ON FINDING THE EHRHART POLYNOMIALS USING A MODIFIED PARTIAL FRACTION METHOD

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Abstract:

A wide variety of topics in pure and applied mathematics involve the problem of counting the number of lattice points inside a polytope. Perhaps the most famous special case is the theory of Ehrhart polynomials, which is the basis structure theorem about this type of counting problem.

We present a modified tool to find the Ehrhart polynomial of a convex polytope, by writing a polytope as a linear system and find the solution of this system using integer programming method with a modification on this method. This method depends on deriving the vector partition function as a partial fraction.

Introduction

We are interested in computing the number of integer solutions to the linear system $x \in R^d_{\geq 0}$ (where $R^d_{\geq 0}$ means d-space of vectors with positive components), Ax=b, where the coefficients of A are non negative (m×d)-integral matrix and $b \in Z^m$.

Let A fixed and study the number of solutions $\Phi_A(b)$ as a function of b, the function $\Phi_A(b)$ often called a vector partition functions, which appears in mathematical areas: Number theory, Discrete Geometry, Commutative Algebra, Algebraic Geometry, Representation Theory, Optimization, as well as applications to Chemistry, Biology, Physics, Computer Science and Economics [1].

Denote the columns of A by $c_1,...,c_d$. For P a d-polytope, Ehrhart studied the particular case of $\Phi_A(b)$ given by the counting function $L(P,t)\#\left(tP \cap Z^d\right)$, for positive integer t, this number of lattice points in the dilation tP of P. before we star, the following lemma is needed which go to Euler, [2].

Properties of the Ehrhart polynomials:

A convex polytope $P \subset R^d$ is the convex hull of infinity many points in R^d. One can define P as the bounded intersection of affine half spaces. A polytope is rational if all vertices have rational coordinates. P° denote the relative interior of P. for a positive integer t, let $L_P(t)$ denote the number of integer points in the dilated polytope $tP = \{tx, x \in P\}$. The fundamental result about the structure of $L_P(t)$ is given by theorems (3) and (4).

Vector partition function:

Let $\phi^\circ_b(b)$ count the integer solutions of x>0, Ax=b, $A \in M_{mxd}$ And $b \in Z^m$. Both $\phi_A(b)$ and $\phi^\circ_b(b)$ are quasi-polynomials, and can hence be algebraically defined for arguments which are not integer vector in the positive span of A. the following identity shows the relationship between the two functions.

Theorem 1, [2]:

The quasi-polynomials $\phi_A(b)$ and $\phi^\circ_b(b)$ satisfy

$$\phi_A(-b) = (-1)^{d-\mathrm{rank}(A)} \phi^\circ_A(b)$$

Corollary 1, [2]:

The quasi-polynomials $\phi_A(b)$ satisfy

$$\phi_A(-b) = (-1)^{d-\mathrm{rank}(A)} \phi_A(-b-r)$$

Lemma 1, [3]:

Let $\Phi_A(b)$ be a vector partition functions for the system Ax=b, $A \in M_{mxd}$

And $b \in Z^m$, then $\Phi_A(b)$ equals the coefficients of $Z^b = z_1^{b_1},...,z_m^{b_m}$ of the function $f(Z) = \dfrac{1}{(1-Z^{c_1})...(1-Z^{c_d})}$ expanded as a power series centered at Z=0. Equivalently, the coefficients of Z^b in f (z) equals the constant term in $\frac{f(Z)}{Z^b}$ denoted by const $\frac{f(Z)}{Z^b}$, so Eulers lemma can be stated as:

$$\phi_A(b) = \text{const}\frac{1}{(1-Z^{c_1})(1-Z^{c_2})\ldots(1-Z^{c_d})Z^b}$$

In a series of articles [4, 5, 6], complex integration of $\frac{f(Z)}{Z^b}$ are used to compute $\Phi_A(b)$ for special case of A.

Here we expand $\frac{f(Z)}{Z^b}$ into partial fractions to compute its constant term, and hence $\Phi_A(b)$.

