Solution of Cauchy's Problem By Using Spline Interpolation

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Abstract

The aim of this paper is to obtain an approximate solution of Cauchy's problem of the second order by using deficient sextic spline which interpolates the lacunary data (0,1,4). Also, we studied the convergence for the approximate solution to the exact solution.

Introduction

Let us consider the Cauchy's initial value problem

$y'' = f(x, y, y')$, $x \in [0,1]$, $y(0) = y_0$, $y'(0) = y'_0$ ...(1.1)

Here we assume that $f(x, y, y') \in C^r[0,1]$, $r \geq 0$, { where $C^r[0,1]$ is the set of all continuously differentiable functions r-times over [0,1]} and satisfies the Lipschitz condition

$\left| f^{(q)}(x, y_1, y'_1) - f^{(q)}(x, y_2, y'_2) \right| \leq L\left(|y_1 - y_2| + |y'_1 - y'_2|\right)$,

$q = 0,1,\ldots,r$ for all $x \in [0,1]$, where L is the Lipschitz constant and y_1, y_2, y'_1 *and* y'_2 are real valued functions. This condition ensures the existence of unique solution of the problem (1.1) [5] .Siddiqi and Akram [6] used quintic spline to find an approximation solution of fourth order boundary-value problems.

In the last several years various authors have used spline functions for finding an approximate solution of initial value problems including (1.1)[1,2,3 ,and 5],that the spline functions of full continuity do not converge to exact solution for arbitrary degrees of the spline (see for example [2] and [3]). For this reason the continuity conditions are relaxed. Saeed[5] used the same idea for finding approximate solution of (1.1)but for the lacunary data (0,2,5). In this paper we consider the same problem but for the lacunary data (0,1,4).

Definition of the approximate values $\bar{Y}_k^{(q)}$

Let $x_k = \frac{k}{m}$; $h = \frac{1}{m}$; $w_{r+2}(h) = \max_{|x - x_1| \leq h} \{ |y^{(r+2)}(x) - y^{(r+2)}(x_1)| \}$, $k=0,1,\ldots,m$

$\bar{Y}_k^{(q)} : \bar{y}_0^{(q)}, \bar{y}_1^{(q)}, \ldots, \bar{y}_m^{(q)}; q = 0,1,\ldots,r+2$, be the approximate values which are an approximate to the exact values $Y^{(q)} : y_0^{(q)}, y_1^{(q)}, \ldots, y_m^{(q)}; q = 0,1,\ldots,r+2$. By using these approximate values we construct a spline function $\bar{S}_\Delta(x)$ which interpolates to the set $\bar{Y}$ on the mesh Δ and approximate the solution $y(x)$ of (1.1). The set $\bar{Y}^{(q)}$ is $\bar{y}_0 = y_0, \bar{y}'_0 = y'_0$, $\bar{y}_0^{(2+q)} = f^{(q)}(x_0, y(x_0), y'(x_0)); q = 0,1,\ldots,r.$

$$\bar{y}_{k+1} = \bar{y}_k + h\bar{y}'_k + \int_{x_k}^{x_{k+1}} \int_{x_k}^{t} f(u, v_k(u), n'_k(u))\,du\,dt$$

$$\bar{y}'_{k+1} = \bar{y}'_k + \int_{x_k}^{x_{k+1}} f(t, v_k(t), n'_k(t))\,dt \quad ,$$

$$\bar{y}_{k+1}^{(2+q)} = f^{(q)}(x_{k+1}, \bar{y}_{k+1}, \bar{y}'_{k+1}),$$

$q = 0,1,\ldots,r$, $k = 0,1,\ldots,m-1$,

$n'_k(x) = y'_k + \int_{x_k}^{x} f(t, v(t), v'(t))\,dt$, and for $x_k \leq x \leq x_{k+1}; k = 0,1,\ldots,m-1$.

$$v_k(x) = \sum_{j=0}^{r+2} \bar{y}_k^{(j)} \frac{(x - x_k)^j}{j!} \quad \text{and}$$

$$v'_k(x) = \sum_{j=0}^{r+1} \bar{y}_k^{(j+1)} \frac{(x - x_k)^j}{j!}$$

The error of the approximates values $\bar{y}_{k+1}^{(j)}$ are estimated by the inequality (1.3) in[1] which is $\left| y_{k+1}^{(j)} - \bar{y}_{k+1}^{(j)} \right| \leq c_j h^{r+3-j} w_{r+2}(h)$,where $k=0,1,\ldots,m-1$ and $j=0,1,\ldots,r+2$, see [1,Lemma 2.2.1 and 2.2.3] where c_j's denote different constants independent of h .

