On The Spectrum Of Some Operators

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Abstract
In this paper we investigate some properties of an operator which imply that either the spectrum of such operator lies on a straight line through the origin or the spectrum is real.

Introduction

Let H be an infinite dimensional complex Hilbert space and let $B(H)$ be the algebra of bounded linear operator on H . the spectrum $\sigma(T)$ of an operator T is the set of all complex number λ for which $T-\lambda I$ is not invertible.

A complex number λ is said to be in the approximate point spectrum $\sigma_{ap}(T)$ of the operator T , if there is a sequence $\{x_n\}$ of unit vectors in H satisfying $(T-\lambda I)x_n \to 0$.

The numerical range $W(T)$ of an operator T is the set $\{\langle Tx, x\rangle : \|x\|=1\}$ [3,P.12].

Two operators A and B are similar if there exists an invertible operator P such that

$$P^{-1}AP = B ; \qquad A, B, P \in B(H)$$

[3,P.42].

In this paper we investigate some properties of an operator T which imply that $\sigma(T)$ is real or lies on a straight line through the origin, in particular, we generalize theorem (1) of [8].

§1 Spectrum of Some Operators

In the following theorem we give conditions that make the spectrum of an operator T real.

Theorem 1
Let T be an element in $B(H)$, if $\exists A \in B(H)$ such that $A(T-\lambda I) = (T-\lambda I)^* A$, $\forall \lambda \in \sigma_{ap}(T) \cup \{0\}$ and $0 \notin \sigma_{ap}(A)$, then the spectrum of T is real.

Proof
Since the boundary of the spectrum of any operator consists of approximate eigenvalues [3,P.43], then to show that the spectrum of T is real, it is enough to show that $\sigma_{ap}(T)$ is real.

Let $\lambda \in \sigma_{ap}(T)$, then there exists a sequence $\{x_n\}$ of unit vectors in H such that $(T-\lambda I)x_n \to 0$.

$$\begin{aligned}|(\bar{\lambda}-\lambda)\langle Ax_n, Ax_n\rangle| &= |\langle \bar{\lambda}Ax_n, Ax_n\rangle - \langle \lambda Ax_n, Ax_n\rangle + \langle ATx_n, Ax_n\rangle - \langle ATx_n, Ax_n\rangle| \\ &= |\langle A(T-\lambda I)x_n, Ax_n\rangle + \langle (\bar{\lambda}I - T^*)Ax_n, Ax_n\rangle| \\ &\le |\langle A(T-\lambda I)x_n, Ax_n\rangle| + |\langle A(T-\lambda I)x_n, Ax_n\rangle| \\ &\le 2\|A\|^2\|(T-\lambda I)x_n\| \to 0\end{aligned}$$

if $\lambda \neq \bar{\lambda}$ then $\langle Ax_n, Ax_n\rangle = \|Ax_n\|^2 \to 0$, which is contradiction to the fact that $0 \notin \sigma_{ap}(A)$. Thus $\lambda = \bar{\lambda}$ and hence $\sigma(T)$ is real.

Recall that an operator T is θ - adjoint if $T^* = e^{i\theta}T$ where $\theta \in R$ [4]. In the following theorem we study the spectrum of an operator T if it is similar to θ adjiont operator.

Theorem 2
Let T be an operator in $B(H)$. If $\exists S \in B(H)$ such that $S^{-1}TS = e^{i\theta}T^*$,and $0 \notin \overline{W(S)}$ then the spectrum of T lies on a straight line through the origin that makes an angle $\theta/2$ with the positive x -axis.

