Strongly C-Compactness

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Abstract

In this paper, we define another type of compactness which is called "strongly c-compactness ". Also, we study some properties of this type of compactness and the relationships with compactness, strongly compactness and c-compactness.

1. Introduction and Preliminaries

A topological space (X,τ) is said to be c-compact space if for each closed set $A \subseteq X$, each open cover of A contains a finite subfamily W such that {cl v: $v \in W$} covers A, [1].

Mashhour et.al.[2] introduced preopen sets, [A subset A of space X is said to be preopen set if $A \subseteq \text{int}(\text{cl}(A))$]. Obviously each open set in (X,τ) is preopen, not conversely. Also, they defined the following concepts:

Let A be a subset of a space X:

i. A is called a preclosed set iff $(X - A)$ is preopen set.

ii. The intersection of all preclosed sets contain A is called the preclosure of A and denoted by pre-clA

iii. The prederived set of A is the set of all elements *x* of X satisfies the condition, that for every preopen set V contains *x*, implies $V \setminus \{x\} \cap A \neq \varnothing$.

Also, they proved some properties, as (the preclosure of a set A is a preclosed set) and (preclosure (B) = B iff B is preclosed set). Pre-open sets are discussed in [3], [4].

Ganster [5] has shown that the family of all preopen sets in X (PO(X)) is a topology on X if closure G is open and $\{x\}$ is preopen for each $x \in$ interior F where $X = F \cup G$.

A space (X,τ) is called strongly compact if every preopen cover of (X,τ) admits a finite subcover.

Strongly compactness is defined in [6] and discussed in [7] and [8].

In this paper we shall introduce a new concept of compactness, which is called a "strongly c-compact space" where [A topological space X is said to be strongly c-compact space if for every preclosed set $A \subseteq X$, each family of preopen sets in X which covers A, there is a finite subfamily W such that {preclosure U :$U \in W$} covers A].

We discuss some properties of this kind of compactness and give some propositions, corollaries, remarks and examples to explain that. After investigating the relationships among compact spaces, c-compact spaces, strongly compact spaces and strongly c-compact spaces are considered.

<u>Proposition (1.1),[1]</u>:

Every compact space is c-compact.

<u>Remark (1.2):</u>

The implication in proposition (1.2) is not reversible, for example: A space $(\mathbf{N},\tau)$ where, $\tau = \{U_n = \{1,2,\ldots,n\} \mid n \in \mathbf{N}\} \cup \{\mathbf{N},\varnothing\}$ is c-compact which is not compact.

<u>Definition (1.3),[9]:</u>

A topological space (X,τ) is said to be a T_3-space iff it is regular and T_1- space.

<u>Proposition (1.4),[1]</u>:

A T_3-c-compact space is compact.

<u>Proposition (1.5), [6], [7],[8]:</u>

Every strongly compact space is compact.

<u>Remark (1.6):</u>

The opposite direction of proposition (1.5) may be false, for example:

Let $X = [0,1]$ as a subspace of $(\mathbf{R},\tau_u)$. Clearly, X is compact, but not strongly compact space, since the preopen cover $C = \{[0,\frac{1}{2})\setminus\{\frac{1}{n}: n \in \mathbf{N}\} \cup \{(\frac{1}{3},1]\} \cup \{(\frac{1}{n}-r_n, \frac{1}{n}+r_n) \mid r_n = \frac{1}{2(n+1)^2} \wedge n > 2\}$ has no finite subcover.

Proposition (1.7),[7], [8]:

If the set of accumulation points of X is finite, then X is strongly compact space, whenever it is compact space.

In proposition (1.8) and remark (1.9) below we discuss the relationship between strongly and c-compact spaces.

Proposition (1.8):

Every strongly compact space is c-compact.

Proof:

Follows directly from propositions (1.5) and (1.1). ■

Remark (1.9):

The opposite direction of proposition (1.8) may be false, see the example in remark (1.2), $(\mathbf{N},\tau)$ is c-compact space which is not strongly compact, since $\{\{1,n\} \mid n \in \mathbf{N}\}$ is a preopen cover for $\mathbf{N}$ which has no finite subcover.

In the following proposition we give some conditions to make the opposite direction of proposition (1.8) true.

Proposition (1.10):

A T_3-c-compact space X is strongly compact, whenever the set of accumulation points of X is finite.

