On Principally Generalized Lifting Modules

Aseel Amer Hassan
Department of Mathematics, College of Science, University of Baghdad, Baghdad-Iraq.
Corresponding Author: aselamer16@gmail.com.

Abstract

In this paper we introduce principally generalized lifting as a generalization of principally lifting modules and we prove under certain conditions some relations between Mj-projective (quasi-discrete) and PGD_1. [DOI: 10.22401/JNUS.20.4.14]

Keywords: lifting modules, principally lifting modules, principally generalized lifting modules.

δ_1 Introduction

Let R be an associative ring with identity and let M be a unital R-module.A sub module L of an R-module M is called small for(short L ≪ M), if K + L ≠ M for any proper sub module K of M. A module M is called hollow, if every proper submodule of M is small in M [1]. A non zero module M is called so- semi hollow, if each proper finitely generated sub module is small in M, and a non zero module M is so- called P-hollow, if each proper cyclic sub module is small in M [5].It is clear that every hollow is semi hollow and every semi hollow is P- hollow. A module M is called lifting (or has the condition D_1), if for every submodule L of M, there is a decomposition M = N ⊕ S such that N ≤ L and S ∩ L ≪ M [2]. It was introduced in [3] that a module M is principally lifting module (or has PD_1), if for all m ∈M, M has a decomposition M = N ⊕ S with N ≤ mR and mR ∩ S ≪ M. M is said to have condition (D_2) in case, if B is a su module of M with M / B is isomorphic to summand of M then B is a summand of M [4]. A module M is called a discrete module, if it has the condition (D_1) and (D_2). M is said to have the condition (D_3) just in case of if M_1 and M_2 are summand.Such that $M_1 + M_2 = M$ then $M_1 \cap M_2$ is a summand of M. A module M is called so- a quasi- discret module, if it has the condition (D_1) and (D_3). [4]

A modul M is so- called a generalized lifting module, if every submodule L of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq L$ and $M_2 \cap L \leq Rad(M)$. As a generalization of Principally lifting module we introduce a principally generalized lifting module (for short PGD_1). Where Rad (M) is the Jacobson radical of M. It is known that Rad (M) equal the sum of all small submodules of M. [4]. In this paper we study the relation between PD_1 and PGD_1 modules and prove some properties of a PGD_1.

δ_2 P-hollows and the condiion (PGD_1)

In this section we introduce PGD_1 module as a generalization of PD_1, that appeared in [3] and we prove results on PGD_1 module.

We start by the following.

Lemma (2.1) [5,2.15]:

Let M be a module then

1. If M is semi- hollow, then each factor modul is semi-hollow.
2. If B ≪ M and M / B is semi-hollow then M is semi-hollow.
3. M is semi-hollow if and only if M is local or Rad(M) = M".

Proposition (2.2) [3]:

The following are equivalent for a module M.

1. M is P- hollow.
2. B ≪ M when M / B is a non Zero cyclic module ".

Remark (2.3):

1- P- hollow modules need no hollow just as is explained in [5] by considering the set Q of all rational as Z- module (Q / Z) is no hollow while is no cyclic for all that proper sub modul K of Q.

2- "hollow module are indecomposable modules then the direct sums of hollow module are not hollows, while according to lemma (2.1), if $M = \oplus_{i \in I} P_i$,where P_i are non-cyclical P-hollows for all i∈I, then M is P – hollow".

Remark (2.4):

Every hollow module is lifting [6].

Definition (2.5):-[5]

A module M is called Principally lifting (or has (PD_1)) if for all m ϵ M, M has a decomposition M = N ⊕ S with N≤ mR and mR ∩ S ≪ M.

As generalization of definition (2.5) we introduce the following:

Definition (2.6):-

M is principally generalizd lifting (or has PGD_1), If for all m ∈ M, M has a dcomposition M = A ⊕ B with A ≤ mR and mR ∩ B ≤ Rad (M).

Note:-

hollow module → lifting module → principally lifting module → principally generalized lifting module.

Example (2.7):-

1. Z_{P^∞} is (PGD_1).
2. Z_4 as Z-module is (PGD_1).
3. Z_p, p is prim number is PGD_1.
4. Z as Z- module is not PGD_1.

Proposition (2.8):-

The condition (PGD_1) is inherited by sum ands.

Proof:

Suppose that M have the condition PGD_1, also K ≤ ⊕ M, if k ∈ K, when M has a decomposition M = A ⊕ B with A ≤ kR and kR ∩ B ≤ Rad(M), it follows that K = A ⊕ (K ∩ B) and kR ∩ (K ∩ B) ≤ kR ∩ B ≤ Rad(M), so kR ∩ (K∩ B) ≤ Rad(K)(due to K ≤ ⊕ M). Therefore K has (PGD_1).

