ON THE TENSOR PRODUCT OF OPERATORS ON HILBERT SPACE

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Abstract:

In this paper we prove that there exist some properties of operators that are invariant under tensor product like posinormal, binormal, pseudo normal operators. But $*-paranormal$has not invariant under tensor product

Introduction

Let H be an infinite dimensional separable complex Hilbert space with inner product $\langle , \rangle$ and let $B(H)$ be the algebra of all bounded linear operators on H, given $A_1, A_2 \in B(H)$,the tensor product $A_1 \otimes A_2$ on the Hilbert space $H \otimes H$ has been considered variously by many of authors(see[1],[3], [12],[14]).

The operation of taking tensor product $A_1 \otimes A_2$ preserves many properties of A_1 and $A_2 \in B(H)$ but by no means all of them .Thus, whereas the normaloid property is invariant under tensor products ,the spectraloid property is not (see [15]p.623 ,[15] p.631) again ,whereas $A_1 \otimes A_2$ is normal if and only if A_1 and $A_2 \in B(H)$are [9] and is similarly for hyponormal ,subnormal ,normaloid ,$\theta-operator$ and U-operator [4-],[5],[8],[9] it was shown in [15]that paranormal is not invariant under tensor product .

The operator A is said to be strongly stable if $\|A^n x\| \to 0$ as $n \to \infty$ for all $x \in H$ [4].

In this paper we prove there exist another operators' properties that are invariant under tensor product like posinormal, pseudo normal, binormal, $\theta-adjo$int operators.

2-We recall $A \in B(H)$ is called posinormal operator if there exists an operator $P \in B(H)$ such that $AA^* = A^*PA$. P is called an interrupter of A. The set of all posinormal operators on H is denoted by $P(H)$. A is called coposinormal if A^* is posinormal operator. [11]

***Theorem* [2-1]**

Let $A_i \in B(H_i)$, $i=1, 2, \dots, n$ and $A_1 \otimes A_2 \otimes \cdots \otimes A_n \neq 0$ on the Hilbert space $H_1 \otimes H_2 \otimes \cdots \otimes H_n$, $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is a posinormal operator if and only if A_i , $i=1,2,\cdots,n$, are all posinormal operators.

Proof :

By induction, it suffices to show that if $A_1 \otimes A_2$ is posinormal if and only if both A_1 and A_2 are.

Let $A_1 \otimes A_2 \neq 0$ is posinormal operator, then:-

$$(A_1 \otimes A_2)(A_1 \otimes A_2)^* = (A_1 \otimes A_2)^*(P_1 \otimes P_2)(A_1 \otimes A_2)$$

$$A_1A_1^* \otimes A_2A_2^* - A_1^*P_1A_1 \otimes A_2^*P_2A_2 = 0$$

If $A_1A_1^*$ and $A_1^*P_1A_1$ are linearly independent then $A_2A_2^* = A_2^*P_2A_2 = 0$ then $A_2 = 0$ contradiction

Hence $A_1A_1^*$ and $A_1^*P_1A_1$ are linearly dependent then $A_1A_1^* = rA_1^*P_1A_1$.

If $A_2A_2^*$ and $A_2^*P_2A_2$ are linearly independent $A_1A_1^* = A_1^*P_1A_1 = 0$ then $A_1 = 0$ contradiction Hence $A_2A_2^*$ and $A_2^*P_2A_2$ are linearly dependent then $A_2A_2^* = r^{-1}A_2^*P_2A_2$.

Now to prove $r = 1$

$$\|A_1\|^2 = \|A_1A_1^*\| = |r|\|A_1^*P_1A_1\| \leq |r|\|A_1^*\|\|A\ \| .$$
$$= |r|\|A_1\|\|A_1\| = |r|\|A_1\|^2$$

hence $1 \leq |r|$ and

$$\|A_2\|^2 = \|A_2 A_2^*\| = |r^{-1}| \|A_2^* P_2 A_2\| \le |r| \|A_2^*\| \|A_2\|$$
$$= |r^{-1}| \|A_2\| \|A_2\| = |r^{-1}| \|A_2\|^2$$

hence $1 \le |r^{-1}|$ it implies that $r = 1$

$\Leftarrow$) t if A_1 and A_2 are posinormal operators then $A_1 A_1^* = A_1^* P_1 A_1$ and $A_2 A_2^* = A_2^* P_2 A_2$

$$(A_1 \otimes A_2)(A_1 \otimes A_2)^* = A_1 A_1^* \otimes A_2 A_2^*$$
$$= A_1^* P_1 A_1 \otimes A_2^* P_2 A_2$$
$$= (A_1 \otimes A_2)^* (P_1 \otimes P_2)(A_1 \otimes A_2)$$

then $A_1 \otimes A_2$ is posinormal.

An operator A on a Hilbert space H is said to be binormal operator if $A^* A$ commutes with AA^*. I.e. $[A^* A, AA^*] = 0$. We denote the class of binormal operators by (BN). [2], [6]

***Theorem* [2-2]**

Let $A_i \in B(H_i), i = 1, 2, \ldots, n$ and $A_1 \otimes A_2 \otimes \cdots \otimes A_n \neq 0$ on Hilbert space $H_1 \otimes H_2 \otimes \cdots \otimes H_n$, $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is binormal operator if and only if each A_i $i = 1, 2, \cdots, n$, is binormal operator.

