THE CYCLIC DECOMPOSITION OF THE FACTOR GROUP CFZ ($Z_{13^{(n)}}$, Z) / $\overline{\overline{R}}$($Z_{13^{(n)}}$)

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Abstract

The main purpose of this work is the determination of cyclic decomposition of the group

$K(Z_{p^{(n)}}) = cf(Z_{p^{(n)}}, Z)/\overline{\overline{R}}(Z_{p^{(n)}})$

(The factor group of all z-valued class functions module the group of Z-valued generalized characters for elementary abelian group $Z_{p^{(n)}}$). Where $p = 13$.For this purpose a recurrence relation is established for the cyclic decomposition of the group above and it is solved. This solution gives the number of $Z_{p^{(i)}}$ for distinct $1 \le i \le n$, in the cyclic decomposition of $K(Z_{p^{(n)}})$. A program for our work is made by C^{++} and is demonstrated by a flow chart in Fig.(1).

Introduction

Let $Z_{p^{(n)}}$ denote the direct sum of n copies groups Z_p of prime p . Let $cf(Z_{p^{(n)}}, Z)$ be the group of all Z-valued class functions on $Z_{p^{(n)}}$ which are constant on the Q-classes (two elements of G are said to be Q-conjugate if the cyclic subgroups they generate are conjugate in G. This defines equivalence relation on G, its classes are called the Q-classes of G), inside this group we have the subgroup $\overline{\overline{R}}(Z_{p^{(n)}})$ of all z-valued generalized characters of $Z_{p^{(n)}}$, we write $K(Z_{p^{(n)}})$ for factor group

$cf(Z_{p^{(n)}}, Z)/\overline{\overline{R}}(Z_{p^{(n)}})$.

The problem of determination of the cyclic decomposition of the group $K(Z_{p^{(n)}})$ leads to the problem of diagonalization of the Z-valued characters table matrix $\equiv^* Z_{p^{(n)}}$.

To simplify the problem of diagonalization of the matrix $\equiv^* Z_{p^{(n)}}$, we partition it to blocks with common properties and we diagonalize the block matrix $\equiv^* Z_{p^{(n)}}$ block wise.

The problem for P=2, 3, 5, 7 and P = 11 has been solved in [4], [5] and [6] respectively, and its solution give thenumber of $Z_{p^{(i)}}$ for distinct $1 \le i \le n$ in the cyclic decomposition of $K(Z_{p^{(n)}})$.

Basic Concept

The cyclic decompositio of $K(Z_{p^n})$:[4]

Our work is specific for p =13 where the diagonalization of the matrix $\equiv^* Z_{p^{(n)}}$ in [1] gives us the cyclic decomposition of the group $K(Z_{p^{(n)}})$.

The cyclic decomposition of $K(Z_{p^{(n)}})$ is determined by determining the invariant factors of the $\equiv^*(Z_{p^{(n)}})$ where

$$\equiv^*(Z_{p^{(n)}}) = \begin{bmatrix} p\equiv^*(Z_{p^{(n-1)}}) & 0 \\ 0 & M_{n-1} \end{bmatrix}$$

Where M_{n-1} is an integral matrix formed by replacing each ω^{p-1} in $\equiv^*(Z_{p^{(n-1)}})$ by

$p-1$ and (–1) else where. [$\omega = e^{\frac{2\pi i}{p}}$ is a primitive nth root of unity] .

To obtain the invariant factor of $\equiv^{*} Z_{p^{(n)}}$ we shall add the number of invariant factor of $\equiv^{*} Z_{p^{(n-1)}}$ after multiplying them by 13 to the number of I.F. of M_{n-1} .