Lemma (2.9):-

The following are equivalent for an indecomposable module M.

1- M has (PGD_1).
2- M is a P-hollow module.

Proof:

(1) ⇒ (2) Suppose that 0≠ mϵ M, Rm is proper submodule of M, then by (1) there exist decomposable M = N⊕S, with N ≤ Rm and Rm ∩ S ≤ Rad (M), because M is indcomposable.

Then either S = 0 or N = 0, if S =0 then M = N, hence M = Rm (Contradiction) (since Rm is proper), hence N = 0. Thus M= S therefor Rm ∩S = Rm ∩ M = Rm ≤ Rad(M) thus Rm ≤ Rad(M) hence mϵ Rad(M), Rm << M.[11].

(2)⇒ (1) Since M is P- hollow then for each proper cyclic sub module mR of M, mR ≪ M. thus M = 0 ⊕ M and 0 ≤ mR, mR ∩ M = mR ≤ Rad (M).

The following definition appeared in [7]

Definition (2.10) :-

[7] Suppose that M is an R-module, if N,L ≤ M and M = N + L, then L is so- called generalized supplement of N just is case N ∩ L ≤ Rad(L). M is called generalized supplemented or (briefly GS) in case each submodule N has a generalized supplement in M.

Example (2. 11):-

[8] Suppose that M is a GS and Rad(M) be Noetherian or M satisfy A.C.C on small sub module, then M is a supplemented module.

Lemma (2.12):-

Suppose that M has (PGD_1), then each cyclic submodule mR has a generalized supplemented S whichever is a summand of M.

Proof:

Let mR ≤ M then there exist N ≤ mR with M = N ⊕ S and mR ∩ S ≤ Rad(M), hence M = mR + S and mR ∩ S ≤ Rad(M), hence S is a GS of M and S ≤ ⊕ M.

Lemma (2.13):-

"The following are equivalent for a module M."

1- M has PGD_1
2- Every one cyclic submodule K of M can be written as K = N⊕ S with N ≤ ⊕ M and S ≤ Rad (M).
3- Each m ∈ M there exist principal ideals I and J of R such that mR = mI ⊕ mJ, where mI ≤ ⊕ M and mJ ≤ Rad(M).

Proof:

(1) ⇒ (2) clear.

(2) ⇒ (1) Let K be a cyclic submodul of M then by(2) K = N ⊕ S with N ≤ ⊕ M and

$S \leq$ Rad (M). Write $M = N \oplus N'$, it follow that $K = N \oplus K \cap N'$.

Let $\pi : N \oplus N' \to N'$ be the natural projection, we have $K \cap N' = \pi(K) = \pi(N \oplus S) = \pi(S) \leq Rad(M)$. hence M has PGD_1.

(2) ⇔(3)Clear.

§3 Results on Mj- projective (quasi-discrete) and PGD_1 modules.

In this section we prove under certain conditions some relations between Mj-projective (quasi- discrete) and PGD_1 module.

We need the definition:

Definition (3.1)[12]:-

Let $M = \oplus_{i\in j} H_i$, then H_i is H_j-projective for each $i \neq j$, if every supplement C of H_i in M is a direct summand.

Lemma (3.2) [9,corollary 4.50]:-

Let $M = \oplus M_i$, where M_i is hollow and Mj-projective whenever $i \neq j$.Then M is a quasi- discrete module.

"It is known that each quasi – discrete module is a direct sum of hollow sub module unique up to isomorphism and is fully relatively projective".

Proposition (3.3):-

Suppose that $M = \oplus_{i\in j} H_i$,where each H_i is a hollow module and is H_j –projective ($j \neq i$). Then M has (PGD_1).

Proof:

Suppose that K is a cyclic sub module of M, and there exists a finite subset F of I that $K \leq^{\oplus} \oplus_{i\in F} H_i$. By lemma (3.2), $\oplus_{i\in F} H_i$ is quasi discrete, thus K can be written as $K = N \oplus S$ wherever $N \leq^{\oplus} \oplus_{i\in F} H_i$,hence $N \leq^{\oplus} M$ and $S \leq Rad(\oplus_{i\in F} H_i)$.Therefore by lemma (2.13) M has PGD_1).

Proposition (3.4) :-

Suppose that M is module with PGD_1, if $M = V + W$ such that $W \leq^{\oplus} M$ and $V \cap W$ is cyclic, then W contains generalized supplemented of V in M.