3. The spline function $\bar{S}_\Delta(x)$

Let $\Delta : x_0 < x_1 < \ldots < x_{m-1} < x_m = 1$ be the uniform partition of the interval [0,1] with $x_k = \frac{k}{m}, k = 0,1,\ldots,m$. We denote by $S_p(6,3,m)$ the class of splines $\bar{S}_\Delta(x)$ such that $\bar{S}_\Delta(x) \in C^1[0,1]$ and $\bar{S}_\Delta(x)$ is polynomial of degree six in $[x_0, x_1]$ and $[x_{m-1}, x_m]$ and of degree five in each subintervals $[x_k, x_{k+1}], k = 1,2,\ldots,m-2$.

Suppose that $\bar{Y}_k^{(q)}, q = 0,1,\ldots,5$ and $k = 0,1,\ldots,m$, be given real numbers. Using these approximate values, we construct a unique lacunary spline function $\bar{S}_\Delta(x)$ of the type (0,1,4) which satisfies the following conditions[4]:

$$\left.\begin{cases} \bar{S}_\Delta(x_k) = \bar{y}_k \\ \bar{S}_\Delta^{(q)}(x_k) = \bar{y}_k^{(q)} \text{ where } q=1,4 \text{ and } k=0,1,\ldots,m \\ \bar{S}'_0(x_0) = \bar{y}'_0 \text{ and } \bar{S}'_{m-1}(x_m) = \bar{y}'_m \end{cases}\right\} \ldots(2.1)$$

The existence and uniqueness of above spline function have been shown in [4], we have

$$\bar{S}_k(x)=\begin{cases}\bar{S}_0(x), & \text{when } x\in[x_0,x_1]\\ \bar{S}_k(x), & \text{when } x\in[x_k,x_{k+1}],\ k=1,\dots,m-2\\ \bar{S}_{m-1}(x), & \text{when } x\in[x_{m-1},x_m]\end{cases}$$

Where

$$\bar{S}_0(x)=\bar{y}_0+\bar{y}'_0(x-x_0)+\bar{y}''_0\frac{(x-x_0)^2}{2!}+\bar{a}_{0,3}h^3+\bar{y}_0^{(4)}\frac{(x-x_0)^4}{4!}+\bar{a}_{0,5}h^5+\bar{a}_{0,6}h^6,\ \dots(2.2)$$

$$\bar{S}_k(x)=y_k+y'_k(x-x_k)+a_{k,2}h^2+\bar{a}_{k,3}h^3+\bar{y}_k^{(4)}\frac{(x-x_k)^4}{4!}+\bar{a}_{k,5}h^5,\ \dots(2.3)$$

$$\bar{S}_{m-1}(x)=\bar{y}_{m-1}+\bar{y}'_{m-1}(x-x_{m-1})+\bar{y}''_{m-1}\frac{(x-x_{m-1})^2}{2!}+\bar{a}_{m-1,3}h^3+\bar{y}_{m-1}^{(4)}\frac{(x-x_{m-1})^4}{4!}+\bar{a}_{m-1,5}h^5+\bar{a}_{m-1,6}h^6\ \dots(2.4)$$

Here

$$\bar{a}_{0,3}=\frac{1}{6h^3}\{18(\bar{y}_1-\bar{y}_0)-(14h\bar{y}'_0+4h\bar{y}'_1+5h^2\bar{y}''_0)\}+\frac{h}{360}(\bar{y}_1^{(4)}-6\bar{y}_0^{(4)}).$$

$$\bar{a}_{0,5}=-\frac{1}{h^5}\{3(\bar{y}_1-\bar{y}_0)-\frac{1}{2}(4h\bar{y}'_0+2h\bar{y}'_1+h^2\bar{y}''_0)\}-\frac{1}{120h}(\bar{y}_1^{(4)}+4\bar{y}_0^{(4)}).$$

$$\bar{a}_{0,6}=\frac{1}{3h^6}\{3(\bar{y}_1-\bar{y}_0)-\frac{1}{2}(4h\bar{y}'_0+2h\bar{y}'_1+h^2\bar{y}''_0)\}+\frac{1}{360h^2}(2\bar{y}_1^{(4)}+3\bar{y}_0^{(4)}).$$

