Numerical Solution for Fractional Order Space-Time Burger's Equation Using Legendre Wavelet - Chebyshev Wavelet Spectral Collocation Method

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Abstract

In this article, a Legendre wavelet- Chebyshev wavelet spectral collocation method is proposed for solving fractional order space-time Burger's equation with the Legendre wavelet and Chebyshev wavelet operational matrices of fractional derivatives. The fractional derivative is described in the Caputo sense. The proposed method is based on Legendre wavelet-Chebyshev wavelet for space and time variables respectively. This method will reduc the problem under consideration to the solution of nonlinear algebraic equations. In order to confirm the efficiency of the proposed method, two numerical examples are implemented and comparing the numerical solution with the exact one, as well as, of other methods in given literatures, we demonstrate the high accuracy and efficiency of the proposed method. [DOI: 10.22401/JNUS.21.1.19]

Keywords: Fractional Burger's equation, Legendre wavelet-Chebyshev wavelet, spectral collocation method.

1. Introduction

In recent years, there has been a growing interest in the field of fractional calculus. Oldham and Spanier [1], Miller and Ross [2], Gorenflo and Mainardi [3], Momani [4,5], Kilbas [6] and Podlubny [7] provide the history and a comprehensive treatment of this subject. Fractional calculus is one of the applied mathematics fields which deals with integrals and derivatives of arbitrary orders.

Fractional calculus has different types of applications in mathematics, physics, and engineering such as electro chemistry, fluid mechanics, viscoelasticity, signalprocessing, biological population, and so on [8-12].

Fractional differential equations have recently a great interest which is caused both by the intensive development of the theory of fractional calculus itself and by its applications in various sciences. For this reason we need a reliable and efficient technique for the solution of fractional differential equations.

Also, the nonlinear partial differential equations (PDEs) arise in many fields of science which are used to describe many complex nonlinear settings in applications, such as vibration and wave propagation, fluid mechanics, plasma physics, quantum mechanics, nonlinear optics, solid state physics, chemical kinematics, physical chemistry, population dynamics, and many other areas of mathematical modeling. One of the famous type of PDEs is the Burger's-type equation. The Burger's equation is used in many fields, such as in the shock waves, mathematical modeling of dynamic fluid and in continuous stochastic processes. The Burger's equation was first introduced in (1915) by [13].

Burger's J.M. [14,15] introduced this equation to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion, therefore it is popularly referred to as “Burger's equation”.

For better understanding, the phenomena that a given nonlinear fractional partial differential equation describes, the solutions of differential equations of fractional order is much involved. The fractional Burger's equation [5] discusses the physical processes of weakly nonlinear acoustic wave through a gas-filled pipe. Fractional derivatives provide more accurate models of real world problems than integer order derivatives do.

The goal of this article is to present the numerical solutions of Burger's-type equations of fractional order of the form:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}-\frac{\partial^{\beta+1}u(x,t)}{\partial x^{\beta+1}}+u(x,t)\frac{\partial u(x,t)}{\partial x}=f(x,t), \quad \text{.....................................}(1)$$

$$a<x<b,\ \ 0<t<\tau,\ \ 0<\alpha,\ \ 0<\ \beta<1$$

subject to one of the two cases of initial (I.C) and boundary (B.C) conditions:

Case 1:

$$u(x,0)=f_0(x),\quad 0<x<1,\ 0<\alpha<1 \quad \text{.....................................}(2)$$

$$\begin{cases} u(0,t)=0, & 0\le t\le\tau \\ u(1,t)=0, & 0\le t\le\tau \end{cases} \quad \text{.........................}(3)$$

Case 2:

$$\begin{cases} u(x,0)=f_0(x), & 0<x<1 \\ \frac{\partial u(x,0)}{\partial t}=\ f_1(x), & 0<x<1 \end{cases},\ 1<\alpha<2 \quad \text{...................................}(4)$$

$$\begin{cases} u(0,t)=0, & 0\le t\le\tau \\ u(1,t)=0, & 0\le t\le\tau \end{cases} \quad \text{......................}(5)$$

where $x\in(0,1)$ and $t\in[0,\tau],\ \tau>0$ are the space and time variables, respectively, the fractional derivative is defined in the Caputo sense and $f(x,t)$ be a source term.