Proof:

Because M has PGD_1 and $V \cap W$ is cyclic we have by lemma (2.13) $V \cap W = N \oplus S$, where $N \leq^{\oplus} M$ and $S \leq$ Rad (M), Since $W \leq^{\oplus} M$, we have $S \leq$ Rad (W). Write $W = N \oplus N_1$. It follows that $V \cap W = N \oplus (V \cap W \cap N_1) = N \oplus (V \cap N_1)$.

Let $\pi : N \oplus N_1 \to N$ be that natural projection. It follows that $V \cap N_1 = \pi(N \oplus (V \cap N_1) = \pi(V \cap W) = \pi(N \oplus S) = \pi(S)$, hence $\pi(S) \leq Rad(M)$, hence $V \cap N_1 \leq Rad(M)$ such that $M = V + N + N_1 = V + N_1$. Therefore N_1 is generalized supplemented of V in M that is contained in W.

Corollary (3.5) :-

Suppose that M is a module with PGD_1 over a principally "ideal ring", if $M = V + mR$, then mR contains a generalized supplemented of V in M.

Proof:

By lemma(2.13) we have $mR = N \oplus S$, wherever $N \leq^{\oplus} M$ and $S \leq Rad(M)$, it follows that $M = V + N$, hence by lemma (2.13) N is cyclic summand of M, hence $V \cap N$ is a cyclic submodule of M and thus apply proposition (3.4).

Lemma (3.6) :-

Suppose that M is module such that PGD_1, then each indcomposable cyclic submodule C of M is either small in M or a sum and of M.

Proof:

"by lemma (2.13) we have $C = N \oplus S$ with $N \leq^{\oplus} M$ and $S \leq Rad(M)$,since C is indecompable either C = S" or C = N, if C = S, then $C \leq$ Rad (M) since C is cyclic, then $C = Rx \leq Rad(M)$, hence $x \in$ Rad (M) imples $C = Rx$ is small in M. If C =N, then $C \leq^{\oplus} M$.

Definition (3.7):-

[4] "A module M is said to be π – projective, if for every two submodule U,V of M with $M = U + V$,there exist $f \in End(M)$ with $Imf \leq U$ and $Im(1- f) \leq V$".

Lemma (3.8):-

[9, 4.47][10, 3.2] let $M = M_1 \oplus M_2$."Then following are equivalent."

1- M_1 is M_2- projective.

2- If $M = N \oplus M_2$, and $N \cap M_2 \leq^{\oplus} N$ hence $M = N_1 \oplus M_2$, wherever $N_1 \leq N$.

Proposition (3.9):-

Let $M = \bigoplus_{i=1} P_i$, where the P_i are local modules for all i, if M has(D_3),"then the following are equivalent".

1- M has PGD_1

2- "M is a quasi-discrete module".

Proof:

(1) ⇒ (2) Because PGD_1 and D_3 are inherited by summand, we have $p_i \oplus p_j$ has PGD_1 and D_3 for all i,j ($i \neq j$).

If $P_i \oplus P_j = K + P_j$, then $P_i \cong (P_i \oplus P_j) / P_j = (K + P_j) / P_j \cong K / (K \cap P_j)$ is a cyclic module. Thus form some $m \in P_i \oplus P_j$

$K = mR + (K \cap P_j)$. By PGD_1 for $P_i \oplus P_j$ and by lemma (2.13) we get $mR = N \oplus S$ with $N \leq^{\oplus} P_i \oplus P_j$,So $S \leq Rad(P_i \oplus P_j)$ hence $P_i \oplus P_j = K \oplus P_j = (N \oplus S) + (K \cap P_j) + P_j = N + P_j$ and by(D_3) for $P_i \oplus P_j$, we have $P_i \oplus P_j = N + P_j$ with $N \leq K$. Hence by lemma (3.8) P_i is P_j–projective for all $i \neq j$, therefor by lemma (3.2), M is quasi- discrete.

(2) ⇒ (1) it is obvious.

Proposition (3.10):-

Suppose that M is a module over a local ring R. If M has PGD_1, then a cyclic submodule of M is either small in M or a summand of M.

Proof:

"The proof follows from lemma (3.6) and the fact that every cyclic module over a local ring is a local module".

Definition (3.11)[3]:-

Suppose that M_1 and M_2 be R-modules M_1 is said to be Pprojective relative to M_2 (or M_1 is M_2- Pprojective), if for each $m_2 \in M_2$ epimorphism $g: m_2R \to m_2R / K$ and each homomorphism $\varphi: M_1 \to m_1R/K$, there exists a homomorphism $f : M_1 \to m_2R$ with $g \circ f = \varphi$.