Proof:

Follows directly from propositions (1.4) and (1.7). ■

2. Strongly c-compactness:

In this section we shall introduce the concept of strongly c-compactness and the relationships among compact, c-compact, strongly compact and strongly c-compact spaces are examined.

Definition (2.1):

A topological space X is said to be "strongly c-compact space" if for each preclosed set $A \subseteq X$, each family $\{V_\alpha:\alpha\in\wedge\}$ of preopen sets in X and covering A there is a finite subfamily W such that $\{\text{pre-cl } V_\alpha : V_\alpha \in W\}$ covers A.

Proposition (2.2):

A strongly compact space is strongly c-compact.

Proof:

Clear. ■

Remark (2.3):

The opposite direction of proposition (2.2) need not be true, see the example of remark (1.2), $(\mathbf{N},\tau)$ is strongly c-compact which is not strongly compact.

Proposition (2.4):

A T_3-strongly c-compact space is strongly compact.

Proof:

Let X be a T_3-strongly c-compact space. If X is not strongly compact, then there is a preopen cover $\{u_\alpha:\alpha\in\wedge\}$ for X which has no finite subcover. Since X is strongly c-compact space, then there is afinite subfamily W of the preopen cover $\{u_\alpha:\alpha\in\wedge\}$ such that $X = \bigcup_{i=1}^{n}\{\text{pre-cl}\, u_{\alpha_i} \mid u_{\alpha_i} \in W\}$. This means, there is $x \in X$, $x \in \text{pre-cl}\, u_{\alpha_i}$ but $x \notin u_{\alpha_i}$ for some $i = 1,2,\ldots,n$. Implies $x \in \text{pre-derived}\, u_{\alpha_i}$. Since X is T_1-space, then $\{x\}$ is a closed set and $x \notin u_{\alpha_i}$, implies $y \notin \{x\}$ $\forall\, y \in u_{\alpha_i}$. Since X is regular, then there are two open sets V_y and V'_y such that $y \in V_y$ and $\{x\} \subseteq V'_y$ and $V_y \cap V'_y = \varnothing$ for each $y \in u_{\alpha_i}$.

Therefore $\{V'_y\}_{y\in u_{\alpha_i}}$ is an open cover for $\{x\}$. But $\{x\}$ is compact, then there is $\{V'_{y_1}, V'_{y_2}, \ldots, V'_{y_n}\}$ covers $\{x\}$.

Let $V' = \bigcap_{i=1}^{n} V'_{y_i}$, then V' is an open set contains x. Let $V = \bigcup_{y\in u_{\alpha_i}} V_y$, then V is an open set contains u_{α_i}, and $V \cap V' = \varnothing$. Since,every open set is a preopen, then V and V' are preopen sets and $x \in V'$, $u_{\alpha_i} \subseteq V$ and $V \cap V' = \varnothing$. Therefore, $x \notin \text{pre-derived}\, u_{\alpha_i}$ which is a contradiction. Then X is a strongly compact space. ■

Corollary (2.5):

A T_3-strongly c-compact space is compact.

Proof:

Follows from propositions (2.4) and (1.5). ■

Remark (2.6):

In general a strongly c-compact space need not be compact, see the example of remark (1.2), (**N**,τ) is strongly c-compact space which is not compact.

On the other hand, a compact space may not be strongly c-compact, for example: The compact space (**N**,τ_I), where τ_I is the indiscrete topology on **N** is not strongly c-compact since $\{\{n\} | n\in \mathbf{N}\}$ preopen cover for **N**, which has no finite subfamily W such that $\{\text{pre-cl } u | u \in W\}$ covers **N**, since pre-cl$\{n\} = \{n\}$ $\forall$ n$\in$ **N**.

In the following proposition we add a condition to make any compact space strongly c-compact space

Proposition (2.7):

If the set of accumulation points of X is finite, X is strongly c-compact space whenever it is a compact space.

Proof:

Follows from propositions (1.7) and (2.2). ■

Proposition (2.8):

A strongly c-compact space is c-compact.

