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NUMERICAL SOLUTION FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS OF TWO-SIDED

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Abstract

In this paper we introduce the numerical solutions of two-sided fractional partial differential equations with or without parameter. The algorithm for the numerical solution for these equations is based on finite difference method. Finally, some numerical examples are provided to show that the numerical method for solving these cases is an effective solution method.

Introduction

The idea has emerged that the fractional partial differential equations, obtained from the standard partial differential equations that replace the standard partial derivatives by fractional partial derivatives, may more accurately describe some physical problems than the corresponding standard partial differential equations [1].

More and more works by researchers from various fields of science and engineering deal with dynamical systems described by fractional-partial differential equations, which have been used to represent many natural processes in physics, finance, and hydrology, [2, 3, 4,9].

Liu F. et al. [5] considered the fractional Fokker-Planck equation and presented its numerical solution. Recently, Liu F. et al. [6] also treated the fractional advection-dispersion equations and derived the complete solution of this equation with an initial condition.

In this paper presents a practical numerical method for solving the two-sided fractional partial differential equation with or without parameter of the form:

- $$\frac{\partial u(x,t)}{\partial t}=\lambda\left[c_+(x,t)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}+c_-(x,t)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}\right]$$

and

- $$\frac{\partial u(x,t)}{\partial t}=c_+(x,t)\frac{\partial^\alpha u(x,t)}{\partial_+ t^\alpha}+c_-(x,t)\frac{\partial^\alpha u(x,t)}{\partial_- t^\alpha}+s(x,t)$$

respectively.

We use a variation on the classical explicit Euler method. We prove this method by using a novel shifted version of the usual grunwaled finite difference an approximation for the non-local fractional derivative operator.

Finite Difference Method for Solving the Two-Sided Fractional partial Differential Equations without Parameter

In this section, we use the finite difference method to solve the two-sided fractional partial differential equations of the form:

$$\frac{\partial u(x,t)}{\partial t}=c_+(x,t)\frac{\partial^\alpha u(x,t)}{\partial_+ t^\alpha}+c_-(x,t)\frac{\partial^\alpha u(x,t)}{\partial_- t^\alpha}+s(x,t) \quad \text{......(1.a)}$$

subject to the initial condition

u (x,0) = f(x), L < x < R .... (1.b)

and the boundary conditions

u (L,t) = k_1(t), $0 \le t \le T$ .... (1.c)
u (R,t) = k_2(t), $0 \le t \le T$

where c_+, c_- and s are known functions of x and t, f is a known function of x, g and k are known function of t and α is a given fractional number.

When α=2 and $c(x,t)=c_+(x,t)+c_-(x,t)$, eq.(1) becomes the following classical parabolic partial differential equation:

$$\frac{\partial u(x,t)}{\partial t}=c(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}+s(x,t)$$

Similarly, when α=1 and $c(x,t)=c_+(x,t)+c_-(x,t)$, eq.(1) reduces to the following classical hyperbolic partial differential equation

$$\frac{\partial u(x,t)}{\partial t}=c(x,t)\frac{\partial u(x,t)}{\partial x}+s(x,t)$$

The finite difference method starts by dividing the x-interval [L,R] into n subintervals to get the grid points x_i= L + iΔx, where $\Delta x=(R-L)/n$ and i=0,1,…,n. Also, the t-interval [0,T] is divided into m subintervals to get the grid points t_j= jΔt, j= 0,1,…,m, where $\Delta t=T/m$.

Next, by evaluating eq.(1) at (x_i,t_j) and use the explicit Euler method one can get:

$$\frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\Delta t}=c_+(x_i,t_j)\frac{\partial^\alpha u(x_i,t_j)}{\partial_+ x^\alpha}+$$
$$c_-(x_i,t_j)\frac{\partial^\alpha u(x_i,t_j)}{\partial_- x^\alpha}+s(x_i,t_j)+O(\Delta t) \quad \text{.................} \quad (2)$$

Then use the shifted Grunwald estimate to the α- the fractional derivative, [7]:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}=\frac{1}{(\Delta x)^\alpha}\sum_{k=0}^{M} g_k u(x-(k-1)\Delta x,t)+O(\Delta x) \quad \cdots \quad (3)$$

where $g_{\alpha,k}=(-1)^k\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$, k=0,1,2,…

Therefore, eq.(2) becomes

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t}=\frac{c_{+i,j}}{(\Delta x)^\alpha}\sum_{k=0}^{i+1} g_k u_{i-k+1,j}+\frac{c_{-i,j}}{(\Delta x)^\alpha}\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j}+s_{i,j},$$
$$i=1,2,...,n-1,\ j=0,1,...,m-1 \quad \text{.......................} \quad (4)$$

where $u_{i,j}=u(x_i,t_j)$, $c_{+i,j}=c_+(x_i,y_j)$, $c_{-i,j}=c_-(x_i,y_j)$ and $s_{i,j}=s(x_i,t_j)$.

