On the Solution of Variational Problems by Walsh Functions

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Abstract

This paper gives a clear procedure for solving the variational problems via Walsh functions. Many types of variational problems are solved by this approach like the simplest variational problem, functional depending on many dependent variables and functional involving derivatives of higher order. An illustrative examples are given for each type of these problems.

Introduction

Variational problems of differential equations can be integrated easily only in exceptional cases. It is therefore essential to find other methods for solving these problems, these methods are called direct methods. The basic idea of the direct methods for solving variational problems is to convert the problem of extremization of a functional into one which involves a finite number of variables.

There are many direct methods for solving the variational problems for example, the direct method of finite differences, Ritz method and Kantorovic method [Elsgolc L., 1962].

(1) A Walsh Direct Method for Solving Variational Problems:

The Walsh direct method for solving variational problems can be described by the following steps:

1- Expand the unknown function by Walsh functions whose coefficients are to be determined.
2- Establishing an operational matrix for performing integration.
3- Finding the necessary condition for extremization.
4- Solving the algebraic equations obtained from the previous steps to evaluate Walsh coefficients.

Because the orthonormal property of powerful Walsh functions, this direct method is simpler in reasoning as well as in calculation.

Next, we use the Walsh direct method to solve the simplest variational problem.

(2) The Simplest Variational Problem:

The regular method for solving the extremization problem of a functional:

$$J = \int_0^1 F(t, y(t), y'(t))dt \qquad (1)$$

is through the Euler equation

$$F_y - \frac{d}{dt}F_{y'} = 0 \qquad (2)$$

However, the differential equation so obtained can be integrated easily only in exceptional case. Therefore, many direct methods have been developed. Ritz's and Kantorovic's method as well known [Elsgolc, 1962]. Here one can use the Walsh's functions to establish direct method unlike other direct methods, beginning with assumption of the unknown variable itself, this method start with the rate variable. In other words, first assume the rate variable $y'(t)$ as Walsh series whose coefficients are to be determined

$$y'(t) = \sum_{i=0}^{\infty} c_i w_i(t) \qquad (3)$$

Taking finite terms as an approximation, we have

$$y'(t) = c_0 w_0(t) + c_1 w_1(t) + c_2 w_2(t) + \dots + c_{m-1} w_{m-1}(t) = C^T W(t) \qquad (4)$$

where $W=[w_0, w_1, \dots, w_{m-1}]$ and $C=[c_0, c_1, c_2, \dots, c_{m-1}]$

By [3], it is known that

$$\int_0^t W(x)\,dx = P\,W(t)$$

where P is defined in [3]

Then the function $y(t)$ can be expressed as:

$$y(t) = \int_0^t y'(x)\,dx + y(0) = C^T P W(t) + y(0) \qquad (5)$$

The other terms in the functional of eq.(1) are known functions of the independent variable t and can be expanded into Walsh series with known coefficients. By substituting $y(t)$, $y'(t)$ and t into the functional given by eq.(1), one can finally get:

$$J = J(c_0, c_1, \dots, c_{m-1}) \qquad (6)$$

The original extremization of functional problem shown in eq.(1) becomes the extremization of the function of finite set of variables in eq.(6).

Taking partial derivatives of J with respect to c_i and setting them equal to zero, that is

$$\frac{\partial J}{\partial c_i} = 0 \quad (i = 0, 1, \dots, m-1).$$

One can have a system of equations which can be solved for c_i, and by substituting the values of c_i into eq.(5), the solution of the above simplest variational problem becomes known.

Let us establish the detailed procedure via this illustrative example

Example (1):

Consider

$$J = \int_0^1 (y'^2 + ty')\, dt$$

with the boundary conditions

$$y(0)=0,\ y(1)=\frac{1}{4}$$

To solve this problem by the Walsh direct method, expand $y'(t)$ as a linear combination of the four Walsh functions yield

$$y'(t) = c_0 w_0(t) + c_1 w_1(t) + c_2 w_2(t) + c_3 w_3(t) \qquad (7)$$

That is

$$y'(t) = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = C^T W(t)$$

where $C = [c_0, c_1, c_2, c_3]$ and $W(t) = [w_0, w_1, w_2, w_3]$.