The diagonalization of block matrix M_{n-1}: **[1]**

The general form of the block matrix M_{n-1} of dimension $p^{n-1} \times p^{n-1}$, $p = 13$ is given by:

$$M_{n-1} \approx \begin{bmatrix} I_{n-2}, A_{n-2}, B_{n-2}, C_{n-2}, D_{n-2}, E_{n-2}, F_{n-2}, \\ G_{n-2}, H_{n-2}, J_{n-2}, K_{n-2}, L_{n-2}, 13I_{n-2} \end{bmatrix}$$

<u>Theorem (1), [4]</u>

$$K(G) = \bigoplus_{i=1}^{r} Z_{d_i}$$

Where $d_i = \pm D_i\left(\equiv^{*}(G)\right) / D_{i-1}\left(\equiv^{*} G\right)$.

<u>Lemm (1),[1] :</u>

If $K\left(Z_{13^{(n-1)}}\right) = \oplus \sum_{i=0}^{n-1} \delta_i Z_{13^i}$

Where δ_i is the number of times the invariant factors (13^i) appears in the invariant factor of $\equiv^{*}\left(Z_{13^{(n)}}\right)$.

Then $K\left(Z_{13^{(n)}}\right) = \oplus \sum_{i=1}^{n} \delta_{i-1} Z_{13^i} \oplus j$,

where J is the direct sum of cyclic Z- modules of orders the distinct invariant factors of M_{n-1} .

<u>Remark (1):</u>

We write the invariant factor as I.F

<u>Example (1):</u>

When n=2, p=13

$$K\left(Z_{13^2}\right) = Z_{13^2} \oplus 13\ Z_{13} \oplus Z_1$$

When n=3 M_2 ~ diag [$I_1, A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1, J_1, K_1, L_1, 13I_1$] .
where $I_1 = M_1$

<u>I_1</u>	<u>A_1</u>	<u>B_1</u>	<u>C_1</u>	<u>D_1</u>	<u>E_1</u>	<u>F_1</u>	<u>G_1</u>	<u>H_1</u>	<u>J_1</u>	<u>K_1</u>	<u>L_1</u>	<u>$13I_1$</u>
1	13	13	13	13	13	13	13	13	13	13	13	13
13	13	13	13	13	13	13	13	13	13	13	13	196
13	13	13	13	13	13	13	13	13	13	13	196	196
13	13	13	13	13	13	13	13	13	13	196	196	196
13	13	13	13	13	13	13	13	13	196	196	196	196
13	13	13	13	13	13	13	13	196	196	196	196	196
13	13	13	13	13	13	13	196	196	196	196	196	196
13	13	13	13	13	13	196	196	196	196	196	196	196
13	13	13	13	13	196	196	196	196	196	196	196	196
13	13	13	13	196	196	196	196	196	196	196	196	196
13	13	13	196	196	196	196	196	196	196	196	196	196
13	13	196	196	196	196	196	196	196	196	196	196	196
13	196	196	196	196	196	196	196	196	196	196	196	196

The number of invariant factors of M_2 is the number of I.F. (13) =78, number of I.F. (196) = 78 then by theorem (1) and Lemma (1) we get

$$K\left(Z_{13^{(3)}}\right) = Z_{13^3} \oplus 91Z_{13^2} \oplus 91Z_{13} \oplus Z_1$$

Matrix of the I.F. of M_{n-1}:

We definea $(p \times n)$rectangular rmatrix $X_{n-1} = (x_{ij})_{p \times n}$, where X_{ij} = number of P^{j-1} in the i-th block of M_{n-1} and the number of the I.F of P^{j-1} in

$$X_{n-1} = \sum_{i=1}^{p} x_{ij} \quad j=1, 2\ldots n \qquad (1)$$

Table (1)
Block matrices of the invariant factor matrix of M_{n-1} and M_n $n \geq 2$.

