G-Cyclicity And Somewhere Dense Orbit

Zeana Zaki Jamil
University of Baghdad – College of Science Dept. of Mathematics

Abstract

Let H be an infinite - dimensional separable complex Hilbert space, and S be a multiplication semigroup of $\mathbb{C}$ with 1. An operator T is called G-cyclic over S if there is a vector $x \in H$ such that $\{\alpha T^n x \mid \alpha \in S, n \geq 0\}$ is norm-dense in H. Bourdon and Feldman have proved that the existence of somewhere dense orbits implies hypercyclicity. We show the corresponding result for G – cyclicity.

Introduction

Let H be an infinite-dimensional separable complex Hilbert space. and $B(H)$ be the Bannach algebra of all linear bounded operator on H. Let S be a multiplication semigroup of $\mathbb{C}$ with 1, an operator $T \in B(H)$, T is called *G-cyclic over* S if there is a vector x in H such that $\{\alpha T^n x \mid \alpha \in S, k \geq 0\}$ is norm-dense in H. In this case x is called a *G-cyclic vector for* T *over* S [3]. Clearly, every hypercyclic operator is G-cyclic and every G-cyclic operator is supercyclic. Bourdon and Feldman [2] prove that every somewhere dense orbit is everywhere dense, and they use this result to give another proof of *Ansari's theorem* " If T is hypercyclic, then for each $n \geq 1$, T^n is hypercyclic, moreover T and T^n share the same collection of hypercyclic vectors" [1]. Also they use their theorem and give another

proof of *Multihypercyclicity Theorem* "If T is multihypercyclic then T is hypercyclic" [4]. Our purpose in this paper is to obtain the corresponding results for *G-cyclic over* S.§1 Somewhere dense orbit is everywhere dense:

The goal of this section is to prove that the existence of somewhere dense orbits implies. G-cyclicity. Next we fix notation required for the discussion.

Notation: Let S be a semigroup of $\mathbb{C}$ with 1, then:

1. $Sorb(T,x) = \{\alpha T^n x \mid \alpha \in S, n \geq 0\}$.
2. $\mathbb{C}orb(T,x) = \{\alpha T^n x \mid \alpha \in \mathbb{C}, n \geq 0\}$.
3. $F(x) = \overline{Sorb(T,x)}$.
4. $U(x) = \text{int}(F(x))$.
5. $X^c = complete\ of\ X\ in\ H$.

Clearly from the definition of *G-cyclic* [3], that every G-cyclic operator is supercyclic operator, so we get :

Proposition 1.1: Suppose that $x \in H$ is such that $Sorb(T,x)$ is somewhere dense in H, then $\mathbb{C}orb(T,x)$ is somewhere dense in H.

Proof: Since S is a semigroup of $\mathbb{C}$ with 1, then $\overline{Sorb(T,x)} \subseteq \overline{\mathbb{C}orb(T,x)}$. Now since $U(x) \neq \phi$ and $U(x) \subseteq \text{int}(\overline{\mathbb{C}orb(T,x)})$. Thus $\mathbb{C}orb(T,x)$ is somewhere dense.

From [2] we get immediately the following two lemmas:

Lemma 1.2: Suppose that $x \in H$ such that $Sorb(T,x)$ is somewhere dense in H. Then T^* may have at most one eigenvalue.

Lemma 1.3: Suppose that $Sorb(T,x)$ is somewhere dense in H, then for each $\alpha \in S, j \in N, \alpha T^{n_j} x$ is a cyclic vector for T.

Peries in [4] proved the following lemma.

Lemma 1.4 [4]: Let P be a complex polynomial, $p(T)$ has a dense range if and only if $p(\lambda) \neq 0$ for every eigenvalue λ of T^*. The next lemma provides the crucial element of the argument.

Lemma1.5: Let $x \in H$, then for every $\lambda \in S$, $U^c(x)$ is invariant under λT.

In addition, $U^c(x)$ is invariant under multiplication by any $\alpha \in S$.

Proof: Since $U(x)$ is nonempty, then there is a positive integer j and a non-zero $\beta \in S$ such that $\beta T^j x$ belongs to $U(x)$ and set $x_j = \beta_j T^j x$. For any $k \in N$, $Sorb(T, T^k x_j)$ is

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dense in $U(x)$, thus x_j is a limit point of $Sorb(T, T^k x_j)$ and $U(x) = U(T^k x_j)$. By lemma (1.5) x_j is cyclic vector for T. i.e. $\{p(T)x_j \mid p \text{ is polynomial}\}$ is dense in H.

Fix $\alpha \in S$, assume that $U^c(x)$ is not λT-invariant, i.e. there is $y \notin U(x)$ but $Ty \in U(x)$. We may assume $y \notin F(x)$, if not, then $y \in \partial F(x)$, also since λT is continues, hence there is a point $y' \in F(x)$ close enough to y and $\lambda Ty'$ is close enough to λTy to keep it in $U(x)$. Thus rename y' as y.

Because $F^c(x)$ is open and $\{p(T)x_j \mid p \text{ ispolynomial}\}$ is dense in H, thus there is a polynomial p so that $p(T)x_j$ is closed enough to y to ensure $p(T)x_j \in F^c(x)$ and $\lambda Tp(T)x_j \in U(x)$. Since $U(x) \subset F(x)$ and $F(x)$ is λT- invariant, then $Sorb(T, \lambda Tp(T)x_j) \subset F(x)$. However $Sorb(T, \lambda Tp(T)x_j) = Sorb(T, p(T)Tx_j)$.