In this article we develop a Legendre wavelet- Chebyshev wavelet collocation method for solving numerically the above problem. The collocation method is considered here for the numerical solution. The approximate solution is expanded as a series with the coefficients of Chebyshev wavelet in time and Legendre wavelet in space with unknown coefficients. By using the collocation technique and the properties of the Legendre wavelet and Chebyshev wavelet, the problem is reduced to the solution of a system of nonlinear algebraic equations. This research paper is organized in following sections:

In section 2 fractional derivative and integration which are necessary for this article are given, in section 3 we give an overview of Legendre wavelet and Chebyshev wavelet and their properties with some theorems needed here after, and in sections 4, the way of establishing the collocation technique for Burger's equation is described by using the above basis. In section 5 the suggested method is applied to Burger's equation, and comparisons are made with the existing exact and numerical solutions that were reported in other methods. Finally this paper is completed with a conclusion in section 6.

2 Defilations and Properties of Fractional Integration and Derivative

In this section, we propose some definitions and preliminary facts of fractional calculus [1,6].

<u>Definition (1):</u>

The vth order Riemann-Liouville fractional integral operator where is defined as

$$I^{v}f(x)=\frac{1}{\Gamma(v)}\int_0^x(x-t)^{v-1}f(t)dt,\ v>0,x>0 \quad \text{.................................}(6)$$

$I^0f(x)=f(x)$

Where $n-1<v\le n,\ \ n\in\mathbb{Z}^+$

<u>Definition (2):</u>

The vth order Riemann-Liouville fractional derivative operator where is defined as

$${}_0D_x^{v}f(x)=\frac{1}{\Gamma(n-v)}\frac{d^n}{dx^n}\int_0^x(x-t)^{n-v-1}f(t)dt, \quad \text{.................................}(7)$$

$v>0,x>0$

where $n\in Z^+$ and $n-1<v\le n$.

<u>Definition (3):</u>

The vth order Caputo fractional derivative operator of where is defined as

$${}^{c}D_x^{v}f(x)=\frac{1}{\Gamma(n-v)}\int_0^x(x-t)^{n-v-1}\frac{d^n}{dx^n}f(t)dt, \quad \text{.................................}(8)$$

$v>0,x>0$

Where $n\in Z^+$ and $n-1<v\le n$.

Caputo fractional order derivative has a useful property:

$$I^{v}\ {}^{c}D_x^{v}f(x)=f(x)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{x^k}{k!} \quad \text{.................................}(9)$$

where $n\in Z^+$ and $n-1<v\le n$.

Also, for the Caputo fractional order derivative, we have

$${}^{c}D_x^{v}x^{\delta}=\begin{cases} 0 & ,for\ \delta<v \\ \frac{\Gamma(\delta+1)}{\Gamma(\delta+1-v)}x^{\delta-v} & ,for\ \delta\ge v \end{cases} \quad \text{................................}(10)$$

3. Legendre and Chebyshev Wavelets:

Wavelets are family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [16]

$$\psi_{a,b}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right)\quad a,b\in\mathbb{R}\quad a\neq 0$$

If we restrict the parameters a and b to discrete values as $a=a_0^{-k}, b=nb_0a_0^{-k},\ a_0>1,\ b_0>0$, where n and k are positive integers, the family of discrete wavelets are defined as

$$\psi_{n,k}(t)=|a_0|^{\frac{k}{2}}\psi\left(a_0^k t-nb_0\right)$$

where $\psi_{n,k}$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0=2$ and $b_0=1$, then $\psi_{n,k}(t)$ forms an orthonormal basis.

3.1 Legendre Wavelets

Legendre wavelet $\psi_{n,k}(x)=\psi(k,n,m,x)$ where $k\in Z^+$, m is the order of Legendre polynomials and x is the normalized time. They are defined over the interval (0,1) by

$$\boldsymbol{\psi_{n,m}(x)}=\begin{cases}\mathbf{2}^{\frac{k+1}{2}}\sqrt{m+\frac{1}{2}}\,L_m(2^{k+1}x-(2n+1)), & \frac{n}{2^k}\le x<\frac{n+1}{2^k}\\ \mathbf{0} & ,\boldsymbol{otherwise}\end{cases}\qquad \text{.......} (11)$$

where $m=0,1,\cdots,M$ and $n=0,1,\cdots,2^k-1, \mathrm{k}\in\mathbb{N}$.

