$\mathcal{G}$-cyclicity

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Abstract

Dimensional separable complex Hilbert space and S be a multiplication semigroup of $\mathbb{C}$ with 1. Generizing the concept of supercyclicity, we define G cyclicity, namely, an operator T is called G – cyclic over S if there is a vector $x \in H$ such that $\{\alpha T^n x \mid \alpha \in S, n \geq 0\}$ is norm- dense in H, such a vector is called G-cyclic vector for T over S. In this paper, we list some basic properties of G –cyclicity. We study necessary and sufficient conditions for an operator to be G – cyclic. Finally we study some of the spectral properties of G – cyclic operators.

Introduction

Let H be an infinite – dimensional separable complex Hilbert space, and let $B(H)$ be the Banach algebra of all linear bounded operators on H. Let T be an element in $B(H)$, T is called "cyclic" if there is a vector x in H such that the closed linear space spanned by the set $\{T^k x \mid k \geq 0\}$ is the whole space [5,p86]. An operator T is " hypercyclic " if there is a vector x in H such that the set $\{T^k x \mid k \geq 0\}$ is norm – dense in H [1, p 71], and T is called " supercyclic " if there exists an $x \in H$ such that $\{\alpha T^k x \mid k \geq 0, \alpha \in \mathbb{C}\}$ is norm – dense in H [8].

In this paper we introduce a concept which unifies these concepts as follows: Let T be an element in $B(H)$, and let $A = A(T)$ be the subalgebra of $B(H)$ generated by the identity operator I and the operator T over the field of complex numbers. It is easily seen that A is the set of all polynomials in T with coefficients in $\mathbb{C}$. Let $\mathcal{G} = \mathcal{G}(T)$ be a multiplication semigroup of A with I. In particular, let S be a multiplication semigroup of $\mathbb{C}$ with 1, and let $\mathcal{G} = \mathcal{G}(S,T)$ be the multiplication semigroup of A consisting of all elements in A of the form $\{\alpha T^k \mid \alpha \in S, k \geq 0\}$ or $\mathcal{G} = A$.

Let x be a vector in H, we say that x is a G-.cyclic vector over S for T if $\{gx \mid g \in \mathcal{G}\}$ is norm- dense in H. We call such phenomena cyclic phenomena, we point out that this term is used in the literature to mean the following three cases:-

1. $\mathcal{G} = A$ (Cyclicity).
2. $S = \mathbb{C}$ and $\mathcal{G} = \mathcal{G}(\mathbb{C}, T)$ (Supercyclicity).
3. $S = \{1\}$ and $\mathcal{G} = \mathcal{G}(\{1\}, T)$ (Hypercyclicity).

In this paper we restrict our study on a multiplication semigroup of $\mathbb{C}$ with 1.Clearly, every hypercyclic operator is G – cyclic, and every G – cyclic operator is supercyclic.

In [9], [10] the authors studied G-cyclicity when, $S = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, $S = \mathbb{D} = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$, $S = \mathbb{B}^c = \{\lambda \in \mathbb{C} \mid |\lambda| \geq 1\}$. An example was given for an operator which G-cyclic over $\mathbb{D}$ but not over $\mathbb{B}^c$ and vise versa.

This paper consists of three sections. In section one; we list basic properties of G – cyclicity. In section two, we give necessary and sufficient conditions for G – cyclicity. In section three, we investigate the spectrum of G – cyclic operators.

Preliminaries

In this section we introduce the definition of G – cyclicity and give some basic properties.

Definition. Let S be a multiplication semigroup of $\mathbb{C}$ with 1, an operator T on a separable complex Hilbert space H is called G – cyclic over S if there is a vector x in H such that the set $\{\alpha T^k x \mid \alpha \in S, k \geq 0\}$ is norm – dense in H. In this case x is called a G – cyclic vector for T over S. It is clear that we may assume $0 \notin S$.

Next we fix notation required for the discussion.