Also write

$$t \approx \tfrac{1}{2}w_0(t) - \tfrac{1}{4}w_1(t) - \tfrac{1}{8}w_2(t) + 0w_3 = h^T W(t) \qquad (8)$$

where $h = \left[\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, 0\right]$

Substituting eq.(7) and eq.(8) in the above variational problem, one can obtain

$$J = \int_0^1 [C^T W(t) W^T(t) C + C^T W(t) W^T(t) h]\, dt \qquad (9)$$

It is easy to check that eq.(9) can be written as

$$J = C^T C + C^T h$$
$$= c_0^2 + c_1^2 + c_2^2 + c_3^2 + \tfrac{1}{2}c_0 - \tfrac{1}{4}c_1 - \tfrac{1}{8}c_2 \qquad (10)$$

since $y(0)=0$, then the solution $y(t)$ is given by

$$y(t) = \int_0^t y'(x)\, dx + y(0) = C^T \int_0^1 W(t)\, dx + 0$$
$$\cong C^T P W(t)$$

but $y(1) = \frac{1}{4}$, that is

$$y(1) = C^T \int_0^1 W(t)\, dx$$

Therefore

$$\frac{1}{4} = C^T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_0$$

substituting $c_0 = \frac{1}{4}$ into eq.(10), the functional J becomes

$$J = \tfrac{3}{16} + c_1^2 + c_2^2 + c_3^2 - \tfrac{1}{4}c_1 - \tfrac{1}{8}c_2$$

For extremization, we take the partial derivatives of J with respect to c_i, i=1,2,3 and set them equal to zero:

$$\frac{\partial J}{\partial c_1} = 0 \Rightarrow 2c_1 - \tfrac{1}{4} = 0 \Rightarrow c_1 = \tfrac{1}{8}$$

$$\frac{\partial J}{\partial c_2} = 0 \Rightarrow 2c_2 - \tfrac{1}{8} = 0 \Rightarrow c_2 = \tfrac{1}{16}$$

$$\frac{\partial J}{\partial c_3} = 0 \Rightarrow 2c_3 = 0 \Rightarrow c_3 = 0$$

Therefore $y'(t) \approx \frac{1}{4}w_0(t) + \frac{1}{8}w_1(t) + \frac{1}{16}w_2(t)$

and Hence $y(t) = \frac{21}{128}w_0(t) - \frac{1}{16}w_1(t) - \frac{1}{32}w_2(t) - \frac{1}{128}w_3(t)$

Note that, the Euler equation is used for the analytic solution, the answer should be

$$y(t) = \tfrac{1}{2}t(1 - \tfrac{1}{2}t),$$

and

$$y'(t) = \tfrac{1}{2}(1 - t)$$

Now, we give the generalization of the Walsh series direct method to include a functional depend on many dependent variables.

(3) Functional Depending On Many Dependent Variables:

To find the extremal points of the functional

$$J = \int_{t_0}^{t_1} F(t, y_1(t), y_2(t), \ldots, y_n(t), y_1'(t), y_2'(t), \ldots, y_n'(t))\, dt \qquad (11)$$

one can use the Euler equation

$$F_{y_i} - \frac{d}{dt} F_{y_i'} = 0,\ i=1,2,\ldots,n \qquad (12)$$

The Walsh direct method starts with the rate variable of each one of its variables. therefore assume that the rate variable $y_i'(t)$, i=1,2,…,n can be expanded as Walsh series whose coefficients are to be determined

$$y_i'(t) = \sum_k c_k^i w^k(t) \qquad (13)$$

and taking finite terms as an approximation to get

$$y_i'(t) \approx c_{i0}w_0(t) + c_{i1}w_1(t) + c_{i2}w_2(t) + \cdots + c_{i,m-1}w_{m-1}(t) \quad (14)$$

also, as seen before

$$y_i(t) = \int_0^t y_i'(x)\,dx + y_i(0) = C^T PW(t) + y_i(0) \quad (15)$$

after substituting $y(t)$, $y'(t)$ and t into eq.(11) one can have:

$$J = J(c_{00}, c_{01}, \ldots, c_{0,m-1}, c_{10}, c_{11}, \ldots, c_{1m}, \ldots, c_{n0}, c_{n1}, \ldots, c_{n,m-1}) \quad (16)$$

Taking partial derivatives of J with respect to c_{ij}, and setting them equal to zero, that is

$$\frac{\partial J}{\partial c_{ij}} = 0 \quad (i=0,1,\ldots,n;\ j=0,1,\ldots,m-1).$$

And by solving the resulting system of equations for c_{ij}, then the solution of the variational problem given in eq.(11) is obtained

To illustrate this method see the following example.