Because x_j is a limit point of $Sorb(T, Tx_j)$, the continuity of $p(T)$ yield $p(T)x_j \in F(x)$. Thus $p(T)x_j \in F(x)$ and its complement, a contradiction. It is easy to prove that $U^c(x)$ is invariant under multiplication under $\alpha \in S$.

Remark: The preceding show that if $y \in Sorb(T, x)$, then $U(x) = U(y)$. Now we will prove the main result.

Somewhere Dense Theorem 1.6: Suppose $T \in B(H)$, and $Sorb(T, x)$ is somewhere dense in H, then T is G-cyclic operator.

Proof: Assume that $\overline{Sorb(T, x)} \neq H$. Since x is cyclic vector for T (1.3), then $\{p(T)x_j \mid p \text{ is polynomial}\}$ is dense in H. Then there is a subcollection Q of polynomial such that $\{q(T)x \mid q \in Q\}$ is dense subset of $U^c(x)$. By (1.5) $U^c(x)$ is λT- invariant for all $\lambda \in S$, so $q(T)orb(T, x) \subset U^c(x)$ for all $q \in Q$, hence, by continuity of T, $q(T)F(x) \subset q(T)orb(T, x) \subset U^c(x)$.

Let W denote the collection of non-zero polynomials not having the (possible) eigenvalue of T^* as a zero and let $p \in W$. Now put $D := U(x) \bigcup \{q(T)x \mid q \in Q\}$, since $\{q(T)x \mid q \in Q\}$ is dense set in $U^c(x)$, hence D is dense set in H. Because $p(T)$ has dense range in H (1.4), therefore $p(T)D$ is dense in H. Suppose, in order to optain a contradiction, that $p(T)x \in \partial U(x)$, hence $p(T)x \notin U(x)$, then $p(T)x \in U^c(x)$. Thus $p(T)U(x) \subset U^c(x)$.

On the other hand, since $U^c(x)$ is λT-invariant for all $\lambda \in S$, thus $p(T)\{q(T)x \mid q \in Q\} \subset U^c(x)$. Therefore $p(T)D \subset U^c(x)$ which contradicting the density of $p(T)D$. Thus $p(T)x \notin \partial U(x)$.

Because $\{p(T)x \mid p \in W\}$ is connected, contains points in $U(x)$ and contains no boundary point of $U(x)$, thus $\{p(T)x \mid p \in W\} \subset U(x)$. Given a coefficient n-tuple c for any polynomial, there is a sequence for coefficient n-tuples of a polynomials in Q converging componentwise to c, and since x is cyclic vector for T, then $\{p(T)x \mid p \in W\}$ is dense in H. Since $\{p(T)x \mid p \in W\} \subset U(x) \subset F(x)$, therefore $F(x) = H$. Thus T is G-cyclic operator.

S2 Applications to the Somewhere Dense Theorem:

In this section we give two applications to the somewhere dense theorem.

First we need the following fact, let X be a topological space and $F_1, F_2, \ldots, F_n$ a finite family of closed subset of X; $X = \bigcup_{i=1}^{n} F_i$, if $\operatorname{int}(F_i) = \phi$, then $X = \bigcup_{i=2}^{n} F_i$ [4].

Proposition 2.1: if T is a *G-cyclic* operator over S then for every positive integer n, T^n is G-cyclic operator over S. Moreover, T and T^n share the same collection of G-cyclic vectors.

Proof: Let x be a G-cyclic vector for T over S, and fixed $n > 1$, then $Sorb(T,x) = \bigcup_{j=0}^{n-1} Sorb(T^n, T^j x)$ will be dense in H, thus $H = \bigcup_{j=0}^{n-1} \overline{Sorb(T^n, T^j x)}$. Thus at least one of the sets $Sorb(T^n, T^j x)$ must be somewhere dense. Therefore by (1.6) T^n is a G-cyclic operator over S. Now because T must have dense range [3], the set $T^{n-j}\left[\overline{Sorb(T^n, T^j x)}\right] = \overline{Sorb(T^n, T^n x)}$ will be dense in H, from which it followed that x is also a G-cyclic vector for T^n.

An operator $T \in B(H)$ is a multi-G-cyclic operator over S provided there is a finite subset $\{x_i\}_1^n$ in H such that $\bigcup_1^n Sorb(T, x_i)$ is dense in H. Clearly every G-cyclic operator is multi-G-cyclic operator. A question arises: **Is the converse true?**

Proposition 2.2: Any multi- -cyclic operator over S is G-cyclic operator over S.

Proof: Let $\{x_j\}$ be a multi-G-cyclic vector for T over S, then $\bigcup_1^n Sorb(T, x_j)$ is dense in H. By [4], there is at least $j; 1 < j < n$, such that $Sorb(T, x_j)$ has somewhere dense in H. Thus by (1.6) T is G-cyclic operator over S.

References

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المستخلص

ليكن H فضاء هلبرت على حقل الاعداد العقدية قابل للفصل غير منتهي البعد و S شبه زمرة جدائية من C تحتوي على 1. يقال للمؤثر الخطي T انه دوري من النمط G اذا وجد متجه $x \in H$ بحيث ان $\{\alpha T^n x | \alpha \in S, n \geq 0\}$ كثيفة في H. بوردن وفيلدمان برهنا وجود مدار كثيف في مكان ما يؤدي الى فوق الدوارية. في هذا البحث اعطينا نتائج مماثلة في حالة دوري من النمط G.