Block matrices of the I.F matrix of M_n-1	Block matrices of the I.F matrix of M_n $p=13, n\geq 2, u=n-2$												
	I_{n-1}	A_{n-1}	B_{n-1}	C_{n-1}	D_{n-1}	E_{n-1}	F_{n-1}	G_{n-1}	H_{n-1}	J_{n-1}	K_{n-1}	L_{n-1}	$13I_{n-1}$
I_u	I_u	A_u	B_u	C_u	D_u	E_u	F_u	G_u	H_u	J_u	K_u	L_u	$13I_u$
A_u	A_u	B_u	C_u	D_u	E_u	F_u	G_u	H_u	J_u	K_u	L_u	$13I_u$	$13A_u$
B_u	B_u	C_u	D_u	E_u	F_u	G_u	H_u	J_u	K_u	L_u	$13I_u$	$13A_u$	$13B_u$
C_u	C_u	D_u	E_u	F_u	G_u	H_u	J_u	K_u	L_u	$13I_u$	$13A_u$	$13B_u$	$13C_u$
D_u	D_u	E_u	F_u	G_u	H_u	J_u	K_u	L_u	$13I_u$	$13A_u$	$13B_u$	$13C_u$	$13D_u$
E_u	E_u	F_u	G_u	H_u	J_u	K_u	L_u	$13I_u$	$13A_u$	$13B_u$	$13C_u$	$13D_u$	$13E_u$
F_u	F_u	G_u	H_u	J_u	K_u	L_u	$13I_u$	$13A_u$	$13B_u$	$13C_u$	$13D_u$	$13E_u$	$13F_u$
G_u	G_u	H_u	J_u	K_u	L_u	$13I_u$	$13A_u$	$13B_u$	$13C_u$	$13D_u$	$13E_u$	$13F_u$	$13G_u$
H_u	H_u	J_u	K_u	L_u	$13I_u$	$13A_u$	$13B_u$	$13C_u$	$13D_u$	$13E_u$	$13F_u$	$13G_u$	$13H_u$
J_u	J_u	K_u	L_u	$13I_u$	$13A_u$	$13B_u$	$13C_u$	$13D_u$	$13E_u$	$13F_u$	$13G_u$	$13H_u$	$13J_u$
K_u	K_u	L_u	$13I_u$	$13A_u$	$13B_u$	$13C_u$	$13D_u$	$13E_u$	$13F_u$	$13G_u$	$13H_u$	$13J_u$	$13K_u$
L_u	L_u	$13I_u$	$13A_u$	$13B_u$	$13C_u$	$13D_u$	$13E_u$	$13F_u$	$13G_u$	$13H_u$	$13J_u$	$13K_u$	$13L_u$
$13I_u$	$13I_u$	$13A_u$	$13B_u$	$13C_u$	$13D_u$	$13E_u$	$13F_u$	$13G_u$	$13H_u$	$13J_u$	$13K_u$	$13L_u$	13(13In)

Example (2):
When n=2

$$M1=\begin{bmatrix} 1 &&&&&&&&&&&& \\ & 13 &&&&&&&&&&& \\ && 13 &&&&&&&&&& \\ &&& 13 &&&&& 0 &&&& \\ &&&& 13 &&&&&&&& \\ &&&&& 13 &&&&&&& \\ &&&&&& 13 &&&&&& \\ &&&&&&& 13 &&&&& \\ &&&&&&&& 13 &&&& \\ &&& 0 &&&&&& 13 &&& \\ &&&&&&&&&& 13 && \\ &&&&&&&&&&& 13 & \\ &&&&&&&&&&&& 13 \end{bmatrix}_{13\times 13}$$

$$X_1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}_{13\times 2} \longrightarrow X_2=\begin{bmatrix} 1 & 12 & 0 \\ 0 & 12 & 1 \\ 0 & 11 & 2 \\ 0 & 10 & 3 \\ 0 & 9 & 4 \\ 0 & 8 & 5 \\ 0 & 7 & 6 \\ 0 & 6 & 7 \\ 0 & 5 & 8 \\ 0 & 4 & 9 \\ 0 & 3 & 10 \\ 0 & 2 & 11 \\ 0 & 1 & 12 \end{bmatrix}_{13\times 3}$$

Then by theorem (1) and $K(Z_{13})=Z_{13} \oplus Z_1$
We obtain

$$K(Z_{13^{(3)}})=Z_{13^3} \oplus 91Z_{13^2} \oplus 91Z_{13} \oplus Z_1$$

$$K(Z_{13^{(2)}})=Z_{13^2} \oplus 13Z_{13} \oplus Z_1$$

Recurrence Relation for the I.F of ($\equiv^*(Z_{13^{(n)}})$)

In this section, we find the recurrence relation for the I.F of $\equiv^*(Z_{13^{(n)}})$.

