A Note on Pointwise Projective Modules

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Abstract

Let R be any ring, and M is a (left) R-module. M is said to be pointwise projective, if for each epimorphism $\alpha : A \to B$, where A and B are any R modules, and for each homomorphism $\beta : M \to B$, then for every $m \in M$, there exists a homomorphism $\varphi : M \to A$, which may depend on m, such that $\alpha \circ \varphi(m) = \beta(m)$. In this paper we study some properties of pointwise projective modules.

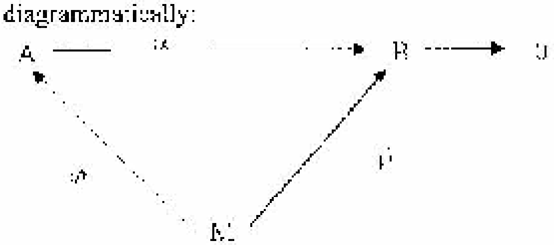
Introduction

Projective modules have been studied extensively, and many generalizations for the concept of a projective module were given, for example, improjective modules [2], quasi-projective modules [10] and nearly-projective modules [7]. Another generalization to the concept of a projective module is the concept of locally projective module, Zimmermann [13], and Azumaya [1]. In this paper we will call a locally projective module, pointwise projective module, and is abreviated by pwp.module, and we study some of the properties of these modules.

In this paper R stands for a commutative ring with 1, unless otherwise stated, and a module means a (left) R-module.

1. Pointwise projective modules: Basic results

Let R be any ring, not necessary commutative, and M is a (left) R-module. We recall that the R-module M is said to be pointwise projective (abbreviated by pwp.), if for each epimorphism $\alpha : A \to B$, where A and B are any R modules, and for each homomorphism $\beta : M \to B$, then for every $m \in M$, there exists a homomorphism $\varphi : M \to A$, which may depend on m, such that $\alpha \circ \varphi(m) = \beta(m)$ diagrammatically:



See [1]. Note that every projective module is pwp., however, the converse is false as we will see later. Azumaya and Zimmermann gave the following characterization for pwp. modules [1] and [13]

1.1. Proposition: Let M be any R-module, the following statements are equivalent:

1. M is a pointwise projective module.
2. Each epimorphism $\alpha : A \to M$, where A is any R-module, is pointwise split (i.e. for each $b \in B$, there exists a homomorphism $\beta : M \to A$, which may depend on b, such that $\alpha \circ \beta(b) = b$).
3. For each $m \in M$, there are $x_1, x_2, \ldots, x_n \in M$ and $\varphi_1, \varphi_2, \ldots, \varphi_n \in M^*$ such that $m = \sum_{i=1}^{n} \varphi_i(m) x_i$.
4. For each $m_1, m_2, \ldots m_t \subset M$, there are $x_1, x_2, \ldots, x_n \in M$ and $\varphi_1, \varphi_2, \ldots, \varphi_n \in M^*$ such that $m_i = \sum_{k=1}^{n} \varphi_k(m_i) x_k, \forall 1 \le i \le t$

Remark: Part (4) in the last proposition is called the Dual-Basis-Lemma for pwp., modules.

Next we give another characterization of pwp. modules.

1.2. Proposition: Let M be any R-module, then M is pwp.if and only if for each free R-module F, every epimorphism $\alpha : F \to M$ is pointwise split.

Proof: ($\Rightarrow$) Clear.

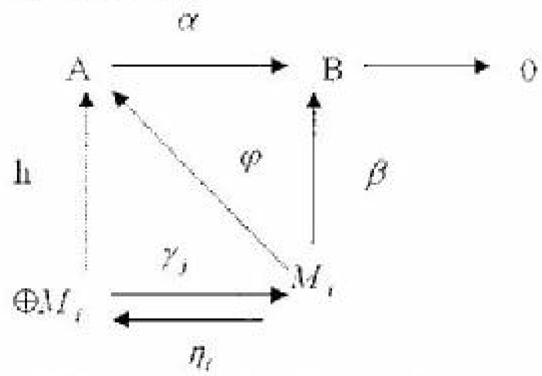
($\Leftarrow$) Let A be any R-module with epimorphism $\beta : A \to M$, let $\{a_i \mid i \in I\}$ be a set of generators for A, and let F be a free R-module with basis $\{z_i \mid i \in I\}$. Define $\gamma : F \to A$ on the basis by $\gamma(z_i) = a_i$. It is clear that γ is an epimorphism. Thus $\beta \circ \gamma : F \to M$ is an epimorphism, hence it is a pointwise split, i.e. for each $m \in M$, there exists a homomorphism $\varphi : M \to F$, φ may depend on m, such that $\beta \circ \gamma \circ \varphi(m) = m$. Hence by (1.1), M is pwp.□

