4] L. M. Delves and J. L. Mohamed" Computational Methods for Linear Integral Equations", Cambridge University Press, 1985.

[5] H. Brunner, "Collocation Methods for Volterra Integral and Related Functional Differential Equations", Cambridge University Press, 2005.

[6] A. Akyuz and M. Sezer," A Chebyshev Collocation Method for Solution of Linear Integro-Differential Equations", Int. J. Compt. Math., Vol. 72, No.4, 1999, pp 491-507.

[7] A. Karamete and M. Sezer," A Taylor Collocation Method for Solution of Linear Integro-Differential Equations", Int. J. Compt. Math., Vol. 79, No. 9, 2002, pp 987-1000.

[8] A. Pedas, E. Tamme, "Discrete Galerkin Method for Fredholm Integro-Differential Equations with Weakly Singular Kernels", J. of Comp.l and Appl. Math., 2007, pp 1-16.

الخلاصة

الهدف من هذا العمل هو استخدام بعض الطرق العددية لحل معادلات فريدهولم–فولتيرا التكاملية التفاضلية ذات الرتبة الاولى الخطية من النوع الثاني. هذه الطرق هي طريقة شبه المنحرف المتكررة وطريقة سمبسون ٣/١ المتكررة. هذه التقنيات تحول المعادلات التكاملية التفاضلية الى نظام من معادلات جبرية .الامثلة العددية اعطيت لتبيان كفاءة ودقة العمل المقدم.