RANGE –KERNEL ORTHOGONALITY OF ELEMENTARY CHORDAL TRANSFORM

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Abstract.

Let $B(H)$ denoted the C^* – algebra of all bounded linear operators on a separable Hilbert space H. For $A,B \in B(H)$, the elementary operator $\Delta_{A,B} : B(H) \to B(H)$ is defined by $\Delta_{A,B}(X) = AXB - X$. We defined the elementary Chordal transform $g_{A,B}$ as an operator on $B(H)$ by $g_{A,B}(X) = (|A^*|^2 + I)^{-1/2} \Delta_{A,B}(X)(|B|^2 + I)^{-1/2}$ for all $X \in B(H)$. In this paper ,the Range –kernel orthogonality of this transform was studied concerning to main rants , the Range – kernel orthogonality of the restrictions of $g_{A,B}$ to Hilbert –Schmidt class and the Range–Kernel orthogonality of restrictions $f_{A,B}$ and $g_{A,B}$ to Schatten p-class.

Keywords: Normal derivation, schatten p-class, unitarily invariant norm, orthogonality, elementary operator, chordal transform.

Introduction

Let $B(H)$ be denoted the C^* – algebra of all bounded linear operators on a separable Hilbert space H. For operators $A,B \in B(H)$, the generalized derivation $\delta_{A,B} : B(H) \to B(H)$ is defined by elementary $\delta_{A,B}(X) = AX - XB$ for all $X \in B(H)$. Also, for $A,B \in B(H)$ define the elementary operator $\Delta_{A,B} : B(H) \to B(H)$ as an operator on $B(H)$ by $\Delta_{A,B}(X) = AXB - X$ for all $X \in B(H)$. Define the Chordal transformation $f_{A,B}$, as an operator on $B(H)$ by $f_{A,B}(X) = \left(|A^*|^2 + I\right)^{-1/2} \delta_{A,B}(X)\left(|B|^2 + I\right)^{-1/2}$ for all A,B and $X \in B(H)$ When $A = B$, we simply write f_A for $f_{A,A}$. We define the elementary Chordal transform $g_{A,B}$ as an operator on $B(H)$ by $g_{A,B}(X) = (|A^*|^2 + I)^{-1/2} \Delta_{A,B}(X)(|B|^2 + I)^{-1/2}$ for all A,B and $X \in B(H)$, when $A = B$, we simply write g_A for $g_{A,A}$. The Chordal transform has some geometric properties that resemble those of the Chordal distance. Recall that the Chordal distance between any two complex numbers a and b is given by $d(a,b) = \frac{|a-b|}{(1+|a|^2)^{1/2}(1+|b|^2)^{1/2}}$. It is easy to see that $d(a,b) \le 1$ for all complex number a and b [22].The orthogonality of the range and the kernel of certain derivations has been extensively studied by several authors (see, e.g., [1],[7],[11],[12],[13] and references therein). In addition to the usual operator norm $\|.\|$, which is defined on all of $B(H)$,we are interested in the general class of unitarily invariant (or symmetric) norms. Each unitarily invariant norm $|||.|||$ is defined on a natural subclass $C_{|||.|||}$ of $B(H)$ called the norm ideal associated with the norm $|||.|||$ and satisfies the invariance property $|||UAV||| = |||A|||$ for all $A \in C_{|||.|||}$ and for all unitary operators $U,V \in B(H)$. While the usual operator norm $\|.\|$ is defined on all of $B(H)$, the other unitarily invariant norms are defined on norm ideals contained in the ideal of compact operators in $B(H)$. For a compact operator A, let $S_1(A) \ge S_2(A) \ge ... \ge 0$ the singular values of A, i.e., the eigenvalues of $|A| = (A^*A)^{1/2}$. There is one – to - one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on norm ideal of operators. More precisely,if $|||.|||$ is a unitarily invariant norm, then there is unique symmetric gauge function ϕ such that $\|A\| = \phi(\{s_j(A)\})$ for all $A \in C_{\|.\|}$. For $1 \le p \le \infty$, define $\|A\|_p = (\sum_{j=1}^{\infty} S_j(A)^p)^{1/p}$,where, by convention, $\|A\|_\infty = S_1(A)$ is the usual operator norm of the compact operator A.These unitarily invariant norms are the well-known Schatten P- norms associated with the Schatten P-lasses. $C_P, 1 \le P \le \infty$.Hence C_1, C_2 and C_∞ are the trace class,the Hilbert – Schmidt class, and the class of compact operators , respectively. The Hilbert – Schmidt class is a Hilbert spaces under the inner product $\langle A, B\rangle = tr\, B^* A = tr\, AB^*$, where '$tr$' denotes the trace functional. So the Hilbert – Schmidt orm is also given by $\|A\|_2 = (\mathrm{tr}\, A^* A)^{1/2} = (\sum_{i,j=1}^{\infty} |\langle Af_j, e_j\rangle|^2)^{1/2}$ where $\{e_j\}$ and $\{f_j\}$ are any orthonormal bases for H [2]. Hirzallah O. and Kittaneh F. studied in [14] the Range -Kernel orthogonality of the Chordal transformation.

