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The Sensitivity of Hoc finite differential Matrix Exponential

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Abstract

In this paper we examine how the matrix exponential e^{At} is affected by perturbations in A, where A comes from using the HOC scheme to find the numerical solution of the initial value problem in parabolic P.D.E. We find that e^{At} has less sensitivity to the rounding error when A is normal it where we use log norms and the Jordan factorizations. Using the condition number of the exponential matrix arising from using HOC finite difference method to problem (1), we made a connection between the condition of the eigensystem of A and the sensitivity of e^{At}.

Introduction

The most well known matrix function is the matrix exponential, which has several applications in control theory. In [5] and [7] the perturbation theory of the matrix exponential and error bounds was discussed , in [9] the sensitivity of the matrix exponential using the ordinary finite difference operator was investigare, in our paper we will use the HOC finite difference method operator to analysis effect of the perturbations in A to the exponential e^{At}.

Consider the constant coefficient diffusion equation in one dimensional space , which has the form

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2},\ 0 < x < X,\ t > 0 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

with $\lambda = 1$,subject to initial conditions

$u(x,0) = g(x),\ 0 \le x \le X$,

and boundary conditions

$u(0,t) = u(X,t) = 0,\ t > 0$,

where $g(x)$ is a continuous function of x .The interval $0 \le x \le X$ is divided into $(N+1)$ subintervals each of width h so that $(N+1)h = X$, and the time variable is discretized with steps of length k. The open region $R = (0 < x < X) \times (t > 0)$, ,with its boundary ∂R ,has been covered by a rectangular mesh where the mesh points having coordinates (mh, nk) with $m = 0,1,\ldots,N+1$ and $n = 0,1,2,\ldots$ The notation $u_m^n \equiv u(mh, nk)$ will be used to denote the numerical solution of (1) at the mesh point (mh, nk) while U_m^n will be used to denote the exact solution of the HOC finite difference scheme to the given problem .

The space derivative in (1). is replaced by the HOC finite difference that is

$$\frac{\partial^2 u}{\partial x^2} = \delta_x^2 u - \frac{h^2}{12}\delta_x^2 \frac{\partial u}{\partial t} + o(h^4)\text{ . and}$$

then substituting in (1) we have

$$\frac{\partial u}{\partial t} = \delta_x^2 u - \frac{h^2}{12}\delta_x^2 \frac{\partial u}{\partial t} + o(h^4)\text{ , and}$$

if we take $\frac{\partial u}{\partial t}$ as a common factor we have

$$\frac{\partial}{\partial t}\left(1 + \frac{h^2}{12}\delta_x^2 u\right) = \delta_x^2 u + o(h^4)\ ,\ldots\ldots\ldots(2)$$

where $\delta_x^2 u$ represent the central difference operator which has the form

$$\delta_x^2 u = \frac{u(x-h,y,t) - 2u(x,y,t) + u(x+h,y,t)}{h^2}.$$

If we apply (2) to all N interior mesh points at time $t = nk\ (n = 0,1,2,\ldots)$, then we have a system of ordinary differential equations of the form

$$\frac{d\,\underline{U}(t)}{dt}B = C\,\underline{U}(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $\underline{U}(t) = \underline{U}^n = (U_1^n, U_2^n, \ldots, U_N^n)^T$ and T denotes the transpose, the matrices B and C are $N \times N$ matrices which have the forms

$$B=\begin{bmatrix} a2 & a1 & & & 0 \\ a1 & a2 & a1 & & \\ & \ddots & \ddots & \ddots & \\ & & a1 & a2 & a1 \\ 0 & & & a1 & a2 \end{bmatrix}$$

$$C=\begin{bmatrix} a3 & a4 & & & 0 \\ a4 & a3 & a4 & & \\ & \ddots & \ddots & \ddots & \\ & & a4 & a3 & a4 \\ 0 & & & a4 & a3 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where $a1=\frac{1}{12}$, $a2=\frac{5}{6}$, $a3=\frac{-2}{h^2}$ and $a4=\frac{1}{h^2}$.