Notation Let S be a multiplication semigroup of $\mathbb{C}$ with 1, and $T \in B(H)$:-

1. $\mathcal{G}C_S(T) = \{x \in H \mid x$ is $\mathcal{G}$-cyclic vector for T over $S\}$
2. $\mathcal{G}C_S(H) = \{T \in B(H) \mid T$ is $\mathcal{G}$-cyclic operator over $S\}$
3. $Sorbt(T,x) = \{\alpha T^k x \mid \alpha \in S, k \geq 0\}$
4. $|S| = \{|\alpha| \mid \alpha \in S\}$.
5. $S^{-1} = \left\{\frac{1}{\alpha} \mid \alpha \in S\right\}$.

Note that $(S^{-1}, \cdot)$ *is a semigroup of* $\mathbb{C}$ *with 1 and* $(S^{-1})^{-1} = S$.

Remarks.

1. $x \in \mathcal{G}C_S(T)$ if and only if $Sorbt(T,x)^{\perp} = \{0\}$.
2. Clearly, from (1.1) every hypercyclic operator (vector) is G – cyclic, and every G – cyclic is supercyclic.

Since the range of a supercyclic operator T, R(T), is dense [3] then we have:-

6. **Proposition.** The range of a G – cyclic operator T on H is a dense in H.

We begin with an easy observation. Compare the following result with [7].

7. **Proposition.** Let $x \in \mathcal{G}C_S(T)$, then $\inf\{\gamma \|T^n x\| \mid n \geq 0, \gamma \in |S|\} = 0$ and $\sup\{\gamma \|T^n x\| \mid n \geq 0, \gamma \in |S|\} = \infty$

Proof. Let $x \in \mathcal{G}C_S(T)$, and assume that $\inf\{\gamma \|T^n x\| \mid n \geq 0, \gamma \in |S|\} = m > 0$. Since $0 \in H$, then there are sequences $\langle \alpha_k \rangle$ in S and $\langle n_k \rangle$ in N such that $|\alpha_k| \|T^{n_k} x_k\| \to 0$. Hence there is $j \in \mathbb{N}$ such that $|\alpha_k| \|T^{n_j} x_k\| < m$ for all $j > k$, a contradiction.

Now assume that $\sup\{\gamma \|T^n x\| \mid n \geq 0, \gamma \in |S|\} = M < \infty$, and let $y \in H$ such that $\|y\| > M$. Since $x \in \mathcal{G}C_S(T)$, then there exist sequences $\langle n_k \rangle$ in $\mathbb{N}$, $\langle \alpha_k \rangle$ in S such that $|\alpha_k| \|T^{n_k} x_k\| \to \|y\|$. Thus we get a sequence of $\{\gamma \|T^n x\| \mid n \geq 0, \gamma \in |S|\}$ that converges to y, hence $\|y\| \leq M$ [12, p57], a contradiction. □

For special case, when S is bounded, one can easily prove the following corollaries.

8. **Corollary.** If S is a bounded semigroup and $x \in \mathcal{G}C_S(T)$, then $\sup\{\|T^n x\| \mid n \geq 0\} = \infty$
9. **Corollary.** Let $T \in B(H)$, and S be a bounded semigroup. If $\|T\| < 1$, then $T \notin \mathcal{G}C_S(H)$.

Remark. The backward shift $B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is not G – cyclic over any bounded semigroup, since $\|B\| = 1$.

One can prove easily the following results that implies that if $x \in \mathcal{G}C_S(T)$, then $\alpha T^n x \in \mathcal{G}C_S(T)$ for all $\alpha \in S$ and $n \geq 0$.

10. **Proposition.** Let $x \in \mathcal{G}C_S(T)$, and let $F \in B(H)$ such that $FT = TF$ and $R(F)$ is dense in H. Then $Fx \in GC_S(T)$.

Terming now to the similarity of G-cyclic operator.