The modified partial fraction method:

This section illustrates the idea of our computation. Our goal is to derive,

$$\phi_A(b) = \text{const}\frac{1}{(1-Z^{c_1})(1-Z^{c_2})\ldots(1-Z^{c_d})Z^b}.$$

We start by expanding $\frac{1}{(1-Z^{c_1})(1-Z^{c_2})\ldots(1-Z^{c_d})Z^b}$ into partial fractions in one of the components of Z, say z_1, therefore,

$$\frac{1}{(1-Z^{c_1})(1-Z^{c_2})\ldots(1-Z^{c_d})Z^b} = \frac{1}{z_2^{b_2}\ldots z_m^{b_m}}\sum_{k=1}^{d}\frac{A_k(Z,b_1)}{1-Z^{c_k}} + \sum_{j=1}^{b_1}\frac{B_j(Z)}{z_1^j}$$

Here A_k and B_j are polynomials in z_1, rational functions in $z_2,\ldots,z_m$, and exponential in b_1. The two sums on the right- hand side correspond to the analytic and the meromorphic part with respect to $z_1 = 0$. The latter does not contribute to the z_1-constant term, whence

$$\phi_A(b) = \text{const}_{z_2\ldots z_m}\left(\frac{1}{z_2^{b_2}\ldots z_m^{b_m}}\text{const}_{z_1}\left(\sum_{k=1}^{d}\frac{A_k(Z,b_1)}{1-Z^{c_k}}\right)\right)$$

$$= \text{const}\left(\frac{1}{z_2^{b_2}\ldots z_m^{b_m}}\sum_{k=1}^{d}A_k(0,z_2,\ldots,z_m,b_1)\right)$$

The effect of one partial fraction is to eliminate one of the variables of the generating function, at the cost of replacing one rational function by a sum of such. It is best to illustrate the above idea through an actual example.

An illustrating example

Consider the constraints

$$x_1 + 2x_2 + = a$$
$$x_1 + x_2 + x_4 = b \qquad , x_1, x_2, x_3, x_4 \geq 0$$

This can be written as Ax=b, where $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$, and b= (a, b), so $\phi_A(a,b)$ counts the integer solutions of the above system, so by Eulers lemma,

$$\phi_A(a,b) = \text{const}\frac{1}{(1-z_1z_2)(1-z_1^2z_2)(1-z_1)(1-z_2)z_1^a z_2^b}$$

We first expand into partial fractions with respect to z_2

$$\frac{1}{(1-z_1z_2)(1-z_1^2z_2)(1-z_2)z_2^b} =$$

$$-\frac{\frac{z_1^{b+1}}{(1-z_1)^2}}{1-z_1z_2} + \frac{\frac{z_1^{2b+3}}{(1-z_1)(1-z_1^2)}}{1-z_1^2z_2} + \frac{\frac{1}{(1-z_1)(1-z_1^2)}}{1-z_2} + \sum_{k=1}^{b}\frac{\ldots}{z_2^k}$$

Taking constant terms gives

$$\phi_A(a,b) = \text{const}_{z_1}\left(\frac{1}{(1-z_1)z_1^a}\text{const}_{z_2}\left(\frac{1}{(1-z_1z_2)(1-z_1^2z_2)(1-z_2)z_2^b}\right)\right)$$

$$= const\left(\frac{1}{(1-z_1)z_1^a}\left(-\frac{z_1^{b+1}}{(1-z_1)^2} + \frac{z_1^{2b+3}}{(1-z_1)(1-z_1^2)} + \frac{1}{(1-z_1)(1-z_1^2)}\right)\right)$$

$$= const\left(-\frac{z_1^{b-a+1}}{(1-z_1)^3} + \frac{z_1^{2b-a+3}}{(1-z_1)^2(1-z_1^2)} + \frac{1}{(1-z_1)(1-z_1^2)z_1^a}\right)$$

Our work is to find the constant term of the above expression with respect to one variable which is z_2, therefore we get,

$$\phi_A(a,b) = const$$
$$\left(\frac{1}{(1-z_1)z_1^a}\left(-\frac{z_1^{b+1}}{(1-z_1)^2} + \frac{z_1^{2b+3}}{(1-z_1)(1-z_1^2)} + \frac{1}{(1-z_1)(1-z_1^2)}\right)\right)$$

Others terms are eliminated since they contents some variables and we wants only constant term, therefore

$$const\left(-\frac{z_1^{b-a+1}}{(1-z_1)^3} + \frac{z_1^{2b-a+3}}{(1-z_1)^2(1-z_1^2)} + \frac{1}{(1-z_1)(1-z_1^2)z_1^a}\right)$$

.... (1)

Now, each term of the three constant terms termed as counting integer solutions to linear systems.

$$\mathrm{const}\left(-\frac{z_1^{b-a+1}}{(1-z_1)^3}\right)=0 \text{ if } b-a+1>0.$$

Equivalently b≥a.