Proof :

By induction, it suffices to show that if $A_1 \otimes A_2$ is binormal if and only if both A_1 and A_2 are.

$\Leftarrow$) Suppose that $A_1 \otimes A_2 \neq 0$ is binormal operator, then

$$(A_1 \otimes A_2)^* (A_1 \otimes A_2)^2 (A_1 \otimes A_2)^* = (A_1 \otimes A_2)(A_1 \otimes A_2)^{*2} (A_1 \otimes A_2)$$
$$A_1^* A_1^2 A_1^* \otimes A_2^* A_2^2 A_2^* - A_1 A_1^{*2} A_1 \otimes A_2 A_2^{*2} A_2 = 0$$

If $A_1^* A_1^2 A_1^*$ and $A_1 A_1^{*2} A_1$ are linearly independent then $A_2^* A_2^2 A_2^* = A_2 A_2^{*2} A_2 = 0$ then $A_2 = 0$ contradiction. Hence $A_1^* A_1^2 A_1^*$ and $A_1 A_1^{*2} A_1$ are linearly dependent then $A_1^* A_1^2 A_1^* = r A_1 A_1^{*2} A_1$

Similarly If $A_2^* A_2^2 A_2^*$ and $A_2 A_2^{*2} A_2$ are linearly independent then $A_1^* A_1^2 A_1^* = A_1 A_1^{*2} A_1 = 0$ hence $A_1 = 0$ contradiction. Hence $A_2^* A_2^2 A_2^*$ and $A_2 A_2^{*2} A_2$ are linearly dependent then $A_2^* A_2^2 A_2^* = r^{-1} A_2 A_2^{*2} A_2$

Now it must be proving that $r = 1$.

$$\|A_1^* A_1^2 A_1^*\| = \|r(A_1 A_1^{*2} A_1)\|$$
$$= |r| \|A_1 A_1^{*2} A_1\|$$
$$= |r| \|(A_1^* A_1^2 A_1^*)^*\|$$

Hence $|r| = 1$

Similarly

$$\|A_2^* A_2^2 A_2^*\| = \|r^{-1}(A_2 A_2^{*2} A_2)\|$$
$$= |r^{-1}| \|A_2 A_2^{*2} A_2\|$$
$$= |r^{-1}| \|(A_2^* A_2^2 A_2^*)^*\|$$

Hence $|r^{-1}| = 1$

implies that $r = 1$ hence A_1 and A_2 are binormal operators.

$\Rightarrow$) on the other hand it is easy to see that if A_1 and A_2 are binormal operators then $A_1 \otimes A_2$ is binormal operator.

We recall $A \in B(H)$. A is called a pseudonormal operator if $Ax = \lambda x$ for some $x \in H, \lambda \in$ ¢ , then $A^* x = \bar{\lambda} x$,i.e. .if x is an eigenvector for A with eigenvalue λ then x is an eigenvector for A^* with eigenvalue $\bar{\lambda}$. [10]

It is clear that if $\sigma_p(A) = \emptyset$ then A is a pseudonormal operator.

***Theorem* [2-3]**

Let $A_i \in B(H_i), i = 1, 2, \ldots, n$ and $A_1 \otimes A_2 \otimes \cdots \otimes A_n \neq 0$ on $H_1 \otimes H_2 \otimes \cdots \otimes H_n$, $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is pseudonormal operator if and only if each A_i, $i = 1, 2, \cdots, n$, is pseudonormal operator.

Proof :

By induction, it suffices to show that $A_1 \otimes A_2 \neq 0$ is pseudo normal if and only if both A_1 and A_2 are.

Let $A_1 \otimes A_2 \neq 0$ be pseudo normal operator

Let $A_1 x = \lambda x$ and $A_2 y = \mu y$

If $(A_1 \otimes A_2)(x \otimes y) = \lambda\mu\,(x \otimes y)$ ……..(2-1)
Then $(A_1 \otimes A_2)^* \ (x \otimes y) \ = \overline{\lambda\mu}\ (x \otimes y)$ …(2-2)

In (2-1) if $A_1 x, \lambda x$ are linearly independent then $A_2 y = \mu y = 0$ contradiction hence $A_1 x, \lambda x$ are linearly dependent $A_1 x = t\lambda x$ but $A_1 x = \lambda x$ then $t = 1$

In (2-1) $A_2 y, \mu y$ are linearly independent then $A_1 x = \lambda x = 0$ contradiction Hence $A_2 y, \mu y$ are linearly independent then $A_2 y = t^{-1}\mu y$ but $A_2 y = \mu y$ then $t^{-1} = 1$

In (2-2)if $A_1^* x$ and $\overline{\lambda} x$ are linearly independent then $A_2^* y = \overline{\mu} y = 0$ then $\mu = 0$ contradiction.

