A Walsh Series Method For Solving Integro-Differential Equation

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Abstract

In this paper, we use the orthonormality properties of the Walsh functions to give a new method for solving the initial value problem associated with the ordinary differential equation (with constant coefficients or with non-constant coefficients). This approach is named a Walsh series method and it is illustrated with some examples.

Introduction

The Walsh functions are initiated by Rademacher [1] and independently developed by Walsh [2] in the early nineteen twenties. In recent years, the Walsh theory has been innovated and applied to various fields in engineering and science[3].

Rademacher and Walsh Functions, [4]:

Rademacher's function $r_i(t)$ is a set of square waves of unit height with period equal to $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, 2^{-i}$ respectively. Alternatively, we state that the number of cycles of the square waves of $r_i(t)$ is 2^{i-1}. It is noted that the set is not complete since, except for $r_0(t)$, the set involves only functions which are odd about $t = \frac{1}{2}$

In 1923, Walsh independently developed a complete set known as Walsh functions. The set of Walsh functions $\varphi_i(t)$ and the set of Radamacher functions have the following relation:

$$\phi_0(t) = r_0(t)$$
$$\phi_1(t) = r_1(t)$$
$$\phi_2(t) = [r_2(t)][r_1(t)]$$
$$\phi_3(t) = [r_2(t)]^1[r_1(t)]$$
$$\phi_4(t) = [r_3(t)]^1[r_2(t)]^0[r_1(t)]^0$$
$$\vdots$$
$$\phi_n(t) = [r_q(t)]^{b_q}[r_{q-1}(t)]^{b_{q-1}} \cdots [r_1(t)]^{b_1}$$

where $q = [\log_2 n] + 1$ in which [.] means taking the greatest integer of ".". Therefore

$n = b_q 2^{q-1} + b_{q-1} 2^{q-2} + \ldots + b_1 2^0$

where $b_q b_{q-1} \ldots b_1$ is the binary expression of n.

The Approach

Consider the first order ordinary integro-differential equation:

$$f'(x) = g(x) + \int_0^1 k(x,y) f(y)\,dy \qquad (1)$$

with $f(0)=\alpha$.

where $g(x)$, $k(x,y)$ are known functions of x and x,y respectively. The problem here is to determine the function $f(x)$.

The approach is based on approximating the first derivative of the unknown solution $f'(x)$ into a Walsh series:

$f'(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x)$

where c_n are the unknown coefficients of the Walsh series of $f'(x)$ that must be determined. Then $f(x)$ is obtained by:

$$f(x) = \int_0^x f'(x) + f(0)$$

$$\Rightarrow f(x) = c_0\int_0^x \phi_0(x)dx + c_1\int_0^x \phi_1(x)dx + \cdots + c_n\int_0^x \phi_n(x)dx + f(0)$$

And by using the same approach described above, one can obtain

$$f(x) = \sum_{i=0}^{n} c_n \left(\sum_{j=0}^{n} b_{ij}\right)\phi_i(x) + f(0)$$

where $\int_0^x \phi_i(x)dx = \sum_{j=0}^{n} b_{ij}\phi_j(x)$ and $\{b_{ij}\}_{i,j=0}^{n}$ are known parameters that can be found similar to the above.

Also, express $g(x)$, x & y appeared in the function $k(x,y)$ as a linear combinations of the Walsh functions with known coefficients. Then by substituting the above functions $f'(x)$, $f(x)$, x, y appeared in the known function $k(x,y)$ and $g(x)$ into eq.(4) and taking the scalar product with $\phi_i(x)$, for all i=0,1,...,n, we obtained a linear system of n equations with n variables $\bar{c} = (c_1, c_2, \ldots, c_n)$. Hence by solving this system by any suitable method, the values of $\bar{c} = (c_1, c_2, \ldots, c_n)$ are computed.

To illustrate this approach, we give the following example:

Example

Consider the first order ordinary integro-differential equation

$$f'(x) = x + \int_0^1 2(1-x) f(y)\,dy \qquad (2)$$

with $f(0)=0$.

Approximate $f'(x)$ as: $f'(x) = c_0\phi_0(x) + c_1\phi_1(x)$

Therefore, $f(x) = \int_0^1 f'(x) + f(0)$

Thus

$$f(x) = \int_0^x [c_0\phi_0(x) + c_1\phi_1(x)]dx = c_0\int_0^x \phi_0(x)dx + c_1\int_0^x \phi_1(x)dx$$

$$= c_0\left(\frac{1}{2}\phi_0(x) - \frac{1}{4}\phi_1(x)\right) + c_1\left(\frac{1}{4}\phi_0(x)\right)$$

$$= \left(\frac{1}{2}c_0 + \frac{1}{4}c_1\right)\phi_0 + \left(-\frac{1}{4}c_0\right)\phi_1$$

Also, write the variable x as $x = \frac{1}{2}\phi_0(x) - \frac{1}{4}\phi_1(x)$

Next, substitute the function $f(x), f'(x)$ & x into eq.(2) to obtain.

$$c_0\phi_0(x) + c_1\phi_1(x) = \frac{1}{2}\phi_0(x) - \frac{1}{4}\phi_1(x) + 2\left(1 - \frac{1}{2}\phi_0(x) + \frac{1}{4}\phi_1(x)\right)\left(\frac{1}{2}c_0 + \frac{1}{4}c_1\right) \qquad (3)$$

and by integrating both sides of eq.(3) from 0 to 1 and use the following facts:-

$$\int_0^1 \phi_0(x)dx = 1 \text{ and } \int_0^1 \phi_i(x)dx = 0, i = 0,1,2,....$$

we obtain:

$$c_0 = \frac{1}{2} + 2\left(1 - \frac{1}{2}\right)\left(\frac{1}{2}c_0 + \frac{1}{4}c_1\right)$$

Hence

$2c_0 - c_1 = 2$

and by multiplying eq.(3) by $\phi_1(x)$ and then integrating from 0 to 1 to give

$-2c_0 + 7c_1 = -2$

and by solving the above equations to give $c_1 = 0$, and then $c_0 = 1$ and hence the solution y(x) = x is the approximate solution of the above example which is agree with the exact solution $y^*(x)=x$.

References

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