About The Two-Parameter Strum-Liouville Eigenvalue Problem

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Abstract

In this work, we study the double eigenvalue problem related to the ordinary differential equation especially for the Strum-Liouville problem which is given by

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]-(\lambda r(x)+\mu s(x)+q(x))y(x)=0, \qquad a\le x\le b$$

with the boundary conditions $y(a)=\alpha$, $y(c)=\beta$ and $y(b)=\delta$

And two methods are presented here to solve this type of the eigenvalue problems, one of them is a numerical method namely the finite difference method and the other is an approximation method, namely the variational method. These methods are illustrated with some examples.

Introduction

The generalized eigenvalue problem (or the one-parameter eigenvalue problem) is the problem of finding the eigenpair (λ, X) which satisfies the equation:

$$AX=\lambda BX \qquad (1)$$

where $A,B:H\to H$ are known linear operators defined on a Hilbert space H. If A and B are n×n matrices then this problem is said to be the algebraic linear eigenvalue problem, [1]. Furthermore if A or B is a differential operator or is an integral operator then this problem is called the continuous linear eigenvalue problem.

On the other hand, the problem of finding the double eigenvalue problem (or the two-parameter eigenvalue problem) is the problem of finding the double eigenvalue (λ,μ) corresponding with the eigenfunction X which satisfies the equation:

$$AX=\lambda BX+\mu CX \qquad (2)$$

where A, B and C are known linear operators defined on a Hilbert space H. Also, if A,B and C are n×n matrices then this problem is said to be the algebraic linear double eigenvalue problem which can be solved by putting $|A-(\lambda+\mu)I|=0$ to calculate the double eigenvalue (λ,μ) and then the eigenvector is obtained by substituting the double eigenvalue into eq.(2) and solving it to find X, [6]. Moreover, if A or B or C is a differential or integral operator then this problem is called the continuous linear double eigenvalue problem.

The one-parameter Strum-Liouville eigenvalue problem consisting of the differential equation:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]-(\lambda r(x)+q(x))y(x)=0 \qquad a\le x\le b$$

with the boundary conditions $y(a)=\alpha$ and $y(b)=\beta$. The finite difference method can be used to convert this continuous eigenvalue problem to an equivalent algebraic eigenvalue problem which can be solved easily. For more details, see [4].

In this work, we use the same method to convert the continuous double Strum-Liouville eigenvalue problem to an equivalent algebraic double eigenvalue problem which can be also solved easily.

The Two-Parameter Strum-Liouville Eigenvalue Problem:

The two-parameter eigenvalue problem consisting of the differential equation:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]-(\lambda r(x)+\mu s(x)+q(x))y(x)=0, \qquad (3)$$

with the boundary conditions

$$y(a)=\alpha$$
$$y(c)=\beta \qquad (4)$$
$$y(b)=\delta,$$

where p, q, r and s are continuous functions of x, c is any given point lies in the interval (a,b) and p, r are positive functions in the closed interval [a,b], [5].

The problem here is to determine the double eigenvalue problem (λ,μ) for which the corresponding nontrivial solution y(x) exists.

Finite Difference Approach:

The finite difference method [3] is one of the most important techniques which is used to convert any continuous eigenvalue problem into an equivalent algebraic one.

This technique is devoted here to solve the two-parameter Strum-Liouville eigenvalue problem given by eq.(3)-(4) and is described by the following steps:-

<u>Step (1):</u> Divide the interval $[a,b]$ into n subintervals such that the point c is one of its mesh points. These points are denoted by x_i, $i=0,1,\ldots,n$ and are given by $x_i = a + ih$, where $h = \frac{b-a}{n}$.

<u>Step (2):</u> Replace the above differential equation (3) into an equivalent difference equation which is given by

$$p_i(y_{i+1} - 2y_i + y_{i-1}) + \frac{h}{2}p'_i(y_{i+1} - y_{i-1}) - h^2(\lambda r_i + \mu s_i + q_i y_i) = 0 \qquad (5)$$

where $p(x_i) = p_i$, $p'(x_i) = p'_i$, $r(x_i) = r_i$, $q(x_i) = q_i$, $s(x_i) = s_i$ and $y(x_i) = y_i$, $i=1,2,\ldots,n-1$.

