Eigenfunction Expansions Method for the Integro-Differential Equations of the First Kind

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Abstract

The main purpose of this paper is to present a new numerical method for solving the integro-differential equation of the first kind, namely the eigenfunction expansions method. This method is discussed for a Hermitian and Non-Hermitian kernels and is required that the eigenfunctions form a complete set.

Introduction

Consider the integro differential equation (IDE) of the first kind:

$$\int k(s,t)x'(t)dt = y(s) \qquad (1)$$

where y is a known function of s, k(s,t) is the kernel of the IDE (1) which is assumed to be known.

The problem here is to determine the unknown function x.

In [4], the eigenfunction expansions method is presented to solve the delay integral equation of the first kind. Here, we use this method to solve the first order ordinary integro-differential equation of the first kind.

Eigenfunction Expansions

This method is described for two cases on the kernel associated with the IDE (1).

Hermitian Kernels

Write the DIE (1) in operator form, that is:

$Kx(t)=y(s)$

where $K= \int k(s,t)\frac{d}{dt}\,dt$

Suppose first the kernel k(s,t) is Hermitian, it has real eigenvalues say λ_i with the corresponding eigenfunctions V_i satisfying

$$\int k(s,t)\frac{dv_i(t)}{dt}dt = \lambda_i v_i(s)$$

or in operator form: $Kv_i = \lambda v_i$

Now, if the set of eigenfunction $\{v_i\}$ form a complete set then any function and in particular the solution x and the function y has an expansion of the form:

$$x(t) = \sum_{i}^{\infty} c_i v_i(t) \qquad (2)$$

and

$$y(s) = \sum_{i}^{\infty} \alpha_i v_i(s) \qquad (3)$$

Substitute eq(2) into eq.(1), we obtain:

$Kx(t)=$

$$\int k(s,t)x'(t)dt = \int k(s,t)\sum_{i} c_i \frac{dv_i(t)}{dt}dt = \sum_{i=1}^{\infty} c_i \int k(s,t)\frac{dv_i(t)}{dt}dt = \sum_{i=1}^{\infty} c_i K v_i = \sum_{i} c_i \lambda_i v_i$$

But $Kx(t) = y(s) = \sum_{i=1}^{\infty} \alpha_i v_i(s)$. Thus

$\sum_{i}^{\infty} \alpha_i v_i(s) = \sum_{i}^{\infty} c_i \lambda_i v_i(s)$ and hence s

Also,

$(y, v_j) = (\sum_{i}^{\infty} \alpha_i v_i, v_j) = \alpha_j$

Thus $x = \sum_{i=1}^{\infty} (y, v_i)\lambda_i^{-1} v_i(t)$ is the solution of the IDE(1)

Note that, we can find the eigenvalues λ_i of the operator K with the corresponding eigenfunction V_i by any suitable method, say expansion method. [2]

Non-Hermitian kernels

If k is not Hermitian then the eigenvalues may not be real and the eigenfunction are not orthogonal in general, [1], [3]. But K^*K is Hermitian, where K^* is the adjoint of the operator K. Thus K^*K has real eigenvalues $\hat{\lambda}_i$ with the corresponding eigenfunction V_i satisfying:

$$K^*K\, V_i = \hat{\lambda}_i V_i \qquad (4)$$

Also, if the set of eigenfunction $\{v_i\}$ form a complete set then any function and in particular the solution x and the function K^*y has the form:

$$x(t) = \sum_{i}^{\infty} c_i v_i(t) \qquad (5)$$

and

$$K^*y(s) = \sum_{i=1}^{\infty} \alpha_i v_i(s) \qquad (6)$$

Now,

$$K^*K\, x(t) = K^*y(s) \qquad (7)$$

Subsitute eq.(5) into eq.(7), we obtain

$$K^*K\,x(t) = K^*K \sum_{i=1}^{\infty} c_i v_i = \sum_{i} c_i K^*K v_i = \sum_{i=1} c_i \lambda_i v_i$$

$$= K^* y(s) = \sum_{i=1} c_i \lambda_i v_i = \sum_{i=1}^{\infty} \alpha_i v_i(s)$$

Thus, $c_i = \dfrac{\alpha_i}{\lambda_i}$.

Also,

$$(K^* y, v_i) = (\sum_{i} \alpha_i v_i, v_i) = \alpha_i$$

Thus $x(t) = \sum_{i=1}^{\infty} (K^* y, v_i)\lambda_i^{-1} v_i(t)$ is the solution of the IDE(1).

Also, note that, we can find the eigenvalues λ_i of the operator K^*K with the corresponding eigenfunction V_i by any suitable method, say expansion method.

Remarks

1. To the best of our knowledge, this method especially for non-Hermitian kernels seems to be new.
2. This method can also be applied for a Fredholm linear integral equation of the first kind (for Hermitian kernels). For more details see [2].
3. It is easy to check that this method can also be applies a Fredholm linear integral equation of the first kind (for non-Hermitian kernels).

References

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الخلاصة

ان الغرض الرئيسي من هذا البحث هو تقديم طريقة جديدة لحل المعادلات التفاضلية التكاملية وهي طريقة توسيع المتجهات الذاتية، هذه الطريقة تتجنب ان تكون المتجهات الذاتية عبارة عن مجموعة تامة، وكذلك نوقشت هذه الطريقة في حالة النواة متناظرة وغير متناظرة.