11. **Proposition.** Let H, K be Hilbert spaces, let $T \in B(H)$ and $F \in B(K)$, and let $X : H \to K$ be a bounded linear transformation such that $R(X)$ is dense in K, and $FX = XT$. If $T \in \mathcal{G}C_S(H)$, then $F \in \mathcal{G}C_S(K)$. In particular, if $T, F \in B(H)$ are similar operators, then $T \in \mathcal{G}C_S(H)$ if and only if $F \in \mathcal{G}C_S(H)$

Proof. Let $y \in \mathcal{G}C_S(T)$, then $Sorbt(T,y)$ is dense in H, thus

$$\overline{[Sorbt(F,Xy)]} = \overline{[\{\alpha F^n(Xy) \mid n \geq 0, \alpha \in S\}]} = \overline{[\{X(\alpha T^n y) \mid n \geq 0, \alpha \in S\}]} = X\overline{(H)} = K \quad \square$$

Next we turn our attention to the direct sum of G-cyclic operators. Compare with [9]. The proof is left to the reader.

12. **Proposition** . Let $\{H_i\}$ be family of Hilbert spaces, let $T_i \in B(H_i)$ for all i. If $\oplus T_i \in \mathcal{G}C_S(\oplus H_i)$, then $T_i \in \mathcal{G}C_S(H_i)$ for all i

1. Necessary and Sufficient Conditions for G-Cyclicity

The goal of this section is to give a characterization for G-cyclic operators. We first characterize the set of all G-cyclic vectors.

2.1. Proposition. Let $T \in B(H)$. Then $\mathcal{G}C_S(T) = \bigcap_k \left(\bigcup_{\alpha \in S} \bigcup_n T^{-n}\left(\frac{1}{\alpha} \mathrm{U}_k\right)\right)$, where $\{\mathrm{U}_k\}_1^\infty$ is a countable base for the topology on H.

Proof. Since H is separable, then let $\{\mathrm{U}_k\}_1^\infty$ be a countable base for the topology on H, $x \in \mathcal{G}C_S(T)$ if and only if $\{\alpha T^n x \mid \alpha \in S, n \geq 0\}$ is dense in H, if and only if for all $k \geq 1$, there is $\alpha \in S, n \in \mathbb{N}$ such that $\alpha T^n x \in \mathrm{U}_k$, if and only if for all $k \geq 1$, there is $\alpha \in S, n \in \mathbb{N}$ such that $T^n x \in \frac{1}{\alpha}\mathrm{U}_k$, if and only if $x \in \bigcap_k \left(\bigcup_{\alpha \in S} \bigcup_n T^{-n}\left(\frac{1}{\alpha} \mathrm{U}_k\right)\right)$ □

Recall that a countable intersection of open sets is G_δ - set.

2.2. Corollary. If the set of G-cyclic vectors over S is not empty, then it is G_δ - set in H.

Proof: Let $x \in \mathcal{G}C_S(T)$. By (1.7) $\alpha T^n x \in \mathcal{G}C_S(T)$ for all $n \geq 0$, $\alpha \in S$. Thus $Sorb(T, x) \subseteq \mathcal{G}C_S(T)$, hence $\mathcal{G}C_S(T)$ is dense in H. Now by (2.1) $\mathcal{G}C_S(T) = \bigcap_k \left(\bigcup_{\alpha \in S} \bigcup_n T^{-n}\left(\frac{1}{\alpha} \mathrm{U}_k\right)\right)$ where $\{\mathrm{U}_k\}_1^\infty$ is a countable base for the topology on H. The desired result follows by the continuity of T and because $\frac{1}{\alpha}\mathrm{U}_k$ is open for all k, and all $\alpha \in S$. □

The following result is a characterization for G-cyclic operators, compare with [4].