Proof:

Let X be a strongly c-compact space, to prove it is c-compact. If not, then there is a closed set A $\subseteq$ X and an open cover $\{u_\alpha:\alpha\in\wedge\}$ for A, such that $A \neq \bigcup_{i=1}^{n} \text{cl}\, u_{\alpha_i}$ $\forall$ n$\in$ **N**. Since, every open set is preopen, then $\{u_\alpha:\alpha\in\wedge\}$ is a preopen cover for A, then there is a finite subfamily $\{u_{\alpha_i}: i = 1,2,\ldots,m\}$ such that $\{\text{pre-cl}\, u_{\alpha_i} : i=1,2,\ldots,m\}$ covers A.

This means, there exists $x \in A$ such that $x \in \text{pre-cl}\, u_{\alpha_i}$ and $x \notin \text{cl}\, u_{\alpha_i}$ for some i=1,2,…,m.

Since $x \notin \text{cl}\, u_{\alpha_i}$, implies $x \notin u_{\alpha_i}$, but $x \in \text{pre-cl}\, u_{\alpha_i}$ then $x \in \text{pre-derived}\, u_{\alpha_i}$.

On the other hand, since $x \notin \text{cl}\, u_{\alpha_i}$, implies $x \notin u_{\alpha_i}$ and $x \notin \text{derived}\, u_{\alpha_i}$. Therefore, there exists an open set V such that $x \in V$ and $V \cap u_{\alpha_i} = \varnothing$.

Now, we get a preopen set V such that $x \in V$ and $V \cap u_{\alpha_i} = \varnothing$, implies $x \notin \text{pre-derived}\, u_{\alpha_i}$ which is a contradiction.

Therefore X is c-compact whenever it is strongly c-compact space. ■

Remark (2.9):

A c-compact space need not be strongly c-compact. As the space (**N**,τ_I).

In the following proposition we add some conditions to make c-compact space to be strongly c-compact.

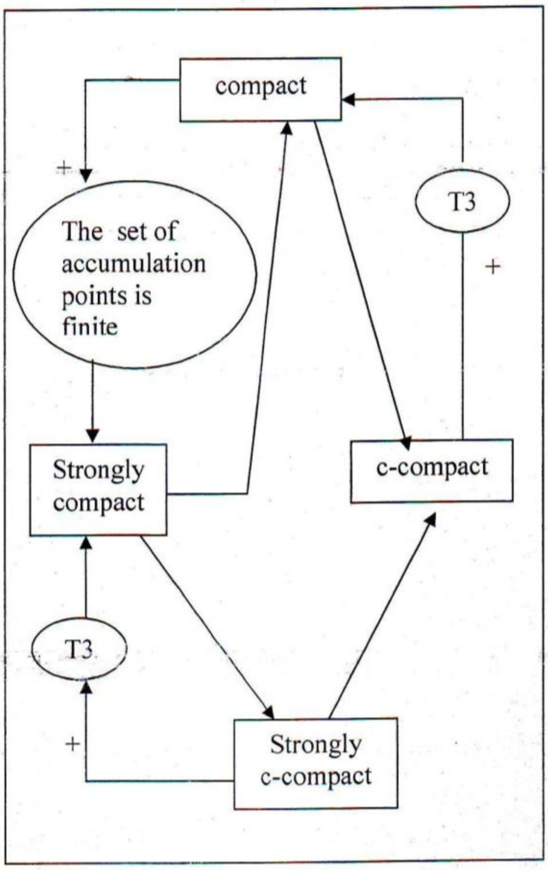
Proposition (2.10):

In a T_3-space (X,τ), if the set of accumulation points of X is finite, then the concepts of c-compactness and strongly c-compactness are concident.

Proof:

Follows from propositions (1.4) and (2.7). ■

The following diagram shows the relationships among the different types of compactness we studied in this section.



3. Certain Fundamental Properties of Strongly c-Compact Space

In this section we shall discuss some properties of strongly c-compact spaces.

Remark (3.1):

Strongly c-compactness is not a hereditary property, as the following example shows;

Let X = **N** ∪ {0},

τ = P(**N**) ∪ {H⊆X | 0∈H∧X – H is finite}.

Now, X is a strongly compact space, implies X is strongly c-compact space (by proposition (2.2)). But, **N** ⊆ X not strongly c-compact since {{n} | n∈ **N** } is a preopen cover for **N** which has no finite subfamily W such that {pre-cl{n}: {n}∈W} cover **N**.