A function $f(x)\in L^2(\mathbb{R})$ defined over (0,1), can be expressed in terms of Legendre wavelets as

$$f(x)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}c_{pq}\psi_{pq}(x),\ \text{.......} (12)$$

where $c_{pq}=\left(f(x),\psi_{pq}(x)\right)$, in which $(.,.)$ denotes the inner product. If the series in (12) is truncated, then it can be written as

$$f(x)=\sum_{p=0}^{2^k-1}\sum_{q=0}^{M}c_{pq}\psi_{pq}(x)=C^T\Psi(x),\qquad \text{.......} (13)$$

where C and $\Psi(\mathrm{x})$ are $2^{\mathrm{k}}(\mathrm{M}+1)\times 1$ matrices given by

$$C=(c_{0,0},c_{0,1},\dots,c_{0,M},c_{1,0},c_{1,1},\dots c_{1M},\dots, c_{(2^k-1),0},c_{(2^k-1),1},\dots,c_{(2^k-1),M})^T\ \text{.......} (14)$$

$$\Psi(x)=(\psi_{0,0},\psi_{0,1},\dots,\psi_{0,M},\psi_{1,0},\psi_{1,1},\dots\psi_{1M},\dots, \psi_{(2^k-1),M},\psi_{(2^k-1),1},\dots,\psi_{(2^k-1),M})^T\ \text{.......} (15)$$

3.1.1 Operational Matrix of Fractional Order Derivative for Legendre Wavelets

Now we present an useful theorem about operational matrix of derivative for Legendre wavelets:

Theorem (3.1) [17]:

Let $\Psi(x)$ be the Legendre wavelet vector defined in Eq. (15), and $\beta>0$, $(N-1<\beta\le N)$ then we have

$$D^\beta\Psi(x)=H^{(\beta)}\Psi(x)\ \text{.......} (16)$$

Where $H^{(\beta)}$ is the $(2^k(M+1))\times(2^k(M+1))$ operational matrix of fractional order derivative in the Caputo sense and its (s,r)-th component is

$$[\boldsymbol{H}^{(\boldsymbol{\beta})}]_{\boldsymbol{sr}}=\begin{cases}\mathbf{0}, & \mathbf{1}\le\boldsymbol{s}\le\lceil\boldsymbol{\beta}\rceil,\\ \mathbf{2}^{\frac{k+1}{2}}\sqrt{q+\frac{1}{2}}\left(\sum_{i=0}^{q}\sum_{j=0}^{i}b_{jr}\binom{i}{j}\frac{(-1)^{q+i}\,(q+i)!\,2^{kj}\,p^{i-j}}{(q+i)!\,(i!)^2}\right), & \lceil\boldsymbol{\beta}\rceil+\mathbf{1}\le\boldsymbol{s}\le\mathbf{2}^{\boldsymbol{k}}(\boldsymbol{M}+\mathbf{1})\end{cases}\ \text{.......} (17)$$

In which b_{jr} are the r[th] coefficients of the Legendre wavelet expansion of the functions

$$f_j(x)=x^j\Upsilon_{\left[\frac{p}{2^k},\frac{p+1}{2^k}\right]},\ j=0,\dots,i$$

and $\Upsilon_{\left[\frac{p}{2^k},\frac{p+1}{2^k}\right]}$ is the characteristic function defined as:

$$\Upsilon_{\left[\frac{p}{2^k},\frac{p+1}{2^k}\right]}(x)=\begin{cases}1, & x\epsilon\left[\frac{p}{2^k},\frac{p+1}{2^k}\right]\\ 0, & otherwise\end{cases}$$

3.2 Chebyshev Wavelets

Chebyshev wavelet $\phi_{n,k}(t)=\phi(k,n,m,t)$ where $k\in Z^+$, m is the order for Chebyshev polynomials and t is the normalized time. They are defined on the interval (0,1) by

$$\phi_{n,m}(t)=\begin{cases}2^{\frac{k+1}{2}}\hat{T}_m(2^kt-(2n+1)) & \frac{n}{2^k}\le t<\frac{n+1}{2^k}\\ 0 & otherwise\end{cases}\ \text{.......} (18)$$

Where $\hat{T}_m(t)=\begin{cases}\frac{1}{\sqrt{\pi}} & , \quad m=0\\ \sqrt{\frac{2}{\pi}}\,T_m, & m>0\end{cases}$ and $m=0,1,\cdots,M-1$ and $n=0,1,\cdots,2^k-1$, $\mathrm{k}\in\mathbb{N}$.