2.3. Theorem. Let $T \in B(H)$. The following statements are equivalent:-

1. $T \in \mathcal{G}C_S(H)$.
2. For each non-empty open sets U,V, there are $\alpha \in S, n \in \mathbb{N}$ such that $T^n(\alpha U) \cap V \neq \varnothing$.
3. For each $x, y \in H$, there are sequences $\langle x_k \rangle$ in H, $\langle n_k \rangle$ in $\mathbb{N}$, $\langle \alpha_k \rangle$ in S such that $x_k \to x$ and $T^{n_k}\alpha_k x_k \to y$.
4. For each $x, y \in H$, and each neighborhood W for zero in H, there are $z \in H$, $n \in \mathbb{N}$, $\alpha \in S$ such that $x - z \in W$ and $T^n \alpha z - y \in W$.

Proof. 1)⇒ 2): Let $\{W_k\}_1^\infty$ be a countable base for the topology on H. By (2.1):
$\mathcal{G}C_S(T) = \bigcap_k \left(\bigcup_{\alpha \in S} \bigcup_n T^{-n}\left(\frac{1}{\alpha} W_k\right)\right)$ for all $k \geq 1$, put $M_k = \bigcup_{\alpha \in S} \bigcup_n T^{-n}\left(\frac{1}{\alpha} W_k\right)$. By (2.2) M_k is dense for all $k \geq 1$. Since U is open, then there are $n \in \mathbb{N}$, $\alpha \in S$ such that $U \cap T^{-n}\left(\frac{1}{\alpha} V\right) \neq \varnothing$ where $V = \cup W_i$; $W_i \in \{W_k\}_1^\infty$. Hence $T^n(\alpha U) \cap V \neq \varnothing$.

2)⇒ 3): Let $x, y \in H$ for all $k \geq 1$, let $B_k = B(x, 1/k)$, $B'_k = B(y, 1/k)$. By (2) we get sequences $\langle n_k \rangle$ in $\mathbb{N}$, $\langle \alpha_k \rangle$ in S, $\langle x_k \rangle$ in H such that $x_k \in B_k$ and $T^{n_k}\alpha_k x_k \in B'_k$ for all $k \geq 1$. Then $\|x_k - x\| < 1/k$ and $\|T^{n_k}\alpha_k x_k - y\| < 1/k$ for all $k \geq 1$. The desired result follows by letting $k \to \infty$.

3)⇒ 4): Let $x, y \in H$, let W be a neighborhood for zero in H. By (3), there are sequences $\langle x_k \rangle$ in H, $\langle n_k \rangle$ in $\mathbb{N}$, $\langle \alpha_k \rangle$ in S such that $x_k \to x$ and $T^{n_k}\alpha_k x_k \to y$. Hence there is $k \in \mathbb{N}$ such that $x_k - x \in W$ and $T^{n_k}\alpha_k x_k - y \in W$. Take $z = x_k$.

4)⇒ 3): Let $x, y \in H$. for all $k \geq 1$, let $B_k = B(0, 1/k)$. By (4) we get sequences $\langle z_k \rangle$ in H, $\langle n_k \rangle$ in $\mathbb{N}$, $\langle \alpha_k \rangle$ in S such that

$z_k - x \in B_k$ and $T^{n_k}\alpha_k z_k - y \in B_k$ for all $k \geq 1$. Therefore $\|z_k - x\| < 1/k$ and $\|T^{n_k}\alpha_k z_k - y\| < 1/k$ for all $k \geq 1$. Let $k \to \infty$.