According to the information in Table (1) which contains the thirteen block matrices of X_{n-1}. Let R(n,d) be the number of I.F. of $(13)^m$, m=1,2,…,n-1 in the nth stage , d represents the position of the block matrices. 13I, L, K, J, H, G, F, E, D, C, B,A, and 13I column-wise in general.

Now we derive a recurrence relation for R(n , d), from table(1) we can see that only the upper left triangular part contributes to R(n,d) when m=1 and d=1,…,13 which represent 13I, L, K, J, H, G, F, E, D, C, B, A and 13I respectively ,where the lower triangular part has no contribution to R(n,d),since the factor 1 dose not appear in the blocks 13I, L, K, J, H, G, F, E, D, C, B,A and 13I in general.

So R(n,d) is equal to the number of I.F (13) in the nth stage,(r-1) position plus the number of I.F (13) in the (n-1)th stage, d position.

So we have:

$$R(n,d) = R(n,d-1) + R(n-1,d)\ . \quad \ldots\ldots (2)$$

$n \geq 3$, $d = 2$, …, 13.

With the boundary conditions

R (2, n) =1 d = 1,..., 12 R(2 , 13)=0

R (n, 1) = R (n-1, 1) = …= R (2, 1) =1

Relation (1) represents the number of I.F. (13) which was computed from the upper triangular part of Table (1).

Table (2)

R (n,d) for I.F (13) in X_{n-1}.

	13I	L	K	J	H	G	F	E	D	C	B	A	I
n \ d	1	2	3	4	5	6	7	8	9	10	11	12	13
2	1	1	1	1	1	1	1	1	1	1	1	1	0
3	1	2	3	4	5	6	7	8	9	10	11	12	12
4	1	3	6	6	15	21	28	36	45	55	66	78	90

Number of the I.F (13) in

$$X_{n-1} = \sum^{13} R(n,d) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now, the value of R (n,d) for I.F (13^m) , m=2,…, n-1 is equal to the number of the I.F. (13^m) in the upper left triangular part plus the number of I.F. (13^{m-1}) in lower part , since this is multiplied by 13 so we have the relation

$$R(n,d) = R(n,d-1) + R(n-1,d) - R(n-1,d-P) \quad \ldots\ldots (4)$$

With the same conditions in (1)

Where $n \geq 3$, $p = 13$ and d=p+1, …, (p-1)m + 1.

R(n - 1, d - p) = 0 when m=1.

The reason for subtracting R(n-1,r-13) in the relation above is that, the blocks in each column are the same blocks as the preceding column except for the last one which is redundant.

From Table (1) and (2).

$$\text{Number of I.F}(p^m) = \sum_{d=(p-1)m-11}^{(p-1)m+1} R(n,d)$$

$$= R(n+1,(p-1)m+1) \quad \ldots\ldots (5)$$

Where $n \geq 3$, m=1, …, n

Recurrence Relation for the I.F of $\equiv^*\left(Z_{13^{(n)}}\right)$ and its general Solutions

In this section, we solve a recurrence relation which is defined in (4) by using the generating function

The generating function for (4) is

$$F_{n(x)} = R_{(n,1)}x + R_{(n,2)}x^2 + \ldots\ldots + R_{(n,d)}x^d + \ldots\ldots$$

We shall find R(n,d) by finding its generating function.