The Chebyshev wavelets $\phi_{n,m}(t)$ form an orthonormal basis for $L^2_{w_n}[0,1]$ with reference to weight function $w_n(t) = w(2^{k+1}t - (2n-1))$, in which $w(t) = \frac{1}{\sqrt{1-t^2}}$.

A function $f(t) \in L^2(\mathbb{R})$ defined over (0,1), can be expressed in the terms of Chebyshev wavelets as

$$f(t) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} c_{pq}\phi_{pq}(t), \quad \text{.................. (19)}$$

where $c_{pq} = \left(f(t), \phi_{pq}(t)\right)$. If the infinite series in (19) is truncated, then it can be written as

$$f(t) = \sum_{p=0}^{2^k-1} \sum_{q=0}^{M} c_{pq}\phi_{pq}(t) = C^T\Phi(t), \quad \text{................................ (20)}$$

where C and $\Phi(t)$ are $2^k(M+1) \times 1$ matrices given by

$$C = (c_{0,0}, c_{0,1}, \dots, c_{0,M}, c_{1,0}, c_{1,1}, \dots c_{1M}, \dots, c_{(2^k-1),0}, c_{(2^k-1),1}, \dots, c_{(2^k-1),M})^T \quad \text{................................ (21)}$$

$$\Phi(t) = (\phi_{0,0}, \phi_{0,1}, \dots, \phi_{0,M}, \phi_{1,0}, \phi_{1,1}, \dots \phi_{1M}, \dots, \phi_{(2^k-1),0}, \phi_{(2^k-1),1}, \dots, \phi_{(2^k-1),M})^T \quad \text{................................ (22)}$$

3.2.1 Chebyshev Wavelet Operational Matrix of Derivative

Now, we present the follwoing favorable theorem about the operational matrix of derivative for Chebyshev wavelets:

Theorem (3.2) [18]:

Let $\Phi(t)$ be the Chebyshev wavelet vector defined in Eq. (22), and $\alpha > 0$, $(N-1 < \alpha \le N)$ then we have

$$D_*^{\alpha}\Phi(t) = K^{(\alpha)}\Phi(t) \quad \text{................................ (23)}$$

where $K^{(\alpha)}$ is the $(2^k(M+1)) \times (2^k(M+1))$ operational matrix of fractional order derivative in the Caputo sense and its (s,r)-th component is

$$\left[\boldsymbol{K}^{(\alpha)}\right]_{sr} = \begin{cases} \mathbf{0}, & \boldsymbol{pM+1 \le s \le pM} + \lceil\boldsymbol{\alpha}\rceil, \\ \mathbf{2}^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi\gamma_m}}\, w_{qr}^{\alpha}, & \boldsymbol{pM} + \lceil\boldsymbol{\alpha}\rceil + \mathbf{1} \le \boldsymbol{s} \le (\boldsymbol{p}+\mathbf{1})\boldsymbol{M} \end{cases} \quad \text{................................ (24)}$$

In which

$$w_{qr}^{\alpha} = \sum_{i=0}^{q}\sum_{j=0}^{q-1} b_{jr}a_{qi}\binom{q-i}{j}(-1)^{q-i-j}\ 2^{kj}\ n^{q-i-j}$$

$$b_{jr} = \left(D_*^{\alpha}\left(t^j\Theta_{\left[\frac{p}{2^k},\frac{p+1}{2^k}\right]}\right), \Phi_q(t)\right)_{w_p}$$

where $\Theta_{\left[\frac{p}{2^k},\frac{p+1}{2^k}\right]}$ is the characteristic function defined as:

$$\Theta_{\left[\frac{p}{2^k},\frac{p+1}{2^k}\right]}(t) = \begin{cases} 1, & t\epsilon\left[\frac{p}{2^k},\frac{p+1}{2^k}\right] \\ 0, & otherwise \end{cases}$$

and a_{qi} is the coefficient of the analytic form for the shifted Chebyshev polynomial expansion $T_{\tau,q}(t) = \sum_{u=0}^{q} a_{qu}\,t^u$, i.e

$$a_{qu} = q(-1)^{q-u}\frac{(q+u-1)!\,2^{2u}}{(q-u)!\,(2u)!}$$

Remark (1):

A function of two variables $u(x,t) \in L^2(\mathbb{R}\times\mathbb{R})$ defined over $[0,1)\times[0,1)$, may be expressed by Legendre wavelets - Chebyshev wavelets basis, as:

$$\mathrm{u}_{\hat{m},\hat{n}}(x,t) = \sum_{i=1}^{\hat{n}}\sum_{j=1}^{\hat{m}} c_{ij}\psi_i(x)\phi_j(t) = \Phi^T(t)C\Psi(x), \quad \text{....................... (25)}$$

where $C = (c_{ij})$ and $u_{ij} = \left(\psi_i(x), \left(u(x,t), \phi_j(t)\right)\right)$, are $(\hat{n}\times\hat{m})$ matrix, and $\hat{n} = \hat{m} = 2^k(M+1)$.

