On the Exact Solution of a Space-Tine Metric Conformal to Minkowskian Manifold

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Abstract

A space-time metric admitting 3-parametric group of isometries with 2-dimensioned space-like conformal to Minkowskian space is investigated subject to perfect fluid conditions. Monge's method may play a very important technique for solving and deriving the exact solution for such physical problem. So, new general exact solutions have been derived and it may be taken as a general solution of conformally perfect fluid distributions. In addition to that, all the perfect fluid distributions with variable and constant energy density have been derived.

Introduction

Perfect fluid distribution described by a metric in the conformally flat form have been a subject of interest realm of Cosmology due to the following advantages

(i) The geometry of light rays inside a star is that of the flat Minkewaski continuum

(ii) Relativistic stellar structure forms a link between special relativity and gravitation kinematically

Many authors have derived such models from a consideration of spherical and plane symmetric in the conformally flat from [1], [5].

In the present article the authors have derived all possible perfect fluid solutions of Einstein field equations by considering a metric in conformally flat forms admitting 3-dimensional group of isometrics with 2-dimensional space-like trajectories.

Basic Equation

Metric admitting a 3-parameter group of isometrics with 2 dimensional space-like can be expressed in the form [2], [4],

[illegible] (1)

Where

$$f(\theta)=\begin{Bmatrix}\sin\theta\\ \sinh\theta\\ \theta\end{Bmatrix} for \begin{Bmatrix}positive\\ negative\\ zero\end{Bmatrix} 2-curvature \tag{2}$$

r = constant

t = constant being space-like

The metric (1) describes flat space-time if we take A= C=1 and B= (mr− nt+ k)^2 provided $m^2 - n^2 = 1, -1, 0$ corresponding to positive, negative and zero 2-curvature respectively. However the constant k can be taken zero or unity without loss of generality provided if m=n=0 and by use of the following transformation [3]

$$\begin{aligned} x &= r\\ y &= \theta\cos\phi\\ z &= \theta\sin\phi\\ t &= t \end{aligned} \tag{3}$$

The metric (1) may transform to

$$ds^2 = B^{-2}(x,t)\left[dt^2 - dx^2 - dy^2 - dz^2\right] \tag{4}$$

which of zero 2-curvature and so is plane symmetric

Field equation

The Einstein field equation for a perfect fluid distribution can be written as [1].

$$Rij - \frac{1}{2}RSij = -\pi Tij = -(a-b)v^iv_j + bSij \tag{5}$$

where

$$a = 8\pi\rho, \quad b = 8\pi P \quad and \quad v^iv_j = 1$$

P, b, and V_i being energy density, pressure and flow vector respectively

The system of equation (5) with reference to the metric (4) gives [4], [5],

$$-8\pi T_1^1 = 3\left(B'^2 - \dot{B}^2\right) + 2B\ddot{B} = -(a+b)V_1V^1 + b \qquad (6)$$

$$-8\pi T_2^2 = -8\pi T_3^3 = 3\left(B'^2 - \dot{B}^2\right) + 2B\left(\ddot{B} - B''\right) = b \qquad (7)$$

$$-8\pi T_4^4 = 3\left(B'^2 - \dot{B}^2\right) - 2BB'' = -(a+b)V_4V^4 + b \qquad (8)$$

$$-8\pi T_4^1 = 8\pi T_1^4 = 2\dot{B}B' = (a+b)V^1V_4 = (a+b)V^4V_1 \qquad (9)$$

The prime and dot denote partial differential with respect to x and t respectively. Eliminating a, b and v_i a money field equation (6-9), we get

$$B''\ddot{B} - \dot{B}'^2 = 0 \qquad (10)$$

which is coupled non-linear partial differential equation to be solved.

New General Exact Solutions

A general solution of the equation (10) is difficult, thus we consider the following assumption with help of Monge's method [6]: which is known from theory of differential equation to solve such physical problem if the intermediate integral exists.

$$B = \alpha x - P(\alpha)t + R(\alpha) \qquad (11)$$

provided

$$x + \overline{P}t + \overline{R} = 0 \qquad (12)$$

Where the bar indicates derivative with respect to α, p and R being the arbitrary functions of the parameter α

Equations (6-8) give

$$a = 3\left(\dot{B}^2 - B'^2\right) \qquad (13)$$

Now from a simple analysis (11-13), we get an expression for B in term of energy density "a", "x" and "t" as

$$B = \left(S^2 - \frac{a}{3}\right)^{\frac{1}{2}} x \quad ST + \phi(a)m \qquad (14)$$

$$\overline{\psi}x + \overline{S}t + \Phi = 0 \qquad (15)$$

where $\psi = \left(S^2 - \frac{a}{3}\right)^{\frac{1}{2}}$, S and Φ being arbitrary function of "a". The bar indicates differentiation with respect to "a"

Let us deduce some particular cases of interest from the system (14)-(15)

(i) $\psi = ms + n$ Equation (15) implies that, the energy density "a" is a function of (mx+t) and B assumes the form

$$B = N(mx+t) + nx \qquad (16)$$

where N is an arbitrary function of (mx+t), n and m are arbitrary constants. The solution so obtained in (16), is matching with Roy et al 1978 [3].

(ii) ϕ =constant : which implies $a = a\left(\frac{x}{t}\right)$

and B assumes the form

$$B = t\,y\left(\frac{x}{t}\right) \qquad (17)$$

where y is arbitrary function of $\frac{x}{t}$

(iii) $\phi = P$ (constant). Equation (15) implies homogeneous energy density, consequently (14) provides the solution

$$B = Px + x(t) \qquad (18)$$

where x is arbitrary function of "t"

The corresponding expression of "a" and "b" are given by

$$b = 3\left(P^2 - \dot{x}^2\right) + 2\left(Px + x\right)\ddot{x} \qquad (19)$$

$$a = 3\left(\dot{x} - P^2\right) \qquad (20)$$

which shows the expanding nature of the model with variable boundary $x - x_0(t) = 0$ such that $b(x_0) = 0$

(iv) case with constant energy density "a"

In this case (14) is recovered with S, ϕ and "a" merely as constants. Consequently we come across a de-sitters universe whose envelope with respect to S comes out to be

$$B = \sqrt{\frac{1}{3}}\, a\left(t^2 - x^2\right) \qquad (21)$$

which describes Einstein universe.

Summing up almost all the solution of the perfect fluid distribution conformal to flat space-time admitting 3-parametric group of isometrics with 2-dimensional space-like, have been obtained. Consequently, the solution so obtained is physically acceptable whether kinematics and dynamic is concern by the authors in the fourth coming paper.

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الخلاصة

تمت دراسة الفضاء الزمكاني (space-time) والموصوف بالمجموعة التي تضم 3-بارامترات متماثلة مع مجموعتين شبيهتين بفضاء منكوفسكي متعلقة بموضوع البحث للموائع الكاملة وكذلك المفاهيم الفيزيائية.

لقد لعبت طرائق مونجي دورا مهما في دراسة هذه المشاكل، لقد تم إيجاد حلول جديدة أكثر عمومية، كذلك تم اشتقاق كل توزيعات الموائع المثالية أخذا بنظر الاعتبار كافة الحالات المتغيرة والثابتة.