$$\bar{a}_{k,2}=\frac{1}{h^2}\{3(\bar{y}_{k+1}-\bar{y}_k)-2h\bar{y}'_k-h\bar{y}'_{k+1}\}+\frac{h^2}{120}(2\bar{y}_{k+1}^{(4)}+3\bar{y}_k^{(4)}).$$

$$\bar{a}_{k,3}=\frac{1}{h^3}\{-2(\bar{y}_{k+1}-\bar{y}_k)+h\bar{y}'_k+h\bar{y}'_{k+1}\}-\frac{h}{120}(3\bar{y}_{k+1}^{(4)}+7\bar{y}_k^{(4)}).$$

$$\bar{a}_{k,5}=\frac{1}{120h}(\bar{y}_{k+1}^{(4)}-\bar{y}_k^{(4)}).$$

Convergence of a spline function

In this section, we find the order of convergence for spline function $\bar{S}_k(x)$ given in section two to the exact solution of problem (1.1). We also prove that it satisfies the differential equation in (1.1) as $m\to\infty$, let $\bar{S}_k(x)$ be the spline function corresponding to the approximate values $\bar{y}_k, k=0,1,\dots,m$ and let $S_k(x)$ be the spline function corresponding to the exact values $y_k, k=0,1,\dots,m$ of problem (1.1). Then we have the following theorems

Theorem 1 The following estimates are valid

(i) $|S_k^{(q)}-\bar{S}_k^{(q)}|\le B_q h^{7-q}w_5(h)$, $q=0,1,\dots5; k=0,1,\dots m-1$

Where B_q denote different constants independent of h.

(ii) $|y^{(q)}(x)-\bar{S}_k^{(q)}(x)|\le E_q h^{7-q}w_5(h)$; $q=0,1,\dots5$ Where $y(x)$ is the exact solution of (1.1) and E_q denote different constants independent of h

Proof of the theorem 1 (i): we have owing to (2.2)

$$S_0(x)-\bar{S}_0(x)=h^3(a_{0,3}-\bar{a}_{0,3})+h^5(a_{0,5}-\bar{a}_{0,5})+h^6(a_{0,6}-\bar{a}_{0,6}),$$

where

$$a_{0,3}-\bar{a}_{0,3}=3h^{-3}(y_1-\bar{y}_1)+\frac{2}{3}h^{-2}(y'_1-\bar{y}'_1)+\frac{h}{360}(y_1^{(4)}-\bar{y}_1^{(4)})$$

Using (1.3) we have

$$|a_{0,3}-\bar{a}_{0,3}|\le\frac{1}{360}(120c_0+240c_1+c_4)h^4w_5(h)$$ where $$=I_0h^4w_5(h)$$

$I_0=\frac{1}{360}(120c_0+240c_1+c_4)$ and c_0, c_1 and c_4 are constants independent of h.

Similarly

$$|a_{0,5}-\bar{a}_{0,5}|\le\frac{1}{120}(360c_0+120c_1+c_4)h^2w_5(h)$$ where $$=I_1h^2w_5(h)$$

$I_1=\frac{1}{120}(360c_0+120c_1+c_4)$,

and $$|a_{0,6}-\bar{a}_{0,6}|\le\frac{1}{180}(180c_0+60c_1+c_4)hw_5(h)=I_2hw_5(h)$$

where

$I_2=\frac{1}{180}(180c_0+60c_1+c_4)$.

Hence

$$|S_0(x)-\bar{S}_0(x)|=h^3|a_{0,3}-\bar{a}_{0,3}|+h^5|a_{0,5}-\bar{a}_{0,5}|+h^6|a_{0,6}-\bar{a}_{0,6}|\le Ih^7w_5(h),$$

where $I=I_0+I_1+I_2$,

and by successive differentiations

$|S_0^{(q)}(x)-\bar{S}_0^{(q)}(x)|\le b_qh^{7-q}w_5(h)$; $q=0,1,\dots,5$. This proves (3.1) for $k=0$ and $x\in[x_0,x_1]$. Further, owing to (2.3)

$$S_k(x)-\bar{S}_k(x)=(y_k-\bar{y}_k)+(y'_k-\bar{y}'_k)(x-x_k)+h^2(a_{k,2}-\bar{a}_{k,2})+h^3(a_{k,3}-\bar{a}_{k,3})+(y_k^{(4)}-\bar{y}_k^{(4)})\frac{(x-x_k)^4}{4}+h^5(a_{k,5}-\bar{a}_{k,5}).$$

