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Range – Kernel Orthogonality of Jordan * - derivation

[1]Buthainah Abdul –[1]Hasan ahmed -[2]Sudad musa Rasheed
[1]Department of Mathematic, College of Science, University of Baghdad
[2]Department of Mathematic, College of Science, University of Sulaimani

Abstract

Let H be a separable infinite dimensional complex Hilbert space and let $B(H)$ be the Banach algebra of bounded linear operators on H into itself. The generalized derivation $\delta_{A,B}: B(H) \to B(H)$ is defined by $\delta_{A,B}(X) = AX - XB$,let $\Delta_{A,B}: B(H) \to B(H)$ be elementary operator is defined by $\Delta_{A,B}(X) = AXB - X$ and let $d_{A,B}$ denotes by either $\delta_{A,B}$ or $\Delta_{A,B}$. In [1],Anderson proved that if $A \in B(H)$ is normal, S is an operator such that $AS = SA$ then $\|\delta_{A,A}(X) + S\| \geq \|S\|$ for all $X \in B(H)$. Hence the range of δ_A is orthogonal to the kernel of δ_A. The jordan*-derivation $J: B(H) \to B(H)$ is defined by $J(X) = J_A(X) = XA - AX^*$,of this paper is to prove a similar orthogonal result for J_A in the usual Hilbert space.

Keywords: Normal derivation, elementary operator, orthogonality result for derivations

Introduction

Let H be an infinite dimensional separable complex Hilbert space and let $B(H)$ denotes the algebra of operators(= bounded linear transformations) on H into itself.Given $A, B \in B(H)$, The generalized derivation $\delta_{A,B}: B(H) \to B(H)$ (elementary operator $\Delta: B(H) \to B(H)$) is defined by $\delta_{A,B}(X) = AX - XB$ (respectively $\Delta_{A,B}(X) = AXB - X$).Let $d_{A,B}$ denotes by either $\delta_{A,B}$ or $\Delta_{A,B}$, the jordan * derivation $J: B(H) \to B(H)$ is defined by $J(X) = J_A(X) = XA - AX^*$ for all $X \in B(H)$ .Recall that M and N are subspaces of a Banach space with norm $\|.\|$.

M is said to be orthogonal to N . If

$\|m + n\| \geq \|n\|$ *for all* $m \in M$ *and* $n \in N$.

The Range Kernel orthogonality of the operator $d_{A,B}$ has been considered by a number of authors in the recent past (see [5],[7],[12],[13]),with the first such result proven by Anderson in [1].Anderson[2]proved that if $A \in B(H)$ is normal, S is an perator such that then $\|\delta_{A,A}(X) + S\| \geq \|S\|$ forall $X \in B(H)$ .This result has a $\Delta_{A,A}$ analogue indeed it is Known that if A and B^* satisfy a normality- like hypothesis and $d_{A,B}(S) = 0$ for some $S \in B(H)$,then $\|d_{A,B}(X) + S\| \geq \|S\|$ for all $X \in B(H)$ (see [5],[13]for further details).The orthogonality of the range and the kernel of certain derivations has been extensively studied by several authors (see, e.g., [1],[7], [12],[13],[14] and references there in).

Orthogonality of the Range and kernel of $J_A(X)$.

We shall prove the Range Kernel orthogonality on Jordan * - derivation from here to the end of this section we assume that H is real separable Hilbert space. First we need the following theorems and propositions to satisfy the Putnam Fuglede Theorem.

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Theorem 1. (Embry 's Theorem) [8]

Let S and T be a pair of commuting normal operators on Hilbert space H. If $AS = TA$ where $A \in B(H)$ and $0 \notin W(A)$, then $S = T$. Recall that an operator A is said to be normaloid if $\|A\| = r(A)$ where $r(A)$ is the spectral radius defined by $r(A) = \sup\{|\lambda| / \lambda \in \sigma(A)\}$. It is known that every hyponormal operator is normaloid [11, p.267].The following theorem appeared in [18].

Theorem 2.

Let N be an operator such that $N - \lambda I$ is normaloid for all complex value of λ. If $AN = N^*A$ for an arbitrary operator A, for which $0 \notin \overline{W}(A)$, then $N = N^*$. Moreover, it was shown in [21] the following.

Theorem 3.

If T is any operator such that $S^{-1}TS = T^*$, where $0 \notin \overline{W}(S)$, then the spectrum of T is real. Moreover, T is

similar to Hermitian operator.Now we are ready to give the following propositions.

Proposition 1. [10]

Let $A, X \in B(H)$. If X is a hyponormal operator such that $X \in \ker J_A$ then $X^* \in \ker J_A$

Proof:

Using [9], we get means that $X^*A = AX$, so $X^* \in \ker J_A$.

Proposition 2.[10]

Let $A \in B(H)$. If $0 \notin W(A)$, then every normal operator in $\ker J_A$ is self-adjoint operator .