4. Legendre Wavelet-Chebyshev Wavelet Spectral Collocation Method for Fractional Order Space–Time Burger's Equation

In this section, we develop the Legendre wavelet-Chebyshev wavelet spectral collocation method to numerically solve the Burger's-type equations of fractional order given by equations (1) with reference to the conditions (2)-(5).

In order to solve equation (1) with respect to the above conditions, we consider $\mathrm{u}_{\hat{m},\hat{n}}$ as an approximate solution of equation (1) namely $u(x,t)$.

$$u_{\hat{m},\hat{n}}(x,t) = \Phi^T(t)C\Psi(x), \quad \text{....................... (26)}$$

where $\Psi(x)$, $\Phi(t)$ are given by equations (15) and (22) respectively and C is a matrix of unknown coefficients.

Using Equations (23) and (16) in Equation (26), we can write

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \Phi^T(t){K^{(\alpha)}}^T C\Psi(x), \quad \text{................... (27)}$$

$$\frac{\partial^{\beta+1}u(x,t)}{\partial x^{\beta+1}} = \Phi^T(t)CH^{(\beta+1)}\Psi(x), \quad \text{............. (28)}$$

$$\frac{\partial u(x,t)}{\partial x} = \Phi^T(t)CH^{(1)}\Psi(x), \quad \text{...................... (29)}$$

Hence the initial and boundary conditions (2)-(5) becomes:

$$u(x,0) = \Phi^T(0)C\Psi(x), \quad (30)$$
$$u(0,t) = \Phi^T(t)C\Psi(0), \quad (31)$$
$$u(1,t) = \Phi^T(t)C\Psi(1), \quad (32)$$
$$\frac{\partial u(x,0)}{\partial t} = \Phi^T(0)D^{(1)^T}C\Psi(x), \quad (33)$$

By using Equation (26) and Equations (27)-(33), implies that equation (1) can be represented in the following matrix form:

$$\Phi^T(t)K^{(\alpha)^T}C\Psi(x) - \Phi^T(t)CH^{(\beta+1)}\Psi(x) + \left(\Phi^T(t)C\Psi(x)\right)\left(\Phi^T(t)CH^{(1)}\Psi(x)\right) = f(x,t), \quad (34)$$
$$\Phi^T(0)C\Psi(x) = f_0(x), \quad (35)$$
$$\Phi^T(t)C\Psi(0) = 0, \quad (36)$$
$$\Phi^T(t)C\Psi(1) = 0. \quad (37)$$

For suitable collocation points (x_i, t_j) where x_i are the shifted Legendre -Gauss-Lobatto nodes ($0 \le i \le \hat{n}$) and the shifted Chebyshev roots t_j ($0 \le j \le \hat{m}$) equations (34)-(37) can be written as:

$$\Phi^T(t_j)K^{(\alpha)^T}C\Psi(x_i) - \Phi^T(t_j)CH^{(\beta+1)}\Psi(x_i) + \left(\Phi^T(t_j)C\Psi(x_i)\right)\left(\Phi^T(t_j)H^{(1)}C\Psi(x_i)\right) = f(x_i,t_j),$$
$$(0 \le i \le \hat{n}-1), (1 \le j \le \hat{m}-1), \quad (38)$$
$$\Phi^T(0)C\Psi(x_i) = f_0(x_i), \ \ 0 \le i \le \hat{n}, \quad (39)$$
$$\Phi^T(t_j)C\Psi(0) = 0, \ \ 0 \le j \le \hat{m}-1, \quad (40)$$
$$\Phi^T(t_j)C\Psi(1) = 0, \ \ 0 \le j \le \hat{m}-1. \quad (41)$$

Case two can be handles in a similar manner as given in case (1) but by adding the following equation

$$\Phi^T(0)K^{(1)^T}C\Psi(x_i) = f_1(x_i), \ 0 \le i \le \hat{n}-1 \quad (42)$$

This generate a nonlinear system of algebraic equations in terms of the unknown coefficients c_{ij}, $i = 0,1,\cdots,\hat{n}$, $j = 0,1,\cdots,\hat{m}$, which can be solved by using Newton's method. So the $u_{\hat{m},\hat{n}}(x,t)$ given in equation. (26) can be solved.