It is known that $\bigoplus_{i \in I} M_i$ is a projective R-module if and only if M_i is a projective R-module $\forall i \in I$ [4;p.118]. A similar result holds for pwp. modules.

1.3. Proposition: $\bigoplus_{i \in I} M_i$ is a projective R-module if and only if M_i is a pwp. R-module $\forall i \in I$.

Proof: Let $\alpha \in \bigoplus_{i \in I} M_i$ be written as $\alpha = [m_i]; m_i \in M_i \forall i \in I$.

(⇒) In the diagram:



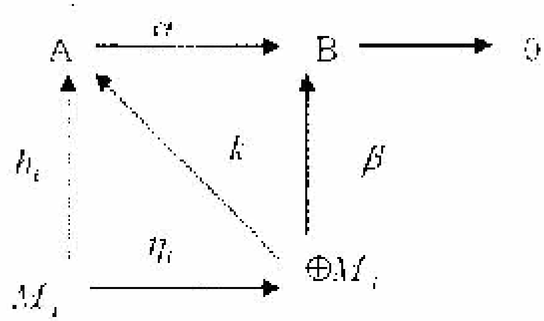
Define $\gamma_j : \bigoplus_{i \in I} M_i \to M_j$ as $\gamma_j([m_i]) = m_j$. By assumption, for all $f \in \bigoplus_{i \in I} M_i$ there exists a homomorphism $h : \bigoplus_{i \in I} M_i \to A$ such that $\alpha \circ h(f) = \beta \circ \gamma_j(f)$. Define $\eta_j : M_i \to \bigoplus_{i \in I} M_i$ as

$$[\eta_i(m)](j) = \begin{cases} m & i = j \\ 0 & i \neq j \end{cases}.$$

Define $\varphi : M_j \to A$ as $\varphi(m) = h \circ \eta_i(m)$. Let $m \in M_i$,

$\alpha \circ \varphi(m) = \beta \circ \gamma_i \circ \eta_i(m) = \beta(m)$, because $\gamma_j \circ \eta_i = I$.

(⇐) In the diagram:



By assumption, for each $m \in M_i$, there exists a homomorphism $h_i : M_i \to A$ such that $\alpha \circ h_i(m) = \beta \circ \eta_i(m)$. Let $f \in \bigoplus_{i \in I} M_i$, define $k : \bigoplus_{i \in I} M_i \to A$ as $k(f) = \sum_{i \in I} h_i f(i)$, clearly k is well defined. Now,

$$\alpha \circ k(f) = \sum_{i \in I} \alpha \circ h_i(f) = \beta\left(\sum_{i \in I} \eta_i f(i)\right) = \beta(f)$$

, by the definition of η_i.□

Recall that an R-module M is called torsionless if $\bigcap Ker f = 0$, where the intersection is taken over all $f \in M^* = Hom(M, R)$. It is known that every projective module is torsionless. The Dual-Basis Lemma for pwp. Modules gives

1.4. Proposition: Let M be a pwp. R-module, then M is torsionless.

Note, form now on we will assume that the ring R is commutative with 1. We shall investigate the behavior of the pwp. property under localization.

1.5. Proposition: Let M be a pwp. R-module, let S be a multiplicatively closed subset of R, then $S^{-1}M$ is a pwp. $S^{-1}R$-module. In particular, M_P is a pwp. R_P-module for each prime ideal P of R.