If b<a, we use Taylor's expansion of $\frac{1}{(1-z_1)^3}$ we get

$\frac{1}{(1-z_1)^3}\sum_{k\geq 0}\binom{k+2}{2}z_1^k$ This gives

$$\mathrm{const}\left(\frac{1}{(1-z_1)^3 z_1^{a-b+1}}\right)=\binom{a-b+1}{2}=\frac{(a-b)^2}{2}+\frac{a-b}{2}$$

Since we get constant term only if the power of z_1 is zero, that is k=b-a+1.

Therefore the first term is,

$$const\left(\frac{1}{(1-z_1)^3 z_1^{a-b+1}}\right)=\begin{cases}0 & if\ b\geq a\\ \frac{(a-b)^2}{2}+\frac{a-b}{2} & if\ b\leq a+1\end{cases}$$

For the second term in (1), we have

$$\mathrm{const}\left(\frac{z_1^{2b-a+3}}{(1-z_1)^2(1-z_1^2)}\right)=0 \quad \text{if } 2b-a+2\geq 0$$

If $a\geq 2b+3$, we expand into partial fractions again, in general

$$\frac{1}{(1-z_1)^2(1-z_1^2)}=\frac{1}{(1-z_1)^2(1-z_1)(1+z_1)}=\frac{1}{(1-z_1)^3(1+z_1)}$$

Taylor expansion for $\frac{1}{z_1^{a-2b+3}}$ about $z_0=1$ is founded, let t=a-2b+3

$$\frac{1}{z_1^t}=1+(-t)(z_1-1)+\frac{(-t)(-t-1)}{2!}(z_1-1)^2+\frac{(-t)(-t-1)(-t-2)}{3!}(z_1-1)^3+\ldots$$

Therefore

$$const_{z_1}\frac{1}{(1-z_1)^3(1+z_1)}\cdot\left(1+(-t)(z_1-1)+\frac{(-t)(-t-1)}{2\,!}(z_1-1)^2+\frac{(-t)(-t-1)(-t-2)}{3\,!}(z_1-1)^3+\ldots\right)$$

..............................(2)

By subtitled

$$\frac{1}{(1-z_1)^3(1+z_1)}=\frac{1/2}{(1-z_1)^3}+\frac{1/4}{(1-z_1)^2}+\frac{1/8}{(1-z_1)}+\frac{1/8}{(1+z_1)}$$

$$\frac{1}{(1-z_1)^2(1+z_1)}=\frac{1/2}{(1-z_1)^2}+\frac{1/4}{(1-z_1)}$$

And

$$\frac{1}{(1-z_1)(1+z_1)}=\frac{1/2}{(1-z_1)}+\frac{1/2}{(1+z_1)}$$

Into (2), and sum the terms with similar denominator, we get

$$const_{z_1}\left(\frac{1/2}{(1-z_1)^3}+\frac{1/4+t}{(1-z_1)^2}+\frac{1/8+t/4+1/4(-t)(-t-1)}{(1-z_1)}+\frac{1/8+t/4+(-t)(-t-1)/4}{4(1+z_1)}\right)$$

By substation t = 2b – a + 3, the constant is

$$\frac{(a-2b)^2}{4}+\frac{2b-a}{2}+\frac{1-(-1)^{a+1}}{8}$$

Similarly to the first constant term computations, this identity is also valid for a=2b+2, 2b+1 and 2b, hence

$$const\left(\frac{z_1^{2b-a+3}}{(1-z_1)^2(1-z_1^2)}\right)=$$

$$\begin{cases}0 & if\ a\leq 2b+2\\ \frac{(a-2b)^2}{4}+\frac{2b-a}{2}+\frac{1-(-1)^{a+1}}{8} & if\ a\geq 2b\end{cases}$$

For the last term, the constant is

$$const\left(\frac{1}{(1-z_1)(1-z_1^2)z_1^a}\right)=$$

$$const_{z_1}\left(\frac{1/2}{(1-z_1)^3}+\frac{1/4+a/2}{(1-z_1)^2}+\frac{1/8+a^2/4+a/2}{(1-z_1)}+\frac{(-1)^a/8}{(1+z_1)}\right)$$

$$=\frac{a^2}{4}+a+\frac{7+(-1)^a}{8}$$

Summing up all terms in (1) gives:

$$\phi_A(a,b)=\begin{cases}\frac{a^2}{4}+a+\frac{7+(-1)^a}{8} & if\ a\leq b\\ ab-\frac{a^2}{4}-\frac{b^2}{4}+\frac{a+b}{2}+\frac{7+(-1)^a}{8} & if\ \frac{a}{2}-1\leq b\leq a+1\\ \frac{b^2}{2}+\frac{3b}{2}+1 & if\ b\leq\frac{a}{2}\end{cases}$$

Also, we can show that, by corollary (1)

$$\phi_A(a,b)=\phi_A(-a-4,-b-3)$$

We also modified the method of finding partial fraction using the following theorem which stat that:

Theorem (2), [7, p.273]:

If a is a simple root of Q(x) so that Q(x)=(x-a)Q1(x), Q1(a)≠0, then the function $\frac{P(x)}{Q(x)}$ can be written in one and only one way in the form $\frac{P(x)}{Q(x)} = \frac{C}{x-a} + \frac{P1(x)}{Q1(x)}$ where C is a constant. C can be calculated by using the form $C = \frac{P(a)}{Q_1(a)} = \frac{P(a)}{Q'(x)}$.