The resulting equation can be explicitly solved for $u_{i,j+1}$ to give

$$u_{i,j+1}=\beta\sum_{k=0}^{i+1} g_k u_{i-k+1,j}+\eta\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j}+\Delta t s_{i,j}+u_{i,j},$$
$$i=1,2,...,n-1,\ j=0,1,...,m-1 \quad \text{........................} \quad (5)$$

where $\beta=c_{+i,j}\frac{\Delta t}{(\Delta x)^\alpha}$ and $\eta=c_{-i,j}\frac{\Delta t}{(\Delta x)^\alpha}$.

Also form the initial condition and boundary conditions one can get

$u_{i,0}=f(x_i)$, i=0,1,…, n
$u_{L,j}=k_1(t_j)$, j=0,1,…, m
$u_{R,j}=k_2(t_j)$, j=0,1,…, m

By evaluating eq.(5) at each $i=1,2,...,n-1$ and $j=0,1,...,m-1$ and using the above three one equations one can get the numerical solutions of eq.(1).

Finite Difference Method for Solving the Two-Sided Fractional Partial Differential Equations with Parameter

In this section, we demonstrate the finite difference method to solve the two-sided fractional partial differential equations of the form:

$$\frac{\partial u(x,t)}{\partial t}=\lambda\left[c_+(x,t)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}+c_-(x,t)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}\right] \quad \text{........} \quad (6)$$

subject to the initial condition

u(x,0) = f(x), L < x < R .... (7)

and the boundary conditions

u (L,t) = k_1(t), ... (8)
u (R,t) = k_2(t), $0\le t\le T$

Where c_+ and c_- are known functions of x and t, f is a known function of x, g and k are known function of t, α is a given fractional member and λ is a scalar parameter. The problem here is to find the eigenpair (λ,u) which satisfy eq.(7)-(8).

This equation can be written as an eigenvalue problem Au=λBu, where

$$A=\frac{\partial}{\partial t},B=c_+(x,t)\frac{\partial^\alpha}{\partial x^\alpha}+c_-(x,t)\frac{\partial^\alpha}{\partial x^\alpha}.$$

When α=2 and $c(x,t)=c_+(x,t)+c_-(x,t)$, eq.(6) becomes the following classical eigenvalue problem

$$\frac{\partial u(x,t)}{\partial t}=\lambda\left[c(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}\right]$$

Similarly, when α=1 and $c(x,t)=c_+(x,t)+c_-(x,t)$, eq.(6) reduces to the following eigenvalue problem

$$\frac{\partial u(x,t)}{\partial t}=\lambda\left[c(x,t)\frac{\partial u(x,t)}{\partial x}\right]$$

By following the same previous steps, eq.(6) reduces to

$$\frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\Delta t}=\lambda\left[c_+(xi,tj)\frac{\partial^\alpha u(x_i,t_j)}{\partial_+ x^\alpha}+\right.$$
$$\left.c_-(x_i,t_j)\frac{\partial^\alpha u(x_i,t_j)}{\partial_- x^\alpha}\right]+O(\Delta t) \quad \text{.........} \quad (8)$$

Also use is made of the shifted Grunwald estimate to the α-th fractional derivative given by eq.(3) to reduce eq.(6) as in the following form

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t}=\lambda\left[\frac{c_{+i,j}}{(\Delta x)^\alpha}\sum_{k=0}^{i+1} g_k u_{i-k+1,j}+\frac{c_{-i,j}}{(\Delta x)^\alpha}\sum_{k=0}^{n-i+1} g_k u_{i+k-1,j}\right],$$
$$i=1,2,...,n-1,\ j=0,1,...,m-1 \quad \text{.........................} \quad (9)$$

where $u_{i,j}=u(x_i,t_j)$, $c_{+i,j}=c_+(x_i,y_j)$, $c_{-i,j}=c_-(x_i,y_j)$ and $g_k=(-1)^k\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$, k=0,1,2,…