Proof:

Let $X \in \ker J_A$ be a normal operator, then $XX^* = X^*X$ and $XA = AX^*$. By Embry 's theorem, $X = X^*$ i.e., X is self-adjoint operator.

Proposition 3.[10]

Let $A \in B(H)$. If $0 \notin W(A)$, then every hyponormal operator in $\ker J_A$ is self-adjoint operator.

Proof:

Let X be a hyponormal operator such that $X \in \ker J_A$ i.e., $XA = AX^*$. Since $0 \notin \overline{W}(A) \supset \sigma(A)$ then A is invertible thus $A^{-1}XA = X^*$, i.e., X is similar to X^*.

So (theorem 3) implies that $\sigma(X)$ is real and so is $Conv\sigma(X)$. It is well- known that $conv\sigma(X) = \overline{W}(X)$, thus $\overline{W}(X) \subset \mathbb{R}$ which implies that X is self-adjoint .

Proposition 4. [10]

Let $A, X \in B(H)$ with $0 \notin W(A)$ and X be a hyponormal operator. If either X or X^* belongs to $\ker J_A$ then X is self- adjoint.

Proof:

Suppose that X is hyponormal operator. If $X \in \ker J_A$ then by (proposition1), $X = X^*$. If $X^* \in \ker J_A$, i.e., $XA = AX^*$ then X satisfies the conditions (theorem2) as every hyponormal is normaloid and $X - \lambda I$ is normaloid and for all complex numbers λ [19]. Consequently $X = X^*$.Now we shall prove the Range Kernel orthogonality of Jordan *-derivation on Hilbert – Schmidt.

Theorem 4.

If $X \in B(H)$ is a normal, $S \in C_2$ is an operator such that $XS = SX^*$, then $\|XA - AX^* + S\|_2^2 = \|XA - AX^*\|_2^2 + \|S\|_2^2$ for all $A \in B(H)$.

proof:

$\|XA - AX^* + S\|_2^2 = (XA - AX^* + S, XA - AX^* + S)$

$= (XA - AX^*, XA - AX^*) + (XA - AX^*, S) + (S, XA - AX^*) + (S, S)$

$= \|XA - AX^*\|_2^2 + \|S\|_2^2 + 2\,\mathrm{Re}(XA - AX^*, S)$

we claim that $(XA - AX^*, S) = 0$. Now, $XS = SX^*$ implies that $X^*S = SX$ and so

$(X^*S)^* = (SX)^* \Rightarrow S^*X = X^*S^*$. Thus

$(XA - AX^*, S) = tr((XA - AX^*)S^*)$

$= tr(XAS^* - AX^*S^*)$ Remark:To

$= tr(XAS^* - AS^*X)$ (since $S^* \in C_2$)

$= 0$.

study Range – Kernel orthogonality of Jordan *-derivatin, it has been shown Let $A \in B(H)$. If

$0 \notin \overline{W}(A)$ then every hyponormal operator in $\ker J_A$ is self-adjoint operator .And thus the study be as usual derivation. So we get $0 \in \overline{W}(A)$ and proof the result in general case.If $X \in B(H)$ is a normal operator, $S \in C2$ is an operator such that $SX = XS^*$ then $\|XA - AX^* - S\|_2^2 = \|XA - AX^*\|_2^2 + \|S\|_2^2$ for all $A \in B(H)$.

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المستخلص

ليكن H فضاء هلبرت القابل للفصل وغير منتهي البعد على حقل الأعداد العقدية وليكن $B(H)$ جبر باناخ لكافة المؤثرات الخطية المقيدة المعرفة على H يعرف مؤثر الاشتقاق المعمم $\delta_{A,B} : B(H) \rightarrow B(H)$ بأنه التطبيق ذو الصيغة $\delta_{A,B}(X) = AX - XB$, $X \in B(H)$ حيث كل من A , B عنصر في $B(H)$ إذا كان $A=B$ يكتب بالصيغة δ_A ويعرف بمؤثر الاشتقاق ويعرف المؤثر الابتدائي Δ على $B(H)$ بأنه التطبيق ذو الصيغة

$$\Delta(X) = \sum_{i=1}^{n} A_i X B_i - X \quad , \quad X \in B(H)$$

حيث A_i و B_i لكل $i = 1,....,n$ عناصر في $B(H)$ ويعرف تطبيق جوردان *الاشتقاق بالشكل $J_A(X) = XA - AX^*$, $X \in B(H)$ لغرض دراسة مدى كل من هذه التطبيقات برهن Anderson انه إذا كان A عنصر في $B(H)$ و S مؤثر سوي فان $\|\delta_A(X) + S\| \geq \|S\|$, $X \in B(H)$ أي أن مدى التطبيق يكون عموديا على نواته بعد ذلك قام عدد من الباحثين بدراسة تعامد المدى مع النواة لكل من $\delta_{A,B}$ و $\Delta_{A,B}$ ، ودرسنا التعامد بين المدى والنواة للتطبيق J_A .