Now if we set $A=CB^{-1}$, we can define the exponential of a real $n\times n$ matrix A by $e^{At}=\sum_{k=0}^{\infty}\frac{(At)^k}{k!}$, and many methods exist for the computation of the above matrix exponential [5]. Our basic approach is to investigate the upper bound for $\phi(t)$ where

$$\phi(t)=\frac{\left\|e^{(A+E)t}-e^{At}\right\|_2}{\left\|e^{At}\right\|_2} \ldots\ldots\ldots\ldots\ldots(5)$$

When A and E commute, the bounding of $\phi(t)$ is trivial since

$$e^{(A+E)t}-e^{At}=e^{At}(e^{Et}-1)=e^{At}(tE)\sum_{k=0}^{\infty}\frac{(Et)^k}{(k+1)!}$$

and thus

$$AE=EA\Rightarrow\phi(t)\le\|E\|_2 te^{|E|t} \ldots\ldots\ldots\ldots(6)$$

If A and E fail to commute, then $e^{(A+E)t}\ne e^{At}e^{Et}$ and we will not consider such case .We will use the following identity which appears in Bellman [2]:

$$e^{(A+E)t}=e^{At}+\int_0^t e^{A(t-s)}Ee^{(A+E)s}ds \ldots\ldots(7)$$

And with simple, manipulation with the above equation we have

$$\phi(t)<\frac{\|E\|}{\|e^{At}\|}\int_0^t\left\|e^{A(t-s)}\right\|\left\|e^{(A+E)s}\right\|ds \ldots\ldots(8)$$

Bounding of the matrix exponential (8), are described in § 2. In §3 we use the result, giving in §2 to find the upper bounds for $\phi(t)$.The bounds indicate that e^{At} is least sensitive when A is normal and thus has a well conditioned eigensystem, however, the results in § 4 do not answer the important question of whether poorly conditioned eigensystems imply sensitivity of the exponential, and in the end of §4 we applied the bounds we mentioned in §2 to one dimensional parabolic problem.

In an attempt to more fully characterize e^{At} sensitivity, the Frechet derivative is useful, recall that when it exists, the Frechet derivative of a map F from $C^{n\times n}$ to $C^{n\times n}$ can not be used but the following map can be used $D(F):C^{n\times n}\to C^{n\times n}$ which satisfying

$$F(A+E)-F(A)=D(F(A))E+O(\|E\|),$$

if we substitute

$$e^{(A+E)t}=e^{At}+\int_0^t e^{A(t-r)}Ee^{(A+E)r}dr,$$

into (7) ,where $c^{n\times n}$ is the set of complex $n\times n$ matrix, we find

$$e^{(A+E)t}=e^{At}+\int_0^t e^{A(t-s)}Ee^{As}+o(\|E\|)$$

Thus, the Frechet derivative , $D(e^{At})$,of the map $F(A)=e^{At}$ is gven by

$$D(e^{At})E=\int_0^t e^{A(t-s)}Ee^{As}ds$$

The norm of the operator $D(e^{At})$ effectively quantifies the rate of change of e^{At} with respect to A. This is a precise measure of sensitivity which can be incorporated in an "exponential condition number" and is the subject of our discussion in § 4.

We will use the following notation. Let $C^{n\times n}$ denote the set of complex $n\times n$ matrices. If $A=(a_{ij})\in C^{n\times n}$, then

$A^*=(a_{ij})$,

$\lambda(A)=\{\lambda|\det(A-\lambda I)=0\}$,

$\|A\|=\max\{|\lambda|\lambda^2\in\lambda(A^*A)\}$,

$\kappa(A)=\|A\|_2\|A^{-1}\|_2\,(0\notin\lambda(A))$,

$\alpha(A)=\max\{\mathrm{Re}(\lambda)|\lambda\in\lambda(A)\}$.

For convenience we will use the 2-norm. However most of the results that we present in this paper, with little modification, is valued for the other norms (e.g. for uniform norm)

Bounding e^{At} .In this section we summarize and compare various approaches for which $\|e^{At}\|$ can be bounded.