Also from the initial condition and boundary condition one can get

$u_{i,0}=f(x_i)$, i=0,1,…, n
$u_{L,j}=k_1(t_j)$, j=0,1,…, m
$u_{R,j}=k_2(t_j)$, j=0,1,…, m

By evaluating eq.(9) at $i=1,2,...,n-1$ and $j=0,1,...,m-1$ one can get a system of algebraic equations which can be solved by any suitable method to get the eigenpair $(\lambda, \{u_{i,j}\}_{\substack{i=1,2,\dots,n-1 \\ j=1,2,\dots,m-1}})$.

Numerical Examples

In this section, two numerical examples are presented, showing the fractional partial differential behaviors of the solution with the parameter α.

Example 1: Consider the two-sided fractional partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \Gamma(0.5)x^{1.5}\frac{\partial^{1.5}u(x,t)}{\partial_+ x^{1.5}} + \Gamma(0.5)(0.4-x)^{1.5} + \frac{\partial^{1.5}u(x,t)}{\partial_- x^{1.5}} + 4xe^t - 0.8e^t$$

subject to the initial condition

u (x,0) = 2x, 0 < x < 0.4

and the boundary conditions

u (0,t) = 0, 0 ≤ t≤ 0.025

u (0.4,t) = 0.8 e^t, 0 ≤ t≤ 0.025

This fractional partial differential equation together with the above initial and boundary condition is constructed such that the exact solution is $u(x,t) = 2xe^t$.

The numerical solution of example (1) by using the finite difference method for $\Delta x = 0.1$ and $\Delta t = 0.0125$

Numerical Solution	Exact Solution	Error
0.17700	0.٢٠٢٥٢	٢.٥٥٢٠٠ E-2
0.39300	0.٤٠٥٠٣	١.٢٠٣٠٠ E-2
0.60600	0.٦٠٧٥٥	١.٥٥٠٠٠ E-3
0.16800	0.٢٠٥٠٦	٣.٧٠٦٠٠ E-2
0.39900	0.٤١٠١٣	١.١١٣٠٠ E-2
0.61000	0.٦١٥١٩	٥.١٩٠٠٠E-3

Example 2: Consider the two - sided fractional partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \lambda\left[\Gamma(0.2)x^{1.8}\frac{\partial^{1.8}u(x,t)}{\partial_+ x^{1.8}} + \Gamma(0.2)(0.2-x)^{1.8}\frac{\partial^{1.8}u(x,t)}{\partial_- x^{1.8}}\right]$$

subject to the initial condition

u (x,0) = x+1.125, 0 < x < 0.2

and the boundary conditions

u (0,t) = e^{-t} +0.125, 0 ≤ t≤ 0.005

u (0.2,t) = e^{-t} +0.325, 0 ≤ t≤ 0.005

This fractional partial differential equation together with the above initial and boundary condition is constructed such that the exact solution is $u(x,t) = x + e^{-t} + 0.125$

The numerical solution of example (2) by using the finite difference method for $\Delta x = 0.05$ and $\Delta t = 0.0025$

Numerical Solution	Exact Solution	Error
٠.٦٢٥00	٠.٦٢٥00	0
١.١٦٩00	١.١٧٢٥٠	٣.٥٠٠٠٠٠E -3
١.٢٢٣00	١.٢٢٢٥٠	٥.٠٠٠٠٠٠E -4
١.٢٧٤00	١.٢٧٢٥٠	١.50000E -3
١.١٦٣00	١.١٧٠٠١	٧.٠١000E -3
١.٢٢١٠٠	١.٢٢٠٠١	٩.٩٠٠٠٠E -4
١.٢٧٣00	١.٢٧٠٠١	٢.٩900٠E -

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الخلاصة

في هذا البحث قدمنا الحل العددي للمعادلات التفاضليةِ الجزئيةِ الكسريةِ ذات اتجاهينِ مع أو بدون متغير. وان خوارزمية الحل العددي لتلك المعادلات قائمة على اساس الفروق المنتهية.

اخيراً قدمنا بعض الامثلة العددية والتي تثبت ان الطرق العددية لحل هذه المعادلات هي طرق ذات حل مؤثر فعال مقبول.