Orthogonality of the Range and kernel of $g_{A,B}$.

It has been shown in [4] that if A and B are contractions, then $S \in C_2$ and $ASB = S$ imply $\| AXB - X + S \|_2^2 = \| AXB - X \|_2^2 + \| S \|_2^2$ for all $X \in B(H)$ .This says that, in the usual Hilbert –Spaces sense, $ran\, \Delta_{A,B} \cap C_2$ is orthogonal to $\ker \Delta_{A,B} \cap C_2$, where $ran\, \Delta_{A,B}$ and $\ker \Delta_{A,B}$ denote the range and the kernel of $\Delta_{A,B}$,respectively.

Moreover, it has been show in [5] that if A, B are normal operators such that $ASB = S$ for some $S \in B(H)$, and if $X \in B(H)$ such that $AXB - X + S \in C_{\|.\|}$, then $S \in C_{\|.\|}$ and $\| AXB - X + S \| \ge \| S \|$.

That is, with respect to the unitarily invariant norm $\|.\|$, $ran\, \Delta_{A,B} \cap C_{\|.\|}$ is orthogonal, in the sense of [1], to $\ker \Delta_{A,B} \cap C_{\|.\|}$, In this result and in the sequel, it is assumed that if $T \notin C_{\|.\|}$,then $\|T\| = \infty$.

Our first orthogonality result for $g_{A,B}$ can be stated as follows.

Theorem

Let $A, S \in B(H)$ such that A is normal, $S \in C_2$ and $ASB = S$

Then $\|g_A(X) + S\|_2^2 = \|g_A(X)\|_2^2 + \|S\|_2^2$ such that X is a tenser prodact.

Proof:

If $g_A(X) + S \notin C_2$, then $g_A(X) \notin C_2$; so $\| g_A(X) + S \|_2^2 = \| g_A(X) \|_2^2 + \| S \|_2^2 = \infty$.

Suppose that $g_A(X) + S \in C_2$.

Then $g_A(X) \in C_2$ and so $S^* g_A(X) \in C_1$. Since A is normal and $ASA = S$, it follows by the Fuglede theorem that $A^* SA^* = S$ and $(A^* SA^*)^* = (S)^* \Rightarrow AS^* A = S^*$. Now

$$S^* g_A(X) =$$

$$S^*[(|A^*|^2 + I)^{-1/2}(AXA - X)(|A|^2 + I)^{-1/2}]$$

$$= S^*(|A^*|^2 + I)^{-1/2} AXA(|A|^2 + I)^{-1/2} -$$

$$S^*(|A^*|^2 + I)^{-1/2} X(|A|^2 + I)^{-1/2},$$

$X = f \otimes h$, where f and g are arbitrary vectors in H and $f \otimes h$ the operator define by $(f \otimes h)x = (x,h)f$ for all X trace.