Remark (3.2):

The continuous image of a strongly c-compact space need not be strongly c-compact. For example:

Let f : (**N**,τ) ⟶ (**N**,τ_I) such that $f(x) = x$ ∀ x ∈ **N** where τ = {U_n | U_n ={1,2,…,n} | n∈ **N** } ∪ {∅, **N** }. Then, f is a continuous function and (**N**,τ) is strongly c-compact space, but (**N**,τ_I) is not strongly c-compact.

Definition (3.3), [10]:

Let f :(X,τ) ⟶ (Y,τ′) be any function, f is said to be a preirresolute function, if and only if the inverse image of any preopen set in Y is a preopen set in X.

Remark (3.4) [10]:

A function f :(X,τ) ⟶ (Y,τ′) is a preirresolute iff the inverse image of any preclosed set in Y is a preclosed set in X.

Lemma (3.5):

A function f :(X,τ) ⟶ (Y,τ′) is a preirresolute if and only if pre cl(f^{-1}(B)) ⊆ f^{1} (pre cl((B)) ∀ B ⊆ Y.

Proof:

Necessity, let f :(X,τ) ⟶ (Y,τ′) be a preirresolute function and let B ⊆ Y.

Since B ⊆ pre cl(B) then f^{-1}(B) ⊆ f^{-1}(pre cl(B)), implies pre cl(f^{-1}(B) ⊆ pre cl(f^{-1}(pre cl((B))). Since f is preirresolute function and pre cl (B) is preclosed set in Y, then f^{-1}(pre cl((B)) is preclosed set in X. So, pre cl (f^{-1}(pre cl((B))) = f^{-1}(pre cl((B)).

Therefore, pre cl(f^{-1} (B)) ⊆ f^{-1}(pre cl((B)).

Sufficiency, suppose pre cl(f^{-1} (B)) ⊆ f^{-1} (pre cl((B)) ∀ B ⊆ Y. To prove f is preirresolute function.

We must prove that if **A** is preclosed set in Y, then f^{-1} (A) is preclosed set in X.

Which means : we must prove that f^{-1} (A) = pre cl (f^{-1} (A)). It is clear that f^{-1} (A) ⊆ pre cl (f^{-1} (A)) ∀ A ⊆ Y.

Now, to prove pre cl (f^{-1} (A)) ⊆ f^{-1} (A). Since A is preclosed set in Y, then pre cl(A) = A and since pre cl (f^{-1} (A)) ⊆ f^{-1}(pre cl((A)). Implies, pre cl (f^{-1} (A)) ⊆ f^{-1} (A).

Therefore, pre cl (f^{-1} (A)) = f^{-1} (A) and f^{-1} (A) is a preclosed set in X. So f is preirresolute function. ■

Prposition (3.6):

The preirresolute image of a strongly c-compact space is a strongly c-compact.

Proof:

Let f :(X,τ) ⟶ (Y,τ′) be a preirresolute onto function and let X be a strongly c-compact space. To prove Y is strongly c-compact space.

Let A be a preclosed subset of Y, {u_α: α∈∧} be a τ′-preopen cover for A. Since f is a preirresolute function, implies {f^{-1}(u_α): α∈∧} is a τ-preopen cover for a preclosed set f^{-1} (A) ⊆ X and since X is strongly c-compact space, then there is a finite family {$u_{\alpha_1}, u_{\alpha_2}, ..., u_{\alpha_n}$} such that {pre cl($f^{-1}$($u_{\alpha_i}$)):i = 1,2,…,n} covers f^{-1} (A). So {f (pre-cl(f^{-1}(u_{α_i}))): i = 1,2,…,n} covers A. In virtue of lemma (3.5), {f (f^{-1} (pre-cl(u_{α_i}))):i = 1,2,…,n} covers A and since f is onto, then { pre-cl((u_{α_i}):i = 1,2,…,n} covers A. Hence, Y is strongly c-compact space. ■

Proposition (3.7), [10]:

Every homeomorphism function is a preirresolute function.

Corollary (3.8):

A strongly c-compactness is a topological property.

Proof:

In virtue of proposition (3.7), then proposition (3.6) is applicable. ■

4. Conclusion and Recommendations:

Our conclusions in this paper, that a strongly c-compact space is c-compact space but not strongly compact space and not compact space. So we have to strive to put another type of compactness which lies between strongly compactness and c-compactness.

For future works, we shall study α-c-compactness, semi-α-c-compactness, semi-p-compactness and semi-p-c-compactness.

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الخلاصة

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