5. Numerical Examples:

In this section, we present two numerical examples to demonstrate the accuracy and applicability of the proposed method.

Example 1:

Consider the following fractional Burger's equation,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^{\beta+1}u(x,t)}{\partial x^{\beta+1}} + u(x,t)\frac{\partial u(x,t)}{\partial x} = f(x,t),$$

$1 < x < 0$, $0 \le t \le \tau$, $0 < \alpha < 1$.
with (I.C)
$u(x,0) = x^2(1-x)^2$, $\quad 0 < x < 1$
and (B.Cs)
$u(0,t) = u(1,t) = 0$, $\quad 0 \le t \le \tau$

In which $0 < \alpha < 1, \tau = 1$. and the source term f is chosen to be:

$$f(x,t) = \left(\frac{8t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{4t^{1-\alpha}}{\Gamma(2-\alpha)}\right)x^2(1-x)^2 + (4t^2-4t+1)\left(\frac{2x^{1-\beta}}{\Gamma(2-\beta)} - \frac{12x^{2-\beta}}{\Gamma(3-\beta)} + \frac{24x^{3-\beta}}{\Gamma(4-\beta)}\right) + (4t^2-4t+1)^2\left(x^2(1-x)^2(2x-6x^2+4x^3)\right)$$

The exact solution of this example is [19]:

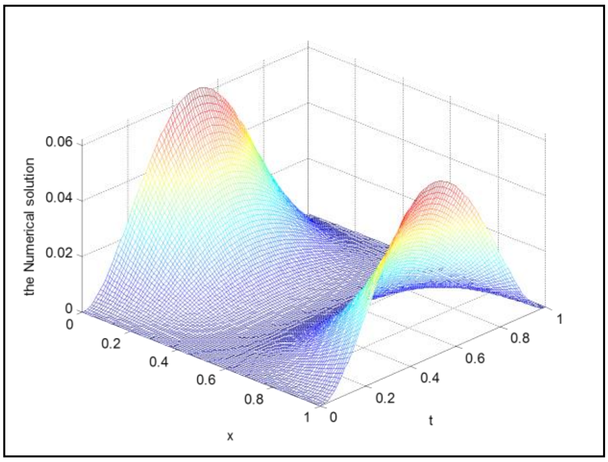
$$u(x,t) = (4t^2-4t+1)x^2(1-x)^2$$

Following Table (1) represents the maximum absolute error (MAE) of the proposed method for Example 1 using different values of $\hat{n}$ and $\hat{m}$ with $\alpha = 0.999$ and $\beta = 0.1$. Also we compares the errors that we have been obtained with those result given in [19].

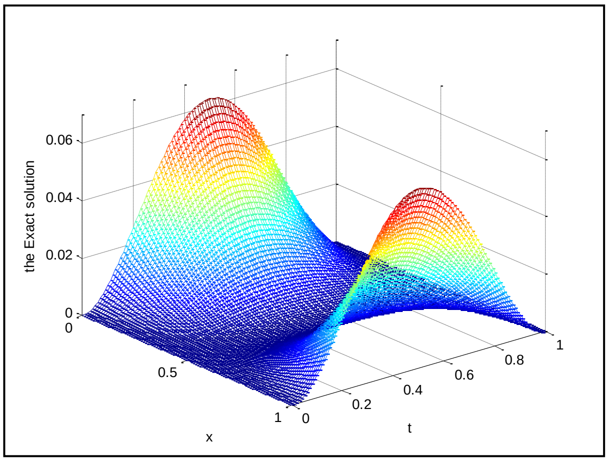
Table (1)
The (MAE) between the Present Method with the Methods in [19] at α = 0.999 and β = 0.1.