$$tr\, S^* g_A(X) =$$

$$trS^*(|A^*|^2 + I)^{-1/2} A(f,h)A(|A|^2 + I)^{-1/2} -$$

$$trS^*(|A^*|^2 + I)^{-1/2} X(|A|^2 + I)^{-1/2}$$

$$= trS^* A(|A^*|^2 + I)^{-1/2}(f,h)A(|A|^2 + I)^{-1/2} -$$

$$trS^*(|A^*|^2 + I)^{-1/2} X(|A|^2 + I)^{-1/2}$$

$$= tr(S^* A(|A^*|^2 + I)^{-1/2} f, A^*(|A|^2 + I)^{-1/2}) -$$

$$trS^*(|A^*|^2 + I)^{-1/2} X(|A|^2 + I)^{-1/2}$$

$$= tr(AS^* A(|A^*|^2 + I)^{-1/2} f, (|A|^2 + I)^{-1/2} h) -$$

$$trS^*(|A^*|^2 + I)^{-1/2} X(|A|^2 + I)^{-1/2}$$

$$= trAS^* A((|A^*|^2 + I)^{-1/2} f, (|A|^2 + I)^{-1/2} h) -$$

$$trS^*(|A^*|^2 + I)^{-1/2} X(|A|^2 + I)^{-1/2}$$

$$= trS^*(|A^*|^2 + I)^{-1/2} X(|A|^2 + I)^{-1/2} -$$

$$trS^*(|A^*|^2 + I)^{-1/2} X(|A|^2 + I)^{-1/2}$$

$$= 0 \quad .$$

Now

$\|g_A(X)+S\|_2^2 =$

$\|g_A(X)\|_2^2 + \|S\|_2^2 + 2\operatorname{Re} tr(S^* g_A(X))$

$= \|g_A(X)\|_2^2 + \|S\|_2^2$.

Corollary

Let A, B and X such that A, B are normal, $S \in C_2$ and $ASB = S$. Then

$\|g_{A,B}(X)+S\|_2^2 = \|g_{A,B}(X)\|_2^2 + \|S\|_2^2$

for all $X \in B(H)$.

Proof: On $H \oplus H$, let

$$L = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad T = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$$

$$, \quad and \quad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Then L is normal (since A and B are normal) and $T \in C_2$ (since $S \in C_2$), $LTL = T$ and

$$g_L(Y)+T = \begin{bmatrix} 0 & g_{A,B}(X)+S \\ 0 & 0 \end{bmatrix}$$

$\|g_L(Y)+T\|_2^2 =$

$\|0\|_2^2 + \|g_{A,B}(X)+S\|_2^2 + \|0\|_2^2 + \|0\|_2^2$

$= \|g_{A,B}(X)+S\|_2^2$.

Now the result follows by applying (Theorem1) to the operators L ,T and Y ,

$\|g_L(Y)+T\|_2^2 = \|g_L(Y)\|_2^2 + \|T\| = \|g_{A,B}(X)\|_2^2 + \|S\|_2^2$.

It has been pointed out that theorem (1) in [5] can be extended to elementary operators induced by any pair of operators (A, B) that satisfies the Fuglede –Putnam property.In the same way, and again by using an argument similar to that used in the proof of Corollary(1),we can obtain the following orthogonality result for $g_{A,B}$.

Theorem

Let A, B and $S \in B(H)$ such that (A, B) satisfies the Fuglede – Putnam property, $S \in C_2$ and $ASB = S$. Then

$\|g_{A,B}(X)+S\|_2^2 = \|g_{A,B}(X)\|_2^2 + \|S\|_2^2$

for all $X \in B(H)$.

For the general class of unitarily invariant norms, we can employ the analysis in [5] to prove the following orthogonality result for $g_{A,B}$.

Theorem

Let A, B and $S \in B(H)$ such thet (A, B) satisfies the Fuglede – Putnam property $S \in C_{\|\|\cdot\|\|}$ and $ASB = S$. Then

$\|| g_{A,B}(X)+S \|| \ge \|| S \||$

for all $X \in B(H)$.

We study in the following theorem the range – kernel orthogonality of the restrictions of $g_{A,B}$ to Hilbert –Schmidt class.

Theorem

Let $A, B and S \in B(H)$ such that $S \in C_2$ .Then

$\|g_{A,B}(X)+S\|_2^2 = \|g_{A,B}(X)\|_2^2 + \|S\|_2^2$ if and only if

for all $X \in B(H)$.

$BS_1^* A = S_1^*$,

where

$S_1 = (|A^*|^2 + I)^{-1/2} S (|B|^2 + I)^{-1/2}$

Proof:

If $BS_1^* A = S_1^*$, then

$B(|B|^2+I)^{-1/2} S^* (|A^*|^2+I)^{-1/2} A$
$= (|B|^2+I)^{-1/2} S^* (|A^*|^2+I)^{-1/2}$. This, together

with the fact that $tr\, YZ = tr\, ZY$ whenever $YZ, ZY \in C_1$, implies that for every $X \in C_2$, we have