Hence $A_1^* x$ and $\overline{\lambda} x$ are linearly dependent $A_1^* x = r\overline{\lambda} x$

Similarly

If $A_2^* y$ and $\overline{\mu} y$ are linearly independent then $A_1^* x = \overline{\lambda} x = 0$ then $\lambda = 0$ contradiction. Hence $A_2^* y$ and $\overline{\mu} y$ are linearly dependent $A_2^* y = r^{-1}\overline{\mu} y$

It is clear $t = r = 1$

$\Rightarrow$) It is clear that If A_1 and A_2 are pseudo normal operators then $A_1 \otimes A_2$ is pseudo normal operator.

3-An operator $A \in B(H)$ is called an $*-paranormal$ operator, if $\|A^* x\|^2 \le \|A^2 x\|$ for every unit vector $x \in H$. [13]

Proposition [3-1]

If A_1 and A_2 are $*-paranormal$ operators then so is $A_1 \otimes A_2$.

Proof :

Since A_1 and A_2 are $*-paranormal$ operators ,then

$$\|A_1^2 x\| \ge \|A_1^* x\|^2 \text{ and } \|A_2^2 y\| \ge \|A_2^* y\|^2$$

$$\|A_{1x}^2\|\|A_2^2 y\| \ge \|A_1^* x\|^2 \|A_2^* y\|^2$$

$$\|(A_1 \otimes A_2)^2 (x \otimes y)\| \quad \ge \|(A_1 \otimes A_2)^* (x \otimes y)\|^2$$

Hence $A_1 \otimes A_2$ is $*-paranormal$.

Remark [3-2]

If $A_1 \otimes A_2$ is $*-paranormal$ operator, then it may not be true that A_1 and A_2 are $*-paranormal$ operators, for example:-

Let $H = \ell_2(¢)$, $A: H \to H$ define as follows:- $A_1(x_1, x_2, \ldots) = (0,0,x_2,0,0,\ldots)$

It is easily checked that

$A_1^*(x_1, x_2, \ldots) = (0, x_3, 0, 0, \ldots)$

Let $x = (x_1, 0, x_3, x_4, \ldots,)$ then $A_1 x = (0,0,0\ldots)$

$A_1^2 x = 0$ and $\|A_1^2 x\| = 0$

Also $A_1^* x = (0, x_3, 0, \ldots)$ and $\|A_1^* x\| = |x_3|^2$

It is clear that A_1 is not $*-paranormal$ operator

Next, Let A_2 be the unilateral shift operator defined on the Hilbert space $\ell_2(¢)$ by

$A_2(y_1, y_2, \ldots) = (0, y_1, y_2, \ldots)$. Recall that $A_2^*(y_1, y_2, \ldots) = (y_2, y_3, \ldots)$.Let $y = (y_1, 0, 0, \ldots)$ one can easily check that

$$\|A_2^* y\|^2 = \|A_2^*(y_1, 0, \ldots)\|^2 = |0|^2 + |0|^2 + \ldots = 0$$

And

$$\|A_2^2 y\| = \|A_2(A_2 y)\| = \|A_2(0, y_1, o, \ldots)\|$$
$$= \sqrt{|o|^2 + |y_1|^2 + |0|^2 \ldots}$$
$$= |y_1| = \|y\|.$$

Suppose y is unit vector, i.e. $\|y\| = 1$, Thus A_2 is $*-paranormal$ operator

Moreover

$$\|A_1^2 \otimes A_2^2\ (x \otimes y)\| = \|A_1^2 x\|\|A_2^2 y\| = 0 \cdot |x_1|^2 = 0$$

and

$$\|A_1^* \otimes A_2^*\ (x \otimes y)\| = \|A_1^* x\|\|A_2^* y\| = |x_3|^2 \cdot 0 = 0$$

Hence $A_1 \otimes A_2$ is $*-paranormal$ operator.

Recall that an operator $A \in B(H)$ is $\theta-adjo$int if $A^* = e^{i\theta} A$ where $\theta \in \Re$ [7]

Theorem [3-3]

Let $A_i \in B(H_i)$, $i = 1, 2, \ldots, n$ and $A_1 \otimes A_2 \otimes \cdots \otimes A_n \ne 0$ on $H_1 \otimes H_2 \otimes \cdots \otimes H_n$ if each A_i, $i = 1, 2, \cdots, n$,is $\theta-adjo$int operator then $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is $\theta-adjo$int operator.

Proof:

By induction, it suffices to show that $A_1 \otimes A_2 \neq 0$ is $\theta - adjo$int if A_1 and A_2 are $\theta - adjo$int operators.

If $A_1^* = e^{i\theta_1} A_1$ and $A_2^* = e^{i\theta_2} A_2$ then $A_1^* \otimes A_2^* = e^{i\theta_1} A_1 \otimes e^{i\theta_2} A_2$ hence $(A_1 \otimes A_2)$ is $\theta_1 + \theta_2 - adjo$int operator.

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الملخص

نبرهن في هذا البحث عن وجود بعض المؤثرات التي تحافظ على خواصها في الجداء التنسوري مثل المؤثر الموجب السويه والمؤثر ثنائي السويه والمؤثر الشبه سوي والمؤثر فوق السوي من النمط *