3)⇒ 1): Since H is separable, then there is a countable base set, say $\{x_j\}_{j \in \mathbb{N}}$. Set $F(j,k) = B(x_j, 1/k)$ for some $j \in \mathbb{N}$, $k \geq 1$. One can show that the collection of all $F(j,k)$ is a base topology on H. Now by the same argument used in the proof of (2.1), we get

$$\mathcal{G}C_S(T) = \bigcap_j \bigcap_k \left(\bigcup_{\alpha \in S} \bigcup_n T^{-n}\left(\frac{1}{\alpha}F(j,k)\right)\right).$$

By Baire's Theorem, it is enough to prove $\bigcup_{\alpha \in S}\bigcup_n T^{-n}\left(\frac{1}{\alpha}F(j,k)\right)$ is dense in H for all $k \geq 1$ and $j \in \mathbb{N}$. For a fixed k, j, let $y \in H$. By (3), there are sequences $\langle z_\ell \rangle$ in H, $\langle \lambda_\ell \rangle$ in S, and $\langle n_\ell \rangle$ in $\mathbb{N}$ such that $z_\ell \to y$ and $T^{n_\ell}\lambda_\ell z_\ell \to x_j$. Thus there is $m > 0$ such that $\|T^{n_\ell}\lambda_\ell z_\ell \to x_j\| < 1/k \ \forall \ell > m$. Hence

$$z_\ell \in \bigcup_{\alpha \in S}\bigcup_n T^{-n}\left(\frac{1}{\lambda_\ell}F(j,k)\right) \forall \ell > m.$$

Therefore there is a sequence $\langle z'_\ell \rangle$ of the sequence $\langle z_\ell \rangle$ such that $z_\ell \in \bigcup_{\alpha \in S}\bigcup_n T^{-n}\left(\frac{1}{\lambda_\ell}F(j,k)\right)$ and $z'_\ell \to y$.

In what follows we give an application of the last theorem.

2.4. **Proposition.** Let $T \in B(H)$ be an invertible operator. $T \in \mathcal{G}C_S(H)$ if and only if $T^{-1} \in \mathcal{G}C_{S^{-1}}(H)$.

Proof. Let $x, y \in H$. Since $T \in \mathcal{G}C_S(H)$, then by (2.3) for all neighborhoods V for zero in H, there are $z \in H$, $n \in \mathbb{N}$, $\alpha \in S$ such that $z - x \in V$ and $\alpha T^n z - y \in V$. Set $u = \alpha T^n z$, then $u - y \in V$ and $\frac{1}{\alpha}T^{-n}u - x \in V$. Thus $T^{-1} \in \mathcal{G}C_{S^{-1}}(H)$ (2.3). ❐

Next we will give another characterization of G-cyclic operators.

2.5. **Proposition.** The operator $T \in \mathcal{G}C_S(H)$ if and only if the set $\{(x, \alpha T^n x) : x \in H, n \geq 0, \alpha \in S\}$ is dense in $H \oplus H$.

Proof. ⇒) Let $(y,z) \in H \oplus H$, and let $\varepsilon > 0$. Since $T \in \mathcal{G}C_S(H)$, then by (2.3) there are $w \in H, n \geq 0, \alpha \in S$ such that $\|w - z\| < \varepsilon/2$ and $\|\alpha T^n w - z\| < \varepsilon/2$. Hence

$$\|(w, \alpha T^n w) - (y,z)\|^2 = \|w - y\|^2 + \|\alpha T^n w - z\|^2 < \varepsilon^2.$$

⇐) Let $z, y \in H$, and let $\varepsilon > 0$. By the conditions there is $t > 0$ and sequences $\langle w_k \rangle$ in H, $\langle \alpha_k \rangle$ in S, and $\langle n_k \rangle$ in $\mathbb{N}$ such that $\|(z,y) - (w_k, \alpha_k T^{n_k} w_k)\|^2 < \varepsilon^2$ for all $k > t$. Hence $\|z - w_k\| < \varepsilon$ and $\|y - \alpha_k T^{n_k} w_k\| < \varepsilon$ for all $k > t$. Then let $k \to \infty$ we get, $w_k \to z$ and $\alpha_k T^{n_k} w_k \to y$. Thus by (2.3) $T \in \mathcal{G}C_S(H)$

Now we will study a condition that implies G-cyclic,

2.6. **Proposition.** Let $T \in B(H)$, let U, V be a nonempty open sets in H, and let W be a neighborhood for zero in H. If there are $n \geq 0$, $\alpha \in S$ such that $T^n \alpha U \cap W \neq \emptyset$ and $T^n \alpha W \cap V \neq \emptyset$, then $T \in \mathcal{G}C_S(H)$.