$trS^* g_{A,B}(X) =$

$trS^*(|A^*|^2+I)^{-1/2}(AXB - X)(|B|^2+I)^{-1/2}$

$= rS^*(|A^*|^2+I)^{-1/2} AXB(|B|^2+I)^{-1/2}$ _

$trS^*(|A^*|^2+I)^{-1/2} X(|B|^2+I)^{-1/2}$,

$X = f \otimes h$, where f and g are arbitrary vectors in H

$tr\, S^* g_{A,B}(X) =$

$trS^*(|A^*|^2+I)^{-1/2} A(f,h) B(|B|^2+I)^{-1/2} -$

$trS^*(|A^*|^2+I)^{-1/2}(f,h)(|B|^2+I)^{-1/2}$

$= tr(S^*(|A^*|^2+I)^{-1/2} Af, B^*(|B|^2+I)^{-1/2} h) -$

$tr(S^*(|A^*|^2+I)^{-1/2} f, (|B|^2+I)^{-1/2} h)$

$= B(|B|^2+I)^{-1/2} S^*(|A^*|^2+I)^{-1/2} Af, h) -$

$tr((|B|^2+I)^{-1/2} S^*(|A^*|^2+I)^{-1/2} f, h)$

$= trB(|B|^2+I)^{-1/2} S^*(|A^*|^2+I)^{-1/2} A(f,h) -$

$tr(|B|^2+I)^{-1/2} S^*(|A^*|^2+I)^{-1/2}(f,h)$

$= tr(|B|^2+I)^{-1/2} S^*(|A^*|^2+I)^{-1/2} X -$

$tr(|B|^2+I)^{-1/2} S^*(|A^*|^2+I)^{-1/2} X = 0$.

Hn H , Now

$$\|g_{A,B}(X)+S\|_2^2 =$$

$$\|g_{A,B}(X)\|_2^2 + \|S\|_2^2 + 2\,\mathrm{Re}\; tr(S^* g_{A,B}(X))$$

$$= \|g_{A,B}(X)\|_2^2 + \|S\|_2^2 \; .$$

Conversely, if

$$\|g_{A,B}(X)+S\|_2^2 = \|g_{A,B}(X)\|_2^2 + \|S\|_2^2$$

for every $X \in C_2$,

then

$$\mathrm{Re}\; trS^* g_{A,B}(X) = 0$$

for every $X \in C_2$.

Replacing X by iX, we get $\mathrm{Im}\, trS^* g_{A,B}(X) = 0$ *for every* $X \in C_2$. But by straight forward computations, we have

$$tr(BS_1^* A - S_1^*)X = trS^* g_{A,B}(X)$$

$$= 0 \quad for \;\; every \;\; X \in C_2 \; .$$

Consequently $BS_1^* A - S_1^* = 0$, and so $BS_1^* A = S_1^*$.

3) Range- Kernel Orthogonality of Restrictions z $f_{A,B}$ and $g_{A,B}$ on C_P .

We study Range–Kernel orthogonality of restrictions $f_{A,B}$ and $g_{A,B}$ to Schatten p-class, and again by using an argument similar to that used in the proof of lemma (3)[8],we gain the following orthogonality of the restrictions result for $f_{A,B}$ and $g_{A,B}$.

Theorem

Let $A \in B(H)$ and $S \in C_P$, $1 < p < \infty$,then

$$\|f_A(X)+S\|_P \ge \|S\|_P$$

for all $X \in B(H)$

with $f_A(X) \in C_P$ if and only if $tr(\tilde{S} f_A(X)) = 0$.

Proof:

We must show that $tr(\tilde{S} f_A(X)) = 0$ .from lemma (3)in [7],

$$tr(\tilde{S} f_A(X)) =$$

$$tr[\tilde{S}(|A^*|^2 + I)^{-1/2}(AX - XA)(|A|^2 + I)^{-1/2}]$$

$$= tr\tilde{S}(|A^*|^2 + I)^{-1/2} AX(|A|^2 + I)^{-1/2} -$$

$$tr\tilde{S}(|A^*|^2 + I)^{-1/2} XA(|A|^2 + I)^{-1/2}.$$

Use Theorem(1)in[13] to complete the proof.

Theorem

Let $A,B \in B(H)$, given $S \in C_P$, $1 < P < \infty$, then $\|f_{A,B}(X)+\hat{S}\|_P \ge \|\hat{S}\|_P$ *for all* $X \in C_P$. if and only if $\tilde{S} \in \ker f_{A,B}$.