Example (2):

Consider

$$J = \int_0^1 (y'^2 + z'^2 - z'y')\,dt$$

with the boundary conditions are

$y(0)=0$, $z(0)=0$

$y(1)=1$, $z(1)=1$

To solve this variational problem by the Walsh direct method, assume that

$$\begin{aligned} y'(t) &\approx c_0w_0(t) + c_1w_1(t) + c_2w_2(t) + c_3w_3(t) \\ z'(t) &\approx a_0w_0(t) + a_1w_1(t) + a_2w_2(t) + a_3w_3(t) \end{aligned} \quad (17)$$

Substituting these equations in the original problem, one can have

$$J = \int_0^1 [C^T W(t) W^T(t) C + A^T W(t) W^T(t) A + C^T W(t) W^T(t) A]\,dt \quad (18)$$

where $C=[c_0, c_1, c_2, c_3]$, $A=[a_0, a_1, a_2, a_3]$ and $W=[w_0, w_1, w_2, w_3]$.

Thus

$$J = C^TC + A^TA + C^TA$$
$$= c_0^2 + c_1^2 + c_2^2 + c_3^2 + a_0^2 + a_1^2 + a_2^2 + a_3^2 + c_0a_0 + c_1a_1 + c_2a_2 + c_3a_3 \quad (19)$$

Also, by using $y(0)=0$, the solution $y(t)$ reduces to

$$y(t) = \int_0^t y'(x)\,dx + y(0) \cong C^T PW(t)$$

But $y(1)=1$, therefore

$$y(1) = C^T \int_0^1 W(t)\,dx \Rightarrow 1 = C^T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_0$$

similarly for $z(t)$ one can find that $a_0=1$.

Subsituting $a_0=1$ and $c_0=1$ into eq.(19) to get

$$J = 3 + c_1^2 + c_2^2 + c_3^2 + a_1^2 + a_2^2 + a_3^2 + c_1a_1 + c_2a_2 + c_3a_3$$

For extremization, take the partial derivatives of J with respect to c_i, a_i $i=1,2,3$ then set them equal to zero

$$\frac{\partial J}{\partial c_1} = 0 \Rightarrow 2c_1 + a_1 = 0$$
$$\frac{\partial J}{\partial c_2} = 0 \Rightarrow 2c_2 + a_2 = 0$$
$$\frac{\partial J}{\partial c_3} = 0 \Rightarrow 2c_3 + a_3 = 0$$
$$\frac{\partial J}{\partial a_1} = 0 \Rightarrow 2a_1 + c_1 = 0$$
$$\frac{\partial J}{\partial a_2} = 0 \Rightarrow 2a_2 + c_2 = 0$$
$$\frac{\partial J}{\partial a_3} = 0 \Rightarrow 2a_3 + c_3 = 0$$

by solving the upper system of equations by using MathCad professional, one can have $c_i=a_i=0$, $i=1,2,3$. Therefore

$$y'(t) = w_0(t), \qquad z'(t) = w_0(t)$$

and the solutions $y(t)$ and $z(t)$ take the form

$$y(t) = z(t) = \tfrac{1}{2}w_0(t) - \tfrac{1}{4}w_1(t) - \tfrac{1}{8}w_2(t) + 0w_3(t)$$

Note that, if the Euler equation is used for the analytic solution, the answer should be

$$y(t) = z(t) = t,$$
$$y'(t) = z'(t) = 1.$$

Next, we give a generalization of the Walsh direct method to include the variational problems depending on one variable but with higher derivatives

(4) Functional Involving Derivatives of Higher Order:

As seen before, the extremal points of the functional

$$J=\int_0^1 F(t,y(t),y'(t),y''(t),\ldots,y^{(n)}(t))dt \qquad (30)$$

can be obtained from its Euler equation

$$F_y-\frac{d}{dt}F_{y'}+\frac{d^2}{dt^2}F_{y''}+\ldots+(-1)^n\frac{d^n}{dt^n}F_{y^{(n)}}=0 \qquad (31)$$

Here, we use the Walsh direct method to find the extremal points of the variational problem given in eq.(20)To do so, write the rate variable $y^{(n)}(t)$ as Walsh series whose coefficients are to be determined

$$y^{(n)}(t)=\sum_{i=0}^{\infty}c_i w_i(t) \qquad (32)$$

and similarly taking finite terms as an approximation,

$$y^{(n)}(t)\approx c_0w_0(t)+c_1w_1(t)+c_2w_2(t)+\cdots+c_{m-1}w_{m-1}(t)$$

Therefore, the solution y(t) can be expressed as:

$$y(t)=\int_0^t\int_0^t\cdots\int_0^t y^n(x)dxdx\ldots dx+y^{n-1}(0)\frac{t^{n-1}}{(n-1)!}+$$
$$y^{n-2}(0)\frac{t^{n-2}}{(n-2)!}+\ldots+y(0)$$
$$=C^TP^nW(t)+y^{n-1}(0)\frac{t^{n-1}}{(n-1)!}+ \qquad (33)$$
$$y^{n-2}(0)\frac{t^{n-2}}{(n-2)!}+\ldots+y(0)$$