Proof
Since the boundary of the spectrum of any operator consists of approximate eigenvalues [3,P.43], it is enough to show that the boundary of $\sigma(T)$ lies on this line. Let $\lambda \in \sigma_{ap}(T)$, then there exists a sequence $\{x_n\}$ of unit vectors in H such that $(T-\lambda I)x_n \to 0$.

$$\left|(\bar{\lambda}-\lambda e^{-i\theta})(S^{-1}x_n,x_n)\right| = \left|(\bar{\lambda}S^{-1}x_n,x_n)-(\lambda e^{-i\theta}S^{-1}x_n,x_n)-(S^{-1}Tx_n,x_n)\right.$$
$$\left. -(S^{-1}x_n,\lambda x_n)-(S^{-1}x_n,Tx_n)+(e^{-i\theta}S^{-1}Tx_n,x_n)-(\lambda e^{-i\theta}S^{-1}x_n,x_n)\right|$$
$$\le \left|(S^{-1}x_n,(T-\lambda I)x_n)\right| + \left|e^{-i\theta}(S^{-1}(T-\lambda I)x_n,x_n)\right|$$
$$\le 2\|S^{-1}\| \, \|(T-\lambda I)x_n\| \to 0$$

Now, if $\bar{\lambda} \neq \lambda e^{-i\theta}$ then $(S^{-1}x_n, x_n) \to 0$, this implies that $0 \in W(S^{-1})$, which is a contradiction [since $0 \notin \overline{W(S)}$ implies that $0 \notin \overline{W(S^{-1})}$ [8]] so $\bar{\lambda} = \lambda e^{-i\theta}$. Put $\lambda = re^{i\alpha}$ then $\bar{\lambda} = re^{-i\alpha}$, $re^{-i\alpha} = re^{i\alpha}e^{-i\theta}$ and hence $\alpha = \frac{1}{2}\theta$.

In the following theorem we will make use of the fact that $W(S)$ is convex [3.P.113] and contains the spectrum of S [3.P.114] to show that T is similar to θ-adjoint operator, namely $S^{-1}TS = e^{i\theta}T^*$.

Theorem 3

Let T be any operator in $B(H)$. If $\exists S \in B(H)$ such that $S^{-1}TS = e^{i\theta}T^*$, and $0 \notin \overline{W(S)}$, then T is similar to θ-adjoint operator.

Proof

Since $\overline{W(S)}$ is convex [3, P.113] and does not contain 0, we can separate 0 from $\overline{W(S)}$ by a half-plane such that $\operatorname{Re} z > \xi$ for some $\xi > 0$ [8].

Let

$$A = \frac{1}{2}(S + S^*)$$

Note that A is self-adjoint so the numerical range of A lies on the real axis [8] to the right of ξ, hence A is positive and invertible and therefore it has a positive square root which is self-adjoint. [8]. Now,

$$TA = \frac{1}{2}T(S+S^*) = \frac{1}{2}\left(e^{i\theta}ST^* + (ST^*)^*\right)$$
$$= \frac{1}{2}\left(e^{i\theta}ST^* + (e^{-i\theta}ST)^*\right)$$
$$= e^{i\theta}\frac{1}{2}(S+S^*)T^*$$
$$= e^{i\theta}AT^*$$

Hence $TA = e^{i\theta}AT^*$ so

$$TA^{1/2}A^{1/2} = e^{i\theta}A^{1/2}A^{1/2}T^*$$
$$TA^{1/2} = e^{i\theta}A^{1/2}A^{1/2}T^*A^{-1/2}$$
$$A^{-1/2}TA^{1/2} = e^{i\theta}A^{1/2}T^*A^{-1/2}$$

Therefore $A^{-1/2}TA^{1/2}$ is θ adjoint.

Recall that if $\theta = 0$, then T is self-adjoint and if $\theta = \pi$, then T is skew self-adjoint [4].

Corollary 4

In the above theorem if $\theta = \pi$ then T is similar to self-adjoint operator and if $\theta = \pi$ then T is similar to skew-self-adjoint operator.

Next we generalize theorem (1).

Theorem 5

Let T be any operator in $B(H)$, if $\exists A \in B(H)$ such that $A(T - \lambda I) = e^{i\theta}(T - \lambda I)^* A$, $\forall \lambda \in \sigma_{ap}(T) \bigcup \{0\}$ and $0 \notin \sigma_{ap}(A)$, then the spectrum of T lies on a straight line through the origin that makes an angle $\theta/2$ with the positive x-axis.