Proof. We will verify (2.3). Let $x, y \in H$. for all $k \geq 1$, Let $B_k = B(x, 1/k)$, $B'_k = B(y, 1/k)$. By our assumption, there exist sequences $\langle n_k \rangle \in \mathbb{N}$, $\langle \alpha_k \rangle \in S$, $\langle w_k \rangle \in W$ and $\langle z_k \rangle \in H$ such that $z_k \in B_k$, $T^{n_k}\alpha_k z_k \in W$ and $T^{n_k}\alpha_k w_k \in B'_k$ for all $k \geq 1$. Therefore

$z_k \to x$, $\quad T^{n_k}\alpha_k z_k \to 0$

$w_k \to 0$, $\quad T^{n_k}\alpha_k w_k \to y$

The proof complete by taking $x_k = z_k + w_k$ for all $k \geq 1$.

2. Spectral Properties of G-cyclic Operators

In this section we discuss the properties of the spectrum of G-cyclic operator. It is known [6] that if T is supercyclic, then T^* has at most one eigenvalue, hence have

Proposition. Let $T \in \mathcal{G}C_S(H)$. Then T^* has at most one eigenvalue with modulus:

1) Greater than one, if S is bounded above.
2) Less than one, if S is bounded below.

Proof.

1) Since $T \in \mathcal{G}C_S(H)$, then T is supercyclic, thus by [6] $\sigma_p(T^*)$ contains at most one non-zero eigenvalue, say λ. Hence there is a unit vector $z \in H$ such that $T^*z = \lambda z$. Let $x \in \mathcal{G}C_S(T)$, it is easy to prove that $\left\{\left\langle \mu T^n x, z\right\rangle \middle| n \geq 0, \mu \in S\right\}$ is dense in $\mathbb{C}$ ...... (*)

Note that for all $n \geq 1$, $\left|\left\langle \mu T^n x, z\right\rangle\right| = |\mu| \left|\left\langle T^n x, z\right\rangle\right|$. Since S is bounded above, then $|\mu| \leq M$ for some M. Now assume that $|\lambda| \leq 1$ Hence $\left|\left\langle \mu T^n x, z\right\rangle\right| < M \left|\left\langle x, z\right\rangle\right|$, a contradiction with (*).

2) Similar.

It would be useful to say something about Weyl-spectrum, $\sigma_w(T)$, of G-cyclic operators.

Corollary. Let $T \in \mathcal{G}C_S(H)$, then Weyl-spectrum of T is the spectrum of T except possibly one element of modulus:

1) Greater than one, if S is bounded above.
2) Less than one, if S is bounded below.

Proof:

1) Since $\sigma(T^*) - \sigma_w(T^*) \subseteq \sigma_p(T^*)$ [1], then by (3.1) either $\sigma_w(T^*) = \sigma(T^*)$ or $\sigma_w(T^*) = \sigma(T^*) - \{\lambda\}; |\lambda| \geq 1$ hence either $\sigma_w(T) = \sigma(T) - \{\bar{\lambda}\};$ $|\lambda| \geq 1$ or $\sigma_w(T) = \overline{\sigma_w(T^*)} = \sigma(T)$.

2) Similar.

For supercyclic operator T, it is shown in [6] that $\sigma(T) \bigcup \partial(rD)$ is connected for some $r \geq 0$. Therefore if T is G cyclic, then $\sigma(T) \bigcup \partial(rD)$ is connected for some $r \geq 0$. A question arises: Is there any restriction on r ?