Proof: Let $n/s \in S^{-1}M$, by (1.1) $n = \sum_{i=1}^{k} \varphi_i(n) x_i$, $x_i \in M$, $\varphi_i \in M^*$. Then $\frac{n}{s} = \sum_{i=1}^{k} \frac{\varphi_i(n)}{s} x_i$. Define $g_i : S^{-1}M \to S^{-1}R$ by $g_i(n/s) = \varphi_i(n)/s, \forall i, 1 \le i \le k$. Note that $\frac{n}{s} = \sum_i g_i(\frac{n}{s}) \frac{x_i}{1}$. Hence by (1.1), $S^{-1}M$ is pwp.□

Remark: The converse of the last proposition is not true, see example (3.5)

2. The Jacobson Radicals of Pointwise Projective Modules

Azumaya [1] proved that if M is a pwp. R-module, then $J(M) = J(R)M$ thus every pwp. module has a maximal submodule. Also he proved that $J(M) = M$ only if $M = 0$, thus we have.

2.1. Proposition: Let M be a pwp. R-module. Then M has a unique maximal submodule if and only if $J(M)$ is a maximal submodule of M.

Proof: Since M is a pwp. then M has a maximal submodule, say U. If U is unique, then $U = J(M)$. Now, if $J(M)$ is maximal in M, then $J(M)$ is a unique maximal submodule□

Recall that a non-zero submodule N of an R-module M is called large if $N \bigcap K \neq 0$ for every non-zero submodule K of M [4;p.106]. A large ideal of R is just a large submodule of the R-module. R Put $Z(M) = \{m \in M \quad ann(m) \text{ is large in } R\}$, $Z(M)$ is called the singular submodule of M [4;p.138]. It is known that if A_i is large in M; $i = 1, 2, \ldots, n$, then $\bigcap_{i=1}^{n} A_i$ is large in M [4;p.109]. We have:

2.2. Proposition: Let M be a pwp. R-module, then $Z(M) = Z(R)M$.

Proof: Let $m \in Z(M)$, by (1.1) $m = \sum_{i=1}^{n} \varphi_i(m) y_i$; $y_i \in M$ and $\varphi_i \in M^*$. Clearly, $ann(m) \subset ann(\varphi_i(m))$, hence $ann(\varphi_i(m))$ is large in R. Thus $\varphi_i(m) \in Z(R)$, therefore $m \in Z(R)M$. Now, let $y \in Z(R)M$, then $y = \sum_{i=1}^{n} r_i m_i$; $r_i \in Z(R)$, $m_i \in M$. Thus $\bigcap_{i=1}^{n} ann(r_i)$ is large in R and it is a subset of $ann(y)$, i.e. $ann(y)$ is large in R. Hence $y \in Z(M)$.□

The following results are used frequently in this work.

2.3. Lemma: Let M be a pwp. R-module, and let I be an ideal of R, such that $M = IM$, then $\forall a \in M$, $ann(a) + I = R$.

Proof: Assume that $ann(a) + I \neq R$, then there exists a maximal ideal A of R such that $ann(a) + I \subseteq A$. Hence $M = IM \subseteq AM$ and thus $M = AM$. By (1.5) and [1] $M_A = J(M_A)$, and by [1] $M_A = 0$. Hence there exists $r \in R - A$ such that $ra = 0$, and this is contradiction.□

2.4. Corollary: Let M be a pwp. R-module, and let I be an ideal of R, such that $M = IM$, then $\forall a \in M$, $a \in Ia$.

Proof: By (2.3) $\forall a \in M$, $ann(a) + I = R$, thus there exist $r \in ann(a)$ and $b \in I$ such that $r + b = 1$, hence $ba = a$.□

§3 The Trace of Pointwise Projective Modules

Recall that the trace of an R-module M is defined by $\sum f(M)$, where the sum is taken over all $f \in M^*$, and is denoted by $T(M)$ (simply T). It is known that if M is a projective R-module, then $TM = M$, $T^2 = T$, and $ann(T) = ann(M)$ [3].

Proposition: **Let M be a pwp. R-module, then:**

1. $TM = M$.
2. $T^2 = T$.
3. $ann(T) = ann(M)$.
4. T is a pure ideal.