Multiplying both sides of (4) by x^d and summing from d= 2 to d = ∞, we obtain:

$$R(n,d)x^d = \sum_{d=2}^{\infty} R(n,d-1)x^d + \sum_{d=2}^{\infty} R(n-1,d)x^d - \sum_{d=2}^{\infty} R(n-1,d-p)x^d$$

Which yields

$$F_n(x) - R(n,1)x = xF_n(x) + F_{n-1}(x)$$

$$- R(n-1,1)x - x^p F_{n-1}(x)$$

$$F_n(x)(1-x) = (1-x^p)F_{n-1}(x)$$

$$F_n(x) = \frac{(1-x^p)}{1-x} F_{n-1}(x)$$

Repeating this relation, we obtain:

$$F_n(x) = \frac{(1-x^p)^2}{(1-x)^2} F_{n-2}(x)$$

$$\vdots$$

$$F_n(x) = \frac{(1-x^p)^{n-1}}{(1-x)^{n-1}} F_1(x)$$

Then

$$F_n(x) = \frac{(1-x^p)^{n-1}}{(1-x)^{n-1}} x \quad n \geq 1 \,.................. \ (6)$$

(Ordinary generating function for (4))

With the boundary condition $F_1(x) = x$.

By using binomial theorem:

We obtain:

$$= \sum_{d=0}^{n-1} (-1)^d \binom{n-1}{d} x^{pd+1} . \sum_{d=0}^{\infty} \binom{n+d-2}{d} x^d$$

$$F_n(x) = x \sum_{r=0}^{n-1} (-1)^d \binom{n-1}{d} x^{pd}$$

$$. \sum_{d=0}^{\infty} \binom{n+d-2}{d} x^d$$

The problem is now solved .To find R (n,d) all that need be done is to read the coefficient of $x^{r-1}, x^{r-(p+1+}, ..., x^{r-((n-1)+p+1)}$ in $F_n(x)$

Thus

$$R(n,d) = \binom{n-1}{0}\binom{n+d-3}{d-1} - \binom{n-1}{1}\binom{n+d-(p+1)}{d-(p-1)}$$

$$+ ...... + (-1)^{n-1} \binom{n-1}{n-1}\binom{n+d-2-(n-1)p+1}{d-(n-1)p+1},$$

$n \geq 2$ .... (7)

$$R(n,d) = \sum_{k=0}^{n-2} (-1)^k \binom{n-1}{k}\binom{n+d-pk-3}{d-(pk+1)}$$

By substituting (5) in (7) we obtain:

Number of the I.F. $(p)^i$

$$= \sum_{k=0}^{i-1} (-1)^k \binom{n}{k}\binom{n+(p-1)i-pk-1}{(p-1)i-pk}$$

Now by theorem (1) we can obtain the cyclic decomposition of $k(Z_p^{(n)})$, $n \geq 2$.

Results

<u>Theorem:</u>

$$K(Z_{p^{(n)}}) = Z_{p^n} \oplus \sum_{i=1}^{n-1} \delta_i Z_{p^i}$$

Where:

$$\delta_i = \sum_{k=0}^{i-1} (-1)^k \binom{n}{k}\binom{n+(p-1)i-pk-1}{(p-1)i-pk}$$

p=3, 5, 7, 11, 13

<u>Example:</u> p=13, n=5

$$k(Z_{13^5}) = Z_{13^5} \oplus \sum_{i=1}^{4} \delta_i Z_{13^i}$$

$$\delta_1 = \binom{5}{0}\binom{16}{12} = 1820$$

$$\delta_2 = \binom{5}{0}\binom{28}{24} - \binom{5}{1}\binom{15}{11} = 13650$$

$$\delta_3 = \binom{5}{0}\binom{40}{36} - \binom{5}{1}\binom{27}{23} + \binom{5}{2}\binom{14}{10} = 13650$$

$$\delta_4 = \binom{5}{0}\binom{52}{48} - \binom{5}{1}\binom{39}{35} + \binom{5}{2}\binom{26}{22}$$