The same example is solved using a modified method by assuming that

$$\frac{P(z_2)}{Q(z_2)} = \frac{1}{(1-z_1z_2)(1-z_1^2z_2)(1-z_2)z_2^b}$$

Write $Q(z_2)$ as

$$Q(z_2) = (z_2 - a_1)(z_2 - a_2)...(z_2 - a_k)$$

therefore,

$$R(z) = -z_1(z_2 - \frac{1}{z_1}).z_1^2(z_2 - \frac{1}{z_1^2}).(z_2 - 1)z_2^b$$

$$= -z_1^3(z_2 - \frac{1}{z_1}).(z_2 - \frac{1}{z_1^2}).(z_2 - 1)z_2^b$$

Then

$$Q(z_2) = (z_2 - \frac{1}{z_1}).(z_2 - \frac{1}{z_1^2}).(z_2 - 1)z_2^b$$

and $P(z_2) = -\frac{1}{z_1^3}$, with constant $C_k = \frac{P(a_k)}{Q'(a_k)}$.

Hence by computation we get,

$$C_1 = \frac{P(a_1)}{Q'(a_1)} = -\frac{z_1^{b+1}}{(1-z_1)^2},$$

$$C_2 = \frac{P(a_2)}{Q'(a_2)} = -\frac{z_1^{3+2b}}{(1-z_1)(1-z_1^2)}$$

and

$$C_3 = \frac{P(a_3)}{Q'(a_3)} = -\frac{1}{(1-z_1)(1-z_1^2)}$$

Those are the constant terms that are founded as before. All other constant can be found using the same way. This makes the solution easier.

Theorem (3), [2]:

If P is a convex rational polytope, then the functions $L_P(t)$ and $L^\circ_p(t)$ are quasi-polynomials in t whose degree is the dimension pf P. if P has integer vertices, then $L_P(t)$ and $L^\circ_p(t)$ are polynomials.

Theorem (4), [2]:

The quasi-polynomials $L_P(t)$ and $L^\circ_p(t)$ satisfy

$$L_P(-t) = (-1)^{\dim P} L^\circ_P(t)$$

Suppose the convex rational polytope $P \subset R^d$ is given by an intersection of half spaces, that is $P = \{x \in R^d : Ax \le b\}$, where $A \in M_{mxd}$ and $b \in Z^m$. We may convert these inequalities into equalities by introducing slack variables.

If P has rational vertices, we can choose A and b in such a way that their entries are integer, without loss of generality nonnegative ones.

The connection to vector partition functions is now evident. Since $tP = \{x \in R^d_{\ge 0} : Ax \le tb\}$, we obtain $L_P(t) = \phi_A(tb)$ as special evaluation of $\phi_A(b)$ as an example, the quadrilateral Q described by

x, y > 0, x+2y≤5

x+y≤4

As special case of the polygons appearing in section (4) with vertices (0,0),(4,0),(3,1) and (0,5/2) has the Ehrhart –quasi polynomial

$$L_Q(t) = \phi_A(5t, 4t) = \frac{23}{4}t^2 + \frac{9}{2}t + \frac{7+(-t)^t}{8}$$

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الخلاصة

حساب حجم متعدد الاضلاع وكذلك حساب عدد النقاط التي احداثياتها اعداد صحيحة في المجال R^D هو موضوع مهم جدا في فروع الرياضيات المختلفة مثل نظرية الاعداد ونظرية التمثيل و متعدد الحدود ايرهارت في التوافيقية والتشفير والنظام الاحصائي.

تم حساب متعدد الحدود ايرهارت باسخدام بعض الطرق. احدى هذه الطرق طورت واستنتجنا مبرهنة لحساب معاملات متعدد الحدود ايرهارت. حيث قمنا بتحويل المسالة الاصلية الى حل منظومة برمجة خطية صحيحة. والطريقة التي استخدمت لحساب المعاملات هي طريقة تجزئة الكسور.