Proposition. Let $T \in B(H)$. then $T \notin \mathcal{G}C_S(H)$ if one of the following statements holds:

1. S is bounded, and $\sigma(T)$ has a component σ such that $\sigma \subset B(0,1)$.
2. S^{-1} is bounded and $\sigma(T)$ has a component σ such that $\sigma \subset \{\lambda \mid |\lambda| > 1\}$.

Proof.

1) Assume $T \in \mathcal{G}C_S(H)$. If $\sigma(T)$ is connected and $\sigma(T) \subset B(0,1)$, then $\lim_{n\to\infty} \|T^n x\| \to 0$ for all $x \in H$ [5]. Thus $\sup\left\{\|T^n x\| \mid n \geq 0\right\} \neq \infty$, a contradiction with (1.5). Now if σ is a component of $\sigma(T)$ such that $\sigma \subset B(0,1)$ Then by Risez decomposition Theorem $T = T_1 \oplus T_2$ such that $\sigma(T_1) = \sigma$. But $T_1 \in \mathcal{G}C_S(H)$ (1.8), hence by the same argument of the first part of this proof we get a contradiction.

2) By using (2.4) and the same argument of the proof of the first part. □

Corollary. Let $T \in \mathcal{G}C_S(H)$

1) S is bounded, then $\sigma(T) \cap rD$ is connected for all $r \geq 1$.
2) S^{-1} is bounded, then $\sigma(T) \cap (rB)^c$ is connected for all $r \geq 1$.

Proof:

1) Let $r \leq 1$, assume $\sigma(T) \cap rD$ is not connected. Then there is a closed and open subset σ of $\sigma(T) \cap rD$, hence $\partial\sigma \neq \emptyset$, thus $\sigma(T) \cap \partial(rD) = \emptyset$, hence $\sigma \subset (rB)$. Since $r \leq 1$, then $\sigma \subset B(0,1)$, a contradiction with (3.3).

2) Similar. □

Let's give a simple application of proposition (3.3).

3.2. Corollary. Let S be a bounded semigroup, then a quasinilpotent operator can not be G-cyclic over S.

3.3. Corollary. A compact operator can not be G-cyclic over any bounded semigroup S.

Proof. Let T be a compact operator which is G-cyclic over S. Since every $\lambda \in \sigma(T)$; $\lambda \neq 0$, is an eigenvalue for T, then $0 \neq \bar{\lambda} \in \sigma_p(T^*)$. But by (3.2) either $\sigma_p(T^*) = \emptyset$ or $\sigma_p(T^*) = \{\lambda\}$;

$|\lambda| > 1$. Hence either $\sigma(T) = \{0\}$ or $\sigma(T) = \{0, \lambda\}$; $|\lambda| > 1$. If $\sigma(T) = \{0\}$, then we get a contraduction with (3.4). If $\sigma(T) = \{0, \lambda\}$; $|\lambda| > 1$, then since λ is an isolated point in $\sigma(T)$, hence $\{0\}$ is a component for $\sigma(T)$, a contradiction with (3.4).

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الخلاصة

ليكن H فضاء هلبرت على حقل الاعداد العقدية قابل للفصل غير منته البعد و S شبه زمرة جدائية من [illegible] وتحتوي على 1. لتعميم مفهوم فوق الدوارية، عرفنا $\mathcal{G}$-دواري، ونعني، يقال للمؤثر الخطي T انه دواري من النمط $\mathcal{G}$ اذا وجد متجه x في H بحيث ان المجموعة $\{\alpha T^n x \mid \alpha \in S, n \geq 0\}$ كثيفة في H. في بحثنا هذا عرضنا بعض الخواص الاولية للمؤثرات الدوارية من النمط $\mathcal{G}$، ودرسنا الشروط الضرورية والكافية لجعل المؤثر الخطي مؤثراً دوارياً من النمط $\mathcal{G}$. وأخيراً درسنا بعض خواص طيف المؤثرات الدوارية من النمط $\mathcal{G}$.