Proof:

On $H \oplus H$, where, let

$$L = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \;\; \hat{S} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix},$$

$$and \;\; Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} ,$$

then $Y \in C_P$. And

$$f_L(Y) + \hat{S} = \begin{bmatrix} 0 & f_{A,B}(X)+S \\ 0 & 0 \end{bmatrix}$$

$$\|f_L(Y)+\hat{S}\|_P = \|f_{A,B}(X)+S\|_P$$

$$\ge \|S\|_P = \|\hat{S}\|_P \; .$$

Recall that C_P , $1 < p < \infty$,is a uniformly convex space, that every non-trivial $\hat{S} \in C_P$ is a Smooth point, and support functional $D_{\hat{S}}$ is given by

$$D_{\hat{S}}(f_{A,B}(X)) = tr\left[\frac{\hat{S}\; f_{A,B}(X)}{\|\tilde{S}\|q}\right]$$

for all $f_{A,B}(X) \in C_P$.

Now $\|f_{A,B}(X)+\hat{S}\|_P \ge \|\hat{S}\|_P$ is satisfied if and only if $D_{\hat{S}}(f_{A,B}(X)) = 0$ or if and only if $tr(\tilde{S} f_{A,B}(X)) = 0$. Choose Y to be the rank one operator $f \otimes g$ for some arbitrary elements f and g in $H \oplus H$. Then $tr(\tilde{S} f_{A,B}(Y)) = 0$ implies that $(f_{A,B}(\tilde{S}) f, g) = 0 \;\Leftrightarrow\; \tilde{S} \in \ker f_{A,B}$. Conversely, suppose that $\tilde{S} \in \ker f_{A,B}$. Since $\tilde{S}Y$ and $\tilde{S} f_{A,B}(Y)$ are trace, $tr(\tilde{S} f_{A,B}(Y)) = 0$ .Then theorem(6) implies that $\|f_{A,B}(X)+\hat{S}\|_P \ge \|\hat{S}\|_P$.

Remark. In the same direction, we have the following characterization of those operators in C_2 which are orthogonal to $rang\, g_{A,B} / C_2$. Invoking the Gateaux differentiability of the Schatten p-norm and the usual operator norm, enables us to characterize these operator that are orthogonal to the $rang\, g_{A,B}$ with respect to these norms.

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الخلاصة

ليكن H فضاء هلبرت القابل للفصل وغير منتهي البعد على حقل الأعداد العقدية وليكن $B(H)$ جبر بناخ لكافة المؤثرات الخطية المقيدة المعرفة على H يعرف مؤثر الاشتقاق المعمم $\delta_{A,B} : B(H) \to B(H)$ بأنه التطبيق ذو الصيغة

$\delta_{A,B}(X) = AX - XB$, $X \in B(H)$ حيث كل من A, B عنصر في $B(H)$ إذا كان $A=B$ يكتب بالصيغة δ_A ويعرف بمؤثر الاشتقاق ويعرف المؤثر الابتدائي $\Delta_{A,B}$ على $B(H)$ بأنه التطبيق ذو الصيغة

$\Delta_{A,B}(X) = AXB - X$, $X \in B(H)$ حيث A و B عنصر في $B(H)$

لغرض دراسة مدى كل من هذه التطبيقات برهن Anderson انه إذا كان A عنصر في $B(H)$ و S مؤثر سوي فأن $\|\delta_A(X) + S\| \geq \|S\|$, $X \in B(H)$. أي أن مدى التطبيق يكون عموديا على نواته .بعد ذلك قام عدد من الباحثين بدراسة تعامد المدى مع النواة لكل من $\delta_{A,B}$ و $\Delta_{A,B}$ كما عرف Fuad Kittaneh التطبيق

$$f_{A,B}(X) = (|A^*|^2 + I)^{-1/2} \delta_{A,B}(X)(|B|^2 + I)^{-1/2} \ , \ X \in B(H)$$

ودرس التعامد بين المدى والنواة لهذا التطبيق وعرفنا التطبيق

$$g_{A,B}(X) = \left(|A^*|^2 + I\right)^{-1/2} \Delta_{A,B}(X)\left(|B|^2 + I\right)^{-1/2} , \ X \in B(H)$$

ودرسنا التعامد بين مدى هذا التطبيق ونواته .والتعامد بين المدى والنواة لقصر التطبيق $f_{A,B}$ و $g_{A,B}$ على C_P