Proof:

1. Clearly $TM \subseteq M$. It follows by (1.1) $M \subseteq TM$.
2. Clearly $T^2 \subseteq T$. Let $t \in T$, then $t = \sum_{i=1}^{k} l_i(m_i)$: $l_i \in M^*$ and $m_i \in M$. By assumption $m_i = \sum_{j=1}^{n} \varphi(m_i) y_j$; $\forall i, 1 \leq i \leq k$, where $y_j \in M$, $\varphi_j \in M^*$. Thus $t = \sum_i \sum_j \varphi_j(m_i) l_i(y_j)$, hence $t \in T^2$.
3. The proof is simple.
4. Let $x \in T$, then $x = \sum_i \varphi_i(a_i)$; $\varphi_i \in M^*$ and $a_i \in M$.

By (2.4) and (1) above $a_i = ya_i$; $y \in T$. Thus

$$x = \sum_i \varphi_i(ya_i) = y(\sum_i \varphi_i(a_i)) = yx .$$

□

Before we give the last proposition in this section we recall some notation. For each $a \in M$, $D_a = ann(ann(a))$, $\bar{D} = \sum_{a \in M} D_a$. It is easy to see that if M is any module, then $T \subseteq D_a$.

3.1. Proposition: Let M be a pwp. R -module, then $T = \bar{D}$.

Proof: Let $m \in M$, $x \in D_m$. By (2.3) and (3.1) there exist $r \in ann(m)$ and $t \in T$ such that $r + t = 1$ thus $xt = x$, hence $D_m \subseteq T$, and $T = \bar{D}$.□

3.2. Example. For the Z -module Q of rationals, $T(Q) = 0$ and $D_a(Q) = Z$ for each $a \in Z$. **This shows that the last proposition is false without Pointwise projectivity.**

4. Projective Modules and Pointwise Projective Modules

As we saw in section one every projective module is a pwp. Module, but the converse is not true as we will see in (5.4). In this section we study conditions under which pwp. Modules are projective modules.

4.1. Proposition: Every finitely generated pwp. R -module is a projective module.

Proof: Let $\{m_1, \ldots, m_t\}$ be a finite set of generators of the R -module M .

By (1.1) $m_j = \sum_{i=1}^{k} \varphi_i(m_j)x_i$; $x_i \in M$, $\varphi_i \in M^* \; \forall j, 1 \le j \le t$. Let $y \in M$, then

$$y = \sum_{j=1}^{t} r_j m_j = \sum_{j=1}^{t}\sum_{i=1}^{k} r_j \varphi_i(m_j)x_i = \sum_i \varphi_i(y)x_i$$

Thus by the Dual-Basis Lemma M is a projective module □

Before we give other conditions under which pwp. Modules are projective modules, we start by the following notation and proposition; let S be the endomorphism ring of an R -module M . Define $M \times M^* \to S$ as $[m, f] = f_m$ where $f_m(a) = f(a)m \; \forall a \in M$. Let Δ be the ideal of S generated by Im [12]. Let $ann_S(m) = \{f \in S \mid f(m) = 0\}$ where $m \in M$.

4.2. Proposition: Let M be a pwp. R -module, then $\forall m \in M$, $ann_S(m) + \Delta = S$.

4.3. Proof: Let $m \in M$, by (1.1) $m = \sum_{k=1}^{n} \varphi_k(m)y_k$; $\varphi_k \in M^*$ and $y_k \in M$. Thus $m = \sum_{k=1}^{n} [y_k, \varphi_k](m)$. Hence $I - \sum_k [y_k, \varphi_k] \in ann_S(m)$ when I is the identity endomorphism of M . Therefore $ann_S(m) + \Delta = S$ □

4.4.

4.5. It is known that an R -module M is finitely generated projective if and only if $S = \Delta$ [9]. By this statement and (4.2) we have,

4.6.

4.7. Proposition: Let M be a pwp. R -module that contains an element which is S -torsion free then M is a finitely generated projective R -module.

5. Regular Modules and Pointwise Projective Modules

Let M be an R -module, M is called Z -regular if for each $x \in M$, there exists $h \in M^*$ such that $x = h(x)x$ [11]. By using (1.1), we have;

5.1. **Proposition:** Every Z -regular module is a pwp. R -module.