$$- \binom{5}{3}\binom{13}{9}$$

$$= 1820$$

$$k(z_{13^{(5)}}) = Z_{13^5} \oplus 1820 Z_{13^4} \oplus 13650 Z_{13^3}$$

$$\oplus 13650 Z_{13^2} \oplus 1820 Z_{13} \oplus Z_1$$

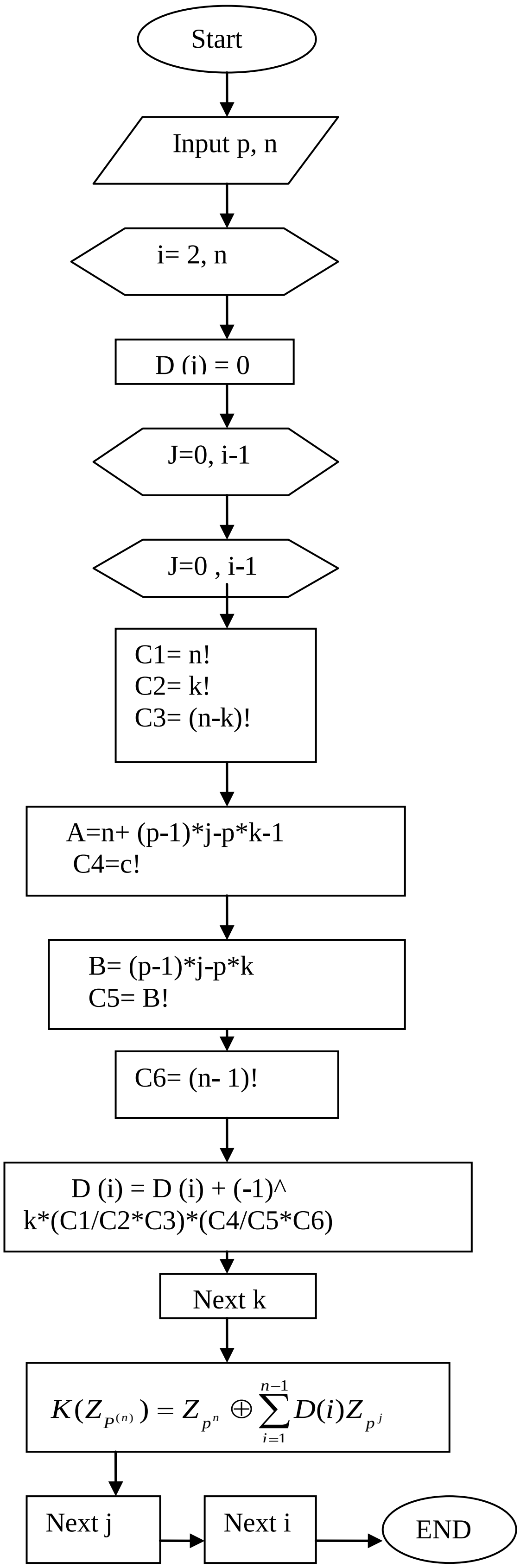

Fig. (1) : Program flowchart for finding the cyclic Decomposition of $K(Z_{13^{(n)}})$.

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الخلاصة

الهدف الأساسي من هذا العمل هو ايجاد التجزئة الدائرية المختلفة للزمرة الأبيلية المنتهية

$$K\left(Z_{p^{(n)}}\right) = cf\left(Z_{p^{(n)}}, Z\right) / \overline{R}\left(Z_{p^{(n)}}\right).$$

(الزمرة الكسرية لكل دوال الصفوف ذات القيم الصحيحة على زمرة الشواخص العمومية ذات القيم الصحيحة للزمرة الإبدالية الأولية Z حيث p=13).

لهذا الغرض تم بناء علاقة مرتدة مع ايجاد الحل العام لهذه العلاقة ولأي n وان هذا الحل يعطينا العوامل الدائرية $Z_{p^{(n)}}$ حيث $1\le i \le n$ ، للزمرة $K\left(Z_{p^{(n)}}\right)$ الذي صيغ بشكل مبرهنة.