We have owing to (1.3) we get

$$a_{k,2}-\bar{a}_{k,2}=h^{-2}\{3(y_{k+1}-y_k)+2h\,y_k'-hy_{k+1}'\}+\frac{h^2}{120}(2y_{k-1}^{(4)}+3y_k^{(4)})-h^{-2}\{3(\bar{y}_{k+1}-y_k)+2h\,y_k'-hy_{k+1}'\}-\frac{h^2}{120}(2\bar{y}_{k-1}^{(4)}+3\bar{y}_k^{(4)})\,.$$

Now

$$|a_{k,2}-\bar{a}_{k,2}|\le\frac{1}{25}(75k_1+k_2)h^3w_5(h)=L_1h^3w_5(h)$$

where $L_1=\frac{1}{25}(75\,k_1+k_2)$,

And k_1 and k_2 are constants independent of h

Similarly

$$|a_{k,3}-\bar{a}_{k,3}|\le L_2h^2w_5(h)$$

and $|a_{k,4}-\bar{a}_{k,4}|\le L_3h^2w_5(h)$

where L_1, L_2 and L_3 are constants independent of h.

Hence $|S_k(x)-\bar{S}_k(x)|\le dh^5w_5(h)$, where $d=L_1+L_2+L_3$

and by successive differentiation we get $|S_k^{(q)}(x)-\bar{S}_k^{(q)}(x)|\le d_q h^{5-q}w_5(h)$, $q=0,1,\ldots,5$. This proves (i) for $k=1,2,\ldots,m-2$.

We can repeate the same manner in above for $k=m-1$. Hence the proof of theorem 1(i) is complete.

Proof of the theorem 1(ii):

$$|y^{(q)}(x)-\bar{S}_k^{(q)}(x)|\le|y^{(q)}(x)-S_k^{(q)}(x)|+|S_k^{(q)}(x)-\bar{S}_k^{(q)}(x)|.$$

From [4 ,theorem 3.1] the following estimates are valid $|S_k^{(q)}(x)-y^{(q)}(x)|\le k_q h^{5-q}w_5(h)$, $q=0,1,\ldots,5$.

Using this estimate and estimate in theorem 1(i) we have

$$|y^{(q)}(x)-\bar{S}_k^{(q)}(x)|\le k_q h^{5-q}w_5(h)+B_q h^{7-q}w_5(h)=(k_q+h^2B_q)h^{5-q}w_5(h)=E_qh^{5-q}w_5(h);$$

$q=0,1,\ldots,5$, where $E_q=k_q+h^2B_q$

Which proves (ii).

The above theorem give error estimate between the approximating spline $S_k(x)$ and the exact solution of (1.1), also for $h=\frac{1}{m}$ the following theorem shows that $S_k(x)$ satisfies the differential equation (1.1) as $m\to\infty$.

Theorem 2: The following estimates is valid

$$|\bar{S}_k^{(2)}(x)-f(x,\bar{S}_k(x),\bar{S}_k'(x))|\le Rh^3w_5(h)$$

where R is constant independent of h.

Proof: We have

$$\bar{S}_k^{(2)}(x)-f(x,\bar{S}_k(x),\bar{S}_k'(x))=\bar{S}_k^{(2)}(x)-y^{(2)}(x)+y^{(2)}(x)-f(x,\bar{S}_k(x),\bar{S}_k'(x))=[\bar{S}_k^{(2)}(x)-y^{(2)}(x)]+[y^{(2)}(x)-f(x,\bar{S}_k(x),\bar{S}_k'(x))].$$

Therefore, owing to the Lipshitz condition

$$|\bar{S}_k^{(2)}(x)-f(x,\bar{S}_k(x),\bar{S}_k'(x))|\le|\bar{S}_k^{(2)}(x)-y^{(2)}(x)|+L\{|y(x)-\bar{S}_k(x)|+|y'(x)-\bar{S}_k'(x)|\},$$

we get the proof of the theorem by using theorem 1(ii) for $q=0,1,2$.

Conclusion

In this work, with [2],[3] and [5] we conclude that the type of the lacunary data has no any role in deciding the order of convergence , but the degree of the spline function has to plays a very important role in deciding the rate of convergence.For this reason, to find a best approximate solution for Cauchy's problem, we must use greatest possible degree for spline function to obtain best accuracy.

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الخلاصة

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