<u>Step (3):</u> Evaluate eq.(5) at $i=1,\ldots,n-1$ and rewrite these equations in the matrix form given by eq.(2) which can be solved easily by using mathcad professional software to give the values of the double eigenvalue (λ,μ) with the corresponding eigenvector $\{y_i\}$, $i=1,\ldots,n-2$

Example (1)

Consider

$$y'' + (\lambda + \mu x)y = 3x^2 + 8x + 4,\ 0 \le x \le 2 \qquad (6)$$

with the boundary conditions

$y(0)=2$
$y(1)=3$
$y(2)=4$.

Divide the interval $[0,2]$ into 6 subintervals to give the mesh points $x_i = ih$, where $h = \frac{1}{3}$ and $i=0,1,\ldots,6$.

Note that $x_0=0$, $x_3=1$ and $x_6=2$ are the mesh points in which the solution at them are given. So the problem here is to determine the double eigenvalue (λ,μ) with the corresponding eigenvector $\{y_i\}$, $i=1,2,4,5$.

After replacing eq.(6) into an equivalent difference equation and evaluating the resulting equation at $=1,2,3,4,5$ one can get the following equations with their solutions (written in the MathCad Software Editor):-

$\lambda := 2 \quad \mu := 3 \quad y1 := \frac{7}{2} \quad y2 := \frac{8}{3} \quad y4 := \frac{10}{3} \quad y5 := \frac{11}{3}$

Given

$$y2 + \left(-2 + \frac{\lambda}{9} + \mu\cdot\frac{1}{27}\right)\cdot y1 = \frac{1}{9}\left(3\cdot\frac{1}{9} + 8\cdot\frac{1}{3} + 4\right) - 2$$

$$\left(-2 + \frac{\lambda}{9} + \mu\cdot\frac{2}{27}\right)\cdot y2 + y1 = \frac{1}{9}\left(3\cdot\frac{4}{9} + 8\cdot\frac{2}{3} + 4\right) - 3$$

$$y4 + \left(-2 + \frac{\lambda}{9} + \mu\cdot\frac{3}{27}\right)\cdot(3) + y2 = \frac{1}{9}\left(3\cdot\frac{9}{9} + 8\cdot\frac{3}{3} + 4\right)$$

$$y5 + \left(-2 + \frac{\lambda}{9} + \mu\cdot\frac{4}{27}\right)\cdot y4 = \frac{1}{9}\left(3\cdot\frac{16}{9} + 8\cdot\frac{4}{3} + 4\right) - 3$$

$$\left(-2 + \frac{\lambda}{9} + \mu\cdot\frac{5}{27}\right)\cdot y5 + y4 = \frac{1}{9}\left(3\cdot\frac{25}{9} + 8\cdot\frac{5}{3} + 4\right) - 4$$

$$\mathrm{Find}(\lambda,\mu,y1,y2,y4,y5) = \begin{pmatrix} 2.061 \\ 2.949 \\ 2.35 \\ 2.683 \\ 3.314 \\ 3.643 \end{pmatrix}$$

Variational Approach

Variational Technique For Solving Any Linear Problem

Consider the linear problem

$$Ly=f \qquad (7)$$

where $L: L^2[a,b] \to L^2[a,b]$ is a bounded linear operator defined on a Hilbert space $L^2[a,b]$ and $f, y \in L^2[a,b]$.

The problem here is to determine the unknown function $y(x)$ in case $f(x)$ is a given function of x.

The problem of finding a solution of eq.(7) is equivalent to finding the critical points of the functional

$$F[y] = \frac{1}{2} < Ly, Ly > - < f, Ly >$$

Where $<Ly,Ly> = \int_a^b (Ly)^2 dx$ *and* $<Ly,f> = \int_a^b (Ly)f(x)dx$. For more details, see [3].

The Variational Formulation of eq.(3)-(4)

First, rewrite eq.(1) as $Ly=f$, where

$$L = \frac{d}{dx}(p(x).) + (q(x,\lambda,\mu) + r(x))$$

Then the solution of eq.(3)-(4) can be found by minimizing its variational formulation which is given by

$$J[y] = \frac{1}{2}\int_a^b (Ly)^2 dx - \int_a^b (Ly) f(x) dx$$
$$= \frac{1}{2}\int_a^b \left(\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x,\lambda,\mu)y(x)\right)^2 dx - \int_a^b \left(\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x,\lambda,\mu)y(x)\right) f(x)dx \quad (8)$$

The Solution For The Vraiational Formulation of eq.(3)-(4)

To solve the above variational problem, assume the solution of eq.(3)-(4) takes the form

$$y(x) = y_h(x) + V(x)$$

where $y_h(x)$ is any function which satisfies the homogeneous boundary conditions and V(x) is any function which satisfies the boundary conditions given by eq.(4).