Proof

Again it is enough to show that the boundary of $\sigma(T)$ lies on this line. Let $\lambda \in \sigma_{ap}(T)$, then there exists a sequence $\{x_n\}$ of unit vectors such that $(T - \lambda I)x_n \to 0$.

$$\left|(\bar{\lambda} - e^{-i\theta}\lambda)(x_n, Ax_n)\right| = \left|(\bar{\lambda}x_n, Ax_n) - (e^{-i\theta}\lambda x_n, Ax_n) - (Tx_n, Ax_n) + (Tx_n, Ax_n)\right|$$
$$\left|((T-\lambda I)x_n, Ax_n) - (e^{-i\theta}A(T-\lambda I)x_n, x_n)\right|$$
$$\le 2\|A\| \, \|(T - \lambda I)x_n\| \to 0$$

Now, if $\bar{\lambda} \neq \lambda e^{-i\theta}$ then $\|Ax_n\| \to 0$, this implies that $0 \in \sigma_{ap}(A)$, which is contradiction so $\bar{\lambda} \neq \lambda e^{-i\theta}$.

Put $\lambda = re^{i\alpha}$ and hence as in the proof of theorem (2) $\alpha = \theta/2$.

It is know that a complex number λ is said to be in the joint approximate point spectrum, $\sigma_{ja}(T)$, of T if $\lambda \in \sigma_{ap}(T)$ such that $(T - \lambda I)x_n \to 0$ implies $(T^* - \bar{\lambda} I)x_n \to 0$.

In general, $\sigma_{ja}(T) \subseteq \sigma_{ap}(T)$ [1], there are many classes of operators T for which

$$\sigma_{ja}(T) = \sigma_{ap}(T) \qquad (1)$$

For example, if T is either normal or hyponormal, then T satisfies (1), [2].

Also recall that an operator T is P - hyponormal $0 \le P \le 1$ if $(TT^*)^p \le (T^*T)^p$ [1]. An invertible operator T is said to be log-hyponormal if $\log(T^*T) \ge \log(TT^*)$ [7].

It is know that (1) holds if T is P - hyponormal or log- hyponormal. [2]

Now for operators that satisfy (1) we can prove the following proposition.

Proposition 6

Let T be an operator in $B(H)$ such that $\sigma_{ja}(T) = \sigma_{ap}(T)$ and let $\lambda, \mu \in \sigma_{ap}(T)$, $\lambda \ne \mu$. If $\{x_n\}$ and $\{y_n\}$ are sequences of unit vectors in H such that $\|(T - \lambda I)x_n\| \to 0$ and $\|(T - \mu I)y_n\| \to 0$ then $\langle x_n, y_n \rangle \to 0$.

Proof

$$|(\lambda-\mu)\langle x_n, y_n\rangle| = |\langle \lambda x_n, y_n\rangle - \langle \mu x_n, y_n\rangle + \langle Tx_n, y_n\rangle - \langle Tx_n, y_n\rangle|$$
$$= |\langle(\lambda I - T)x_n, y_n\rangle - \langle x_n, (T^* - \bar{\mu} I)y_n\rangle|$$
$$\le \|(\lambda I - T)x_n\|\|y_n\| + \|x_n\|\|(T^* - \bar{\mu} I)y_n\|$$
$$\le \|(\lambda I - T)x_n\| + \|(T^* - \bar{\mu} I)y_n\| \to 0$$

Since $\lambda \ne \mu$ then $\langle x_n, y_n \rangle \to 0$.

Recall that a family $\{M_\alpha\}, \alpha \in I$ of subspaces in H is said to form an orthogonal family if $\langle M_\alpha, M_\beta \rangle = 0$ for all $\alpha, \beta \in I, \alpha \ne \beta$ [5, P.152].

Corollary 7: The eigen space of an operator T with $\sigma_{ap}(T) = \sigma_{ap}(T)$ form an orthogonal family.

Now we prove the following theorem.