$\hat{n} = \hat{m}$	*Our Method*	Δt	*Method[19]*
8	7.445286×10^{-3}	$\frac{1}{5}$	1.33732×10^{-2}
12	5.43691×10^{-3}	$\frac{1}{10}$	7.00853×10^{-3}
16	3.69124×10^{-3}	$\frac{1}{20}$	3.50049×10^{-3}
20	1.64525×10^{-3}	$\frac{1}{40}$	1.74046×10^{-3}
24	6.66764×10^{-4}	$\frac{1}{80}$	8.67665×10^{-4}
28	3.56293×10^{-4}	$\frac{1}{160}$	4.32965×10^{-4}

We represent in Fig.(1) a comparison between the exact and numerical solution given by the proposed method for $\hat{m} = \hat{n} = 12(k = 2, M = 3)$ with α =0.9 and β = 0.999.



(a)



(b)

Fig.(1): Result of Example 1:
(a) Numerical solution, (b) Exact solution.

Example 2:

Consider the following fractional Burgers equation,

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} - \frac{\partial^{\beta+1}u(x,t)}{\partial x^{\beta+1}} + u(x,t)\frac{\partial u(x,t)}{\partial x} = f(x,t),$$
$$0 < x < 1,\ \ 0 \le t \le \tau,\ \ 1 < \alpha < 2.$$

with (I.Cs)

$$u(x,0) = x^2(1-x)^2,$$
$$\frac{\partial u}{\partial t}(x,0) = -4x^2(1-x)^2, \qquad 0 < x < 1,$$

and (B.Cs)

$$u(0,t) = u(1,t) = 0,\ \ 0 \le t \le 1$$

In which $1 < \alpha < 2,\ \tau = 1$. and the source term f is chosen as:

$$f(x,t) = \left(\frac{8t^{2-\alpha}}{\Gamma(3-\alpha)}\right)x^2(1-x)^2$$
$$+(4t^2-4t+1)\left(\frac{2x^{1-\beta}}{\Gamma(2-\beta)} - \frac{12x^{2-\beta}}{\Gamma(3-\beta)} + \frac{24x^{3-\beta}}{\Gamma(4-\beta)}\right)$$
$$+(4t^2-4t+1)^2\left(x^2(1-x)^2(2x-6x^2+4x^3)\right).$$

Following Table (2) represent the maximum absolute error (MAE) of the proposed method for Example (2) using different values of $\hat{n}$ and $\hat{m}$ with $\alpha = 1.1$ and $\beta = 0.8$ also we compares the errors that we have been obtained with result given in [19].

Table (2)
The (MAE) between the Present Method with the Methods in [19] at α = 1.1 and β = 0.8.

$\hat{n} = \hat{m}$	*Our Method*	*h*	*Method[19]*
8	9.6522×10^{-5}	$\frac{1}{4}$	2.24697×10^{-2}
12	8.9193×10^{-5}	$\frac{1}{8}$	1.82814×10^{-2}
16	6.5716×10^{-6}	$\frac{1}{16}$	1.159734×10^{-2}
20	5.7428×10^{-6}	$\frac{1}{32}$	6.505725×10^{-3}
24	4.4875×10^{-7}	$\frac{1}{64}$	4.807568×10^{-3}

Following Fig.(2) shows the numerical solutions of Example 2 by fix β, say, $\beta = 0.999$, and decrease α from 1.0 to 0.1, and $\hat{n} = \hat{m} = 10(k = 1, M = 4)$

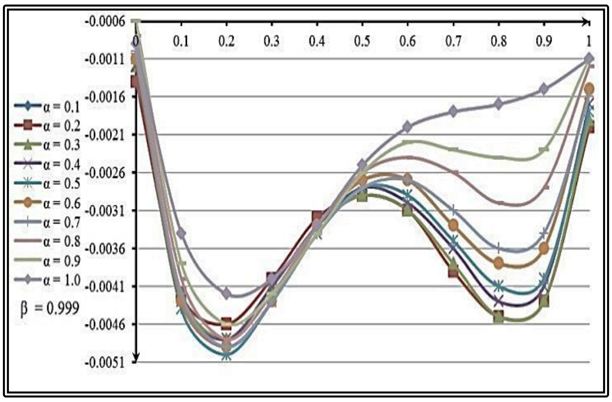


Fig.(2): The Numerical Solution with β = 0.999, α = 0.1,…, 1.0 and $\boldsymbol{\hat{n} = \hat{m} = 10}$ $(\boldsymbol{k = 1, M = 4})$

6. Conclusions

In this article, a modified numerical technique algorithm based on spectral collocation technique, this approach was employed for solving Burger's equation of space-time fractional order. The Legendre wavelet and Chebyshev wavelet with the collocation method are used to transform the suggested problem to the solution of nonlinear system of algebraic equations. The obtained result shows that the proposed method is more accurate than the method in [19].