By substituting $y(t), y'(t), y''(t),\ldots y^{n}(t)$ and t into eq.(20) then J becomes

$$J=J(c_0,c_1,\ldots,c_{m-1}) \qquad (24)$$

The original extremization of functional problem shown in eq.(20) becomes the extremization of the function of finite set of variables in eq.(24). By putting $\frac{\partial J}{\partial c_i}=0$ (i=0,1,…,m-1) and solving the resulting system, the values of c_i are obtained.

Example (3):

Consider

$$J=\int_0^1(1+y''^2)\,dt$$

with the boundary conditions

$$y(0)=0,\quad y(1)=1,$$
$$y'(0)=0 \qquad y'(1)=1$$

To solve this problem by the Walsh direct method, assume that For

$$y''(t)\approx c_0w_0(t)+c_1w_1(t)+c_2w_2(t)+c_3w_3(t) \qquad (25)$$

and after simple computations J takes the form

$$J=1+c_0^2+c_1^2+c_2^2+c_3^2 \qquad (36)$$

and by using y(0)=0 and y'(0)=0, the functions y(t) and y'(t) became

$$y'(t)=\int_0^t y''(x)\,dx+y'(0)=C^T\int_0^t w(x)\,dx+1$$
$$y(t)=\int_0^t y'(x)\,dx+y'(0)=C^T\int_0^t w(x)\,dx+1$$
$$0\cong C^TPW(t)+y'(0)t+y(0)$$
$$=C^TPW(t)+t+0$$

The solution y(t) and y'(t) must satisfy the boundary conditions

$$y(1)=C^TP\int_0^1 w(x)\,dx+1$$
$$y'(1)=C^T\int_0^1 w(x)\,dx+1$$

Therefore $1=C^T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+1=c_0+1 \Rightarrow c_0=0$

and $1=C^TP\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+1=(\tfrac{c_0}{2}+\tfrac{c_1}{4}+\tfrac{c_2}{8})=0 \Rightarrow \tfrac{c_1}{4}+\tfrac{c_2}{8}=0$

by substituting these values into eq.(36), J is written as

$$J=1+c_1^2+c_2^2+c_3^2$$

and by putting

$$\frac{\partial J}{\partial c_i}=0 \Rightarrow 2c_i=0 \Rightarrow c_i=0$$

$$\frac{\partial J}{\partial c_2} = 0 \Rightarrow 2c_2 = 0 \Rightarrow c_2 = 0$$

$$\frac{\partial J}{\partial c_3} = 0 \Rightarrow 2c_3 = 0 \Rightarrow c_3 = 0$$

one can get $y''(t) = 0$ and hence $y'(t) = w_0(t)$ finally

$$y(t) = \tfrac{1}{2} w_0(t) - \tfrac{1}{4} w_1(t) - \tfrac{1}{4} w_2(t)$$

also, if the Euler equation is used for the analytic solution, the answer should be the same.

Remarks

(1) The above direct method can be also similarly used to solve the general case:

$$J = \int_0^1 F(t, y_1, y_1', \ldots, y_1^{(n)}, y_2, y_2', \ldots, y_2^{(n)}, \ldots, y_m, y_m', \ldots, y_m^{(n)})\,dt$$

(2) If the limits of integration for the variational problems are a and b then the Walsh direct method can be also used by transforming the interval [a,b] into [0,1]. To do this use the transform $x = \frac{t-a}{b-a}$ for any real numbers a & b

References

1. Rademacher H., "Einige Satze uber Reihen von allgemeinen orthogonalfunctionen", Math. Ann., Vol 87, pp 112-138.
2. Elsgolc L., "Calculus of Variation", Pergamon Press, London, (1962).
3. Khaleel A.J. & Ayyoub A., (2002), " A Walsh Series Method For Solving Linear System", to be appeared .

الخلاصة

يعطى في هذا البحث طريقة واضحة لحل المسائل التغايرية عن طريق استعمال دوال والش.

هذه الطريقة استعملت لحل عدة صيغ للمسائل التغايرية منها المسائل التغايرية البسيطة، المسائل التغايرية ذات الدوال المعتمدة على عدة متغيرات مستقلة والمسائل التغايرية ذات الدوال التي تحتوي على مشتقات ذات رتب عليا.

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