Theorem 8

Let T be an operator in $B(H)$ such that $\sigma_{ja}(T) = \sigma_{ap}(T)$. Assume there exists $S \in B(H)$ s.t $0 \notin \overline{W(S)}$ and $\exists \theta \in R$ such that $S^{-1}TS = e^{i\theta}T^*$. Let $\lambda \in \sigma_{ap}(T)$ s.t. $\lambda \ne e^{i\theta}\bar{\lambda}$ (i.e. $\arg\lambda \ne \theta/2$). Then $\sigma(T)$ is real.

Proof

Again it is enough to show that $\sigma_{ap}(T)$ is real. Assume there exists $\lambda \in \sigma_{ap}(T)$ such that $\lambda \ne \bar{\lambda}$. It is clear that $\lambda \ne 0$, so there exists a sequence $\{x_n\}$ of unit vectors such that $(T - \lambda I)x_n \to 0$. Since $\sigma_{ja}(T) = \sigma_{ap}(T)$ then $(T^* - \bar{\lambda} I)x_n \to 0$ Now

$$\|(T^* - \lambda I)x_n\| = |e^{i\theta}|\|(T^* - \bar{\lambda} I)x_n\|$$
$$= \|(e^{i\theta}T^* - e^{i\theta}\bar{\lambda} I)x_n\|$$
$$= \|(S^{-1}TS - e^{i\theta}\bar{\lambda} I)x_n\|$$
$$= \|(S^{-1}(T - e^{i\theta}\bar{\lambda} I)Sx_n\| \to 0$$

Since $0 \notin \overline{W(S)}$ implies $0 \notin \overline{W(S^{-1})}$, this relation implies that $\|(T - e^{i\theta}\bar{\lambda} I)Sx_n\| \to 0$.

Hence $\langle x_n, Sx_n \rangle = \langle S^{-1}Sx_n, Sx_n \rangle \to 0$ by proposition (6) since $\lambda \ne e^{i\theta}\bar{\lambda}$.

Put $y_n = \frac{Sx_n}{\|Sx_n\|}$, then $\|y_n\| = 1$ and $\langle S^{-1}y_n, y_n \rangle \to 0$ i.e $0 \in \overline{W(S^{-1})}$ a contradiction. This completes the proof of the theorem.

It was shown in [6] that if T is hyponormal and $\sigma(T)$ is real then T is self – adjoint therefore we get the following corollary.

Corollary 9: Let T be a hyponormal operator such that $S^{-1}TS = e^{i\theta}T^*$, for some $\theta \in R$, and $S \in B(H)$ with $0 \notin \overline{W(S)}$. If for each $\lambda \in \sigma_{ap}(T)$, $e^{i\theta}\bar{\lambda} \ne \lambda$ then T is self-adjoint.

References

1. Aluthge, A. (1990) "On P-Hyponormal Operators For 0<P<1", Integr. Equat. Oper. Th., 13, P.307-315.
2. Aluthge, A. And. Wang, D. (2002). "The Joint Approximate Point Spectrum Of An Operator", Hokkaido Math. Journal, 31, P. 187-197.

3. Halmos, P. R. (1982). "A Hilbert Space Problem Book", Springer Velag.
4. Hamed, N. (2002). "Jordan"-Derivations On B(H)", Ph. D. Thesis, College Of Science, University Of Baghdad.
5. Kreyszg,E. (1978). "Introductory Functional Analysis With Applications", John Wily & Sons.
6. Stampfli, J. G. (1965). "Hyponormal Operators And Spectral Density", Trans. Amer. Math. Soc. 117p.469-476.
7. Tanahashi K., (1999)."On Log-Hyponormal Operators", Integr.Equat. Oper. Th., 34P. 364-373.
8. Williams, J. P. (1969). "Operators Similar To Their Adjoint", Proc. Amer. Math. Soc. 20 121-123.

الخلاصة

في هذا البحث درسنا المؤثر التي ينتج عنها كون طيف المؤثر يقع على خط مستقيم يمر بنقطة الأصل أو كون طيف المؤثر حقيقيا.