Remark (3.12) [3]:-

Cleary every M- projective module is M- P projectiv, if M is a cyclic module then each M- Pprojective modul is M - projective module, there are R-modules M_1 and M_2, where M_1 is M_2- Pprojective whilist M_1 is no M_2-projective. Example M_1 = Q (the set of all rational number) R = Z and $M_2 = \bigoplus_{i \in I} Z$, where $f : \bigoplus_{i \in I} Z \to Q$ is an epimorphism (as Q is a homomorphic image of a free Z-module). Clearly Q is $\bigoplus_{i \in F} Z$- projective for every finite subset F of I, hence Q is $(\bigoplus_{i \in I} Z)$-P projective, while Q is not $(\bigoplus_{i \in I} Z)$–projective, since f does not split (due to Q not a projective Z-module).

Lemma (3.13):-

Let $M = M_1 \oplus M_2$ be an R-module. Then the following are equivalent".

1- M_1 is M_2 –Pprojective

2- M_1 is m_2R- projective for all that $m_2 \in M_2$

For all $m_2 \in M_2$, if $M_1 \oplus m_2R = m_2R + Y$, then there is $L \leq Y$ such that $M_1 \oplus m_2R = L \oplus m_2R$.

Proof:

(1)⇒ (2) by definition of relative Pprojective

(2)⇒ (3) by lemma (3.8)

(3)⇒ (1) by lemma(3.8)

Corollary (3.14):-

Let $M = M_1 \oplus M_2$ a module over local ring R- module M_1 and M_2 are relatively Pprojective in that case M has PGD_1, if and only if every one M_1 and M_2 have PGD_1.

Proof:

⇐) Suppose that C are arbitrary cyclic submodule of M then $C = (m_1 + m_2)R$, where $m_1 \in M_1, m_2 \in M_2$, since M_1 and M_2 have PGD_1,then we have nothing to prove either $m_1 = 0$ or $m_2 = 0$.

Now to avoid triviality we may consider C is not a small submodule of M since $C = (m_1+m_2) R \leq m_1R + m_2R$, we have m_1R or m_2R is not small in M. Without loss of generality we may assume m_1R is no small in M, hence it is not small in M_1 by pro position (3.10), m_1R is a summand of M_1 and hence m_1R is M_2-Pprojective hence m_1R is m_2R-projective.

Since $m_1R \oplus m_2R = (m_1 + m_2)R + m_2R$, we have by lemma (3.13) that there is $N \leq (m_1 + m_2)R$ with $m_1R \oplus m_2R = N \oplus m_2R$.It follows that $(m_1 + m_2)R = N \oplus [(m_1 + m_2)R \cap m_2R]$. "Since C is a local module and m_2R is not contained in C, we have that C = N.To show that N is a summand of M.

It is clear that" $m_1R \oplus M_2 = N + M_2$ and hence $N \cap M_2 = N \cap (N \oplus m_2R) \cap M_2 = (m_1R \oplus m_2R) \cap M_2 \cap N = m_2R \cap N = 0$ (since N = C).As $m_1R \leq^{\oplus} M_1$, where $N \oplus M_2 = m_1R \oplus M_2 \leq^{\oplus} M$ $C = N \leq^{\oplus} M$. Therefore $C \oplus L = M$. The converse follows from proposition (2.8).

References

[1] Fleuery P. “Hollow modules, and local Endmorphism Ring, Pac”. J. Math, (53), 379-385, 1974.

[2] Keskin D. “On lifting Modules, Comm. Algbra” 28(7), 3427-3440, 2000.

[3] Kamal M. and yousef A. “on principally lifting module” V.2, 127- 137, 2007.

[4] Wisbauer R. “Foundations of module and ring theory, Gordon and Breagh, Reading”, 1991.

[5] Clark J. Lomp C. Vanaja N. and wisbauer R. “lifting modules, Brikhauser-Basel, Ist Edition”, 2006.

[6] Ben A. laroussi hamdouni, “On lifting modules, University of Baghdad, The College of Science”, A Thesis of MCS, 2001.

[7] Xue W. “Characterizations of Semi perfect and Perfect rings, Publications Matematiques”, 40115-125, 1996

[8] Wang Y. and Ding N. “Generalized supplemented Modules”, *Taiwanese journal of Math*, 10 (6), 1589-1601, 2006

[9] Mohamed S. and Muller B.J. “Continuous and discrete modules, Cambridge University Press”, 1990.

[10] Kamal M. and yousef A. “On supplementation and generalized projective modules” to appear in J, Egyptain Math. Soc, 2013.

[11] Goodearl K.R. “Ring Theory, Pure and Applied Math.”, Marcel-Dekker.(33), 1976.

[12] Anderson F. W. and Fuller K.R. “Ring and Categories of Modules”, Springer – Verlage, New York. 1992.