Approximate the unknown function $y_h(x)$ as a linear combination of the elements of a basis for $L^2[a,b]$, say $\phi_i(x)$, i=1,2,...,n, for which $\phi_i(a) = \phi_i'(a) = 0 = \phi_i(b) = \phi_i'(b) = \phi_i''(c) = \phi_i(c)$. Then the solution of eq.(3)-(4) takes the form

$$y(x) = \sum_{i=1}^{n} a_i\phi_i(x) + V(x)$$

where $\{a_i\}$, i=1,2,...,n are the unknown parameters to be determined.

By substituting this solution into the functional given by eq.(8), one can get

$$J[\bar{a}] = \frac{1}{2}\int_a^b \left(\frac{d}{dx}\left(p(x)\left[\sum_i a_i\phi_i'(x) + V'(x)\right]\right) + q(x,\lambda,\mu)\left[\sum_i a_i\phi_i(x) + V(x)\right]\right)^2 dx - \int_a^b \left(\frac{d}{dx}\left(p(x)\left[\sum_i a_i\phi_i'(x) + V'(x)\right]\right) + q(x,\lambda,\mu)\left[\sum_{i=1}^{n} a_i\phi_i(x) + V(x)\right]\right) f(x)dx \quad (9)$$

Then the values of $\bar{a}$ can be found by minimizing the functional $J[\bar{a}]$ with respect to $\bar{a}$ (to do so, any suitable method in unconstrained optimization method can be used or one can use the MathCad professional software) and hence the solution of eq.(3)-(4) is obtained.

Example (2)

Consider

$$y''(x) + (\mu^2 xe^{-\lambda} + \mu)y(x) = x^2 + x$$

with the boundary conditions

$$y(0) = 0$$
$$y'(1) = 1$$
$$y(\tfrac{1}{2}) = \tfrac{1}{2}$$

Rewrite the above differential equation as Ly=f, where $L = \frac{d^2}{dx^2} + (\mu^2 xe^{-\lambda} + \mu)$ and $f(x) = x^2 + x$

Approximate the solution of this example as a polynomial of degree three, i.e.,

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Then this solution must satisfy the above boundary conditions and after simple computations this solution reduces to

$$y(x) = \left(1 + \frac{2}{3}a_3\right)x + \left(-\frac{11}{6}a_3\right)x^2 + a_3 x^3$$

where a_3 is the unknown parameter that must be determined.

By substituting this solution into the functional defined in eq.(8) and solving the resulting minimization problems one can get the value of a_3 and hence the solution of the above example is obtained. To do so, see the following MathCad professional editor.

$$y(x,a3) := \left(1 + \frac{2}{3}a3\right)x + \left(\frac{-11}{6}a3\right)x^2 + a3\,x^3$$

$$g(a3,\lambda,\mu) := \int_0^1 0.5\left[\left(\frac{d^2}{dx^2}y(x,a3)\right) + \left(e^{-\lambda}x\mu^2 + \mu\right)y(x,a3)\right]^2 dx$$

$$k(a3,\lambda,\mu) := \int_0^1 \left[\left(\frac{d^2}{dx^2}y(x,a3)\right) + \left(e^{-\lambda}x\mu^2 + \mu\right)y(x,a3)\right](x^2+x)\,dx$$

$$f(a3,\lambda,\mu) := g(a3,\lambda,\mu) - k(a3,\lambda,\mu)$$

a3 := 0.1 λ := 0.1 μ := 0.8

p := Minimize(f, a3, λ, μ) p = [illegible]

the exact solution of the differential equation

$$y''(x) + (\mu^2 xe^{-\lambda} + \mu)y(x) = x^2 + x$$

with the boundary conditions

$$y(0) = 0$$
$$y'(1) = 1$$
$$y(\tfrac{1}{2}) = \tfrac{1}{2}$$

is y(x)=x with the exact eigenvalue (λ,μ)=(0,1). On the other hand the approximated solution using the variational technique is y(x)=0.9999x with the approximated double eigenvalue $(\lambda,\mu) = (1.194 \times 10^{-5}, 1)$

References

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الخلاصة

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