Zimmermann [13] proved that every pwp. Module over a regular ring is a Z -regular module. We will obtain this result by a different way. But first recall that an R -module M is called F -regular if every submodule of M is pure [5]. It is known that if M is a projective F -regular R -module, then M is Z -regular [5]. Let us note that a Z -regular module may not be projective [11]. However we have

5.2. Proposition: Let M be an R -module. Then M is an F -regular pwp. Module if and only if M is Z -regular.

Proof: ($\Leftarrow$) It follows by (5.1) and [5].

($\Rightarrow$) Let M be an F -regular module, let $x \in M$. By (1.1) $x = \sum_{i=1}^{n} h_i(x)y_i$; $y_i \in M$, $h_i \in M^*$. But Rx is a pure submodule of M then $x = \sum_i h_i(x)z_i$; $z_i \in Rx$. Thus

$\exists r_i \in R$ such that $x = \sum_{i=1}^{n} h_i(x) r_i x$. Put $h = \sum_{i=1}^{n} r_i h_i \in M^*$, then $x = h(x)x$. □

It is known that every module over a regular ring is F-regular [8]. Thus we have.

5.3. Proposition: Let R be a regular ring, then every pwp. R-module, is Z-regular.

Now we are ready to give examples of pwp. Modules that are not projective.

5.4. Example: Let K be a field, let $i \in I$ where I is an infinite countable set. And let $K_i = K$. Put $R = \prod_{i \in I} K_i$ with the usual operations, R is a ring. Since K is a field then K is a regular ring, and hence R is a regular ring. Let $P = \bigoplus_{i \in I} K_i$. Clearly, P is an ideal of R, also P is a Z-regular R-module [11]. By (5.1) P is a pwp R-module. We claim that P is not projective R-module. In fact, it can be easily shown that P is not a direct summand of R, thus it is not a direct summand of $\bigoplus_{i \in I} R_i$, $R_i = R$. Therefore P is not a direct summand for any free R-module. Hence by [4,p.265] P is not projective.

The following example shows that there are locally pwp. modules which are not pwp. modules. The example appeared in [5] for another purpose.

5.1. Example: Let R be a regular ring which has no finitely generated maximal ideal, and thus no maximal ideal which is a direct summand (For example, R could be the infinite direct product of copies of a field F). Let $\{P_\alpha \mid \alpha \in \Lambda\}$ be a family of Maximal ideals of R, and let $M = \coprod_\alpha (R / P_\alpha)$. Since R is regular, then R is locally a field. Hence M_{P_α} is a vector space over the field R_{P_α}. Thus M_{P_α} is a free R_{P_α}-module. Therefore M is a locally pwp. module. Now, we show that $M^* = 0$. In fact, let $f : R / P_\alpha \to R$ be an R-homomorphism. Since R / P_α is simple, then either $f = 0$ or f is $1-1$. If f is $1-1$, then R / P_α can be identified with an ideal A_α in R. But $ann(A_\alpha) = P_\alpha$, and P_α is a pure ideal in R (Because R is regular), hence $P_\alpha \cap A_\alpha = P_\alpha . A_\alpha = 0$. Beside this, by the maximality of P_α, $P_\alpha + A_\alpha = R$. Hence P_α is a direct summand of R which is a contradiction. Thus $(R / P_\alpha)^* = 0$, and this implies that $M^* = 0$. Now assume that M is a pwp. R-module, thus by (1.1) $M = 0$ which is absurd.

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الخلاصة

لتكن R حلقة و M موديول على R . يقال للموديول M ... بي الإسقاط إذا كان لكل تشاكل شامل $\alpha : A \to B$ حيث A و B موديولات على R , ولكل تشاكل $\beta : M \to B$, ولكل $m \in M$ يوجد تشاكل $\varphi : M \to A$ الذي قد يعتمد على m , بحيث أن $\alpha \circ \varphi(m) = \beta(m)$. في هذا البحث درسنا بعض خواص الموديولات نقطية الإسقاط.