References

[1] Oldham B., Spanier J.; “The Fractional Calculus”; Academic Press, New York, 1974.

[2] Miller K.S., Ross B.; “An Introduction to the Fractional Calculus and Fractional Differential Equations”; John Wiley & Sons Inc., New York, 1993.

[3] Gorenflo R., Mainardi F.; “Fractional Calculus: Integral and Differential Equations of Fractional Order”; in: Carpinteri A., Mainardi F. (Eds.), Fractals and Fractional Calculus, New York, 1997.

[4] Momani S., Odibat Z.; “Analytical Approach to Linear Fractional Partial Differential Equations Arising In Fluid Mechanics”; Physics Letters A 355, Issues 4-5, 271–279, 2006.

[5] Momani S.; “Non-Perturbative Analytical Solutions of The Space and Time-Frational Burgers Equations”; Chaos Solitons and Fractals 28, 930–937, 2006.

[6] Kilbas A. A., Srivastava M. H.; “Trujillo JJ. Theory and Applications of Fractional Differential Equations”. San Diego: Elsevier; 2006.

[7] Podlubny I.; “Fractional Differential Equations”; Academic Press, New York, 1999.

[8] Mohammed O. H., Fadhel F. S., AL-Safi M. G.; “Numerical Solution for the Time-Fractional Diffusion-Wave Equations by Using Sinc-Legendre Collocation Method”; Mathematical Theory and Modeling, 5(1), PP.49-57, 2015.

[9] Mohammed O. H., Fadhel F. S. and AL-Safi M. G. S.; “Shifted Jacobi Tau Method for Solving The Space Fractional Diffusion Equations”; IOSR Journal of Mathematics (IOSR-JM), 10(3), PP. 34-44. 2015.

[10] AL-Safi M. G. S., Hummady L. Z.; “Approximate Solution for Advection Dispersion Equation of Time Fractional Order by Using the Chebyshev Wavelets-Galerkin Method”; Iraqi Journal of Science. 58(3B), 1493-1502, 2017.

[11] Mohammed O. H., Fadhel F. S. and AL-Safi M. G. S.; “Sinc-Jacobi Collocation Algorithm For Solving The Time-Fractional Diffusion-Wave Equations”, International Journal of Mathematics and Statistics Studies UK, Vol 3, No.(1), PP 28-37, 2015.

[12] AL-Safi M. G. S, Joey F. L., Abbas M. S.; “Numerical Solution for Telegraph Equation of Space Fractional Order sing Legendre Wavelets Spectral Tau Algorithm”; Australian Journal of Basic and Applied Sciences, 10(12), PP. 383-391, 2016.

[13] Bateman H., “Some Recent Researches on The Motion of Fluids”, Mon. Weather Rev., Vol. 43, pp. 163-70, 1915.

[14] Burgers J. M., “Mathematical Example Illustrating Relations Occurring in The Theory of Turbulent Fluid Motion”, Trans. Roy. Neth. Acad. Sci. Amsterdam, Vol. 17, PP. 1-53, 1939.

[15] Burgers J. M. “A Mathematical Model Illustrating The Theory of Turbulence”, Adv. in Appl. Mech., Vol. 1, PP. 171-99, 1948.

[16] Gu J.S., Jiang W.S., “The Haar Wavelets Operational Matrix of Integration”, Int. J. Syst. Sci., 27: 623-628, 1996.

[17] Mohammadi F., Hosseini M.M., Mohyud-Din S.T, “A New Operational Matrix for Legendre Wavelets and its Applications for Solving Fractional Order Boundary Values Problems”, International Journal of the Physical Sciences, 6(32): 7371-7378, 2011. Available online at http://www.academicjournals.org/IJPS.

[18] Mohammadi F., “Numerical Solution of Bagley-Torvik Equation Using Chebyshev Wavelet Operational Matrix of Fractional Derivative”, Int. J. Adv. Appl. Math. and Mech. 2(1), 83-91, 2014.

[19] Li C., Zhao Z., Chen Y. Q., “Numerical Approximation of Nonlinear Fractional Differential Equations With Subdiffusion and Superdiffusion”, Comput. Math. Appl 62, 855-875, 2011.