(a) Power series. By taking the norm of $e^{At}=\sum_{k=0}^{\infty}\frac{(At)^k}{k!}$ we trivially obtain

$$\|e^{At}\| \le e^{\|E\|t} \qquad \text{.......(9)}$$

(b) Log norms. Dahlquist [1] has shown that if $\mu(A)=\max\{\mu \mid \mu\in\lambda((A^*+A)/2)\}$, Then

$$\|e^{At}\| \le e^{\|E\|t} \qquad \text{.......(10)}$$

The scalar $\mu(A)$ is the log norm of A with respect to the 2-norm. A general discussion of log norms may be found in Strom [9]. If the matrix A can be putted as $A=YBY^{-1}$ then $e^{At}=Ye^{Bt}Y^{-1}$ and thus we have

$$A=YBY^{-1}\rightarrow \|e^{At}\| \le \kappa(Y)e^{\mu(B)t} \qquad \text{.......(11)}$$

(c) Jordan canonical form. Recall the Jordan decomposition theorem which states that if $A\in C^{n\times n}$ then there exists an invertible matrix $X\in C^{n\times n}$ such that

$$X^{-1}AX=J_{m_1}(\lambda_1)\oplus\ldots\oplus J_{m_p}(\lambda_p)\equiv J \qquad \text{.......(12)}$$

where

$$J_k=J_{m_k}(\lambda_k)=\begin{bmatrix}\lambda_k & 1 & & 0\\ & \lambda_k & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_k\end{bmatrix}\in C^{m_k\times m_k},\ k=1,\ldots,p \qquad \text{.......(13)}$$

The matrix X is not unique ,it is well known, [8], that if the Jordan canonical form (JCF) of A is specified by (12) and (13) then

$$e^{At}=X[e^{J_1t}\oplus\ldots\oplus e^{J_pt}]X^{-1}, \qquad \text{.......(14)}$$

where

$$e^{J_kt}\equiv e^{\lambda_kt}\begin{bmatrix}1 & t & t^2/2 & & t^r/r!\\ & 1 & t & & \\ & & \ddots & \ddots & \\ & & & & t\\ 0 & & & & 1\end{bmatrix},$$

$$r=m_k-1 \qquad \text{.......(15)}$$

Using the fact that if $\|B\|\le q\max|b_{ij}|$,for $B\in C^{q\times q}$, then from (15) we can show that

$$\|e^{J_kt}\|\le m_k\, e^{\lambda_kt}\max_{0\le j\le m_k-1} t^j/j!$$

Taking the norms to (14) and define $m=\max\{m_1,\ldots,m_p\}$ we obtain

$$\|e^{At}\|\le m\kappa(X)e^{\alpha(A)t}\max_{0\le r\le m_k-1} t^r/r! \qquad \text{.....(16)}$$

To illustrate how to compute the above bounds we will reduce the parabolic equation, (1), using HOC finite difference method to a system of linear ordinary differential equation $\frac{dX(t)}{dt}-CB^{-1}X(t)=AX(t)$, see equation (3), where

$$A=\begin{bmatrix}-1+\varepsilon & 0 & 0\\ -2 & -1-\varepsilon & 0\\ 6 & 4 & -2+\varepsilon\end{bmatrix},\ \varepsilon=10^{-10}$$

then using (9), (16), (10), and (11) we have

(a) $\|e^{At}\|\le e^{7.8273t}$,

(b) $\|e^{At}\|\le 1.0149\times10^{11}e^{(-1+\varepsilon)t}$

(c) $\|e^{At}\|\le e^{1.668t}$

(d) $\|e^{At}\|\le 2e^{0.4128t}$,

In table (1) we compare these bounds for a selected values of t.

Table(1)

t	$\|e^{At}\|$	Power(a)	Jordan(b)	Log Norm (c)	Log Norm(d)
0	3.00E+00	1	1.01E+11	1.00E+00	2.00E+00
5	1.07E+13	9.93E-16	6.84E+08	4.60E+03	1.58E+01
10	1.14E+26	9.85E+25	4.61E+06	7.12E+07	1.24E+02
15	1.22E+39	9.76E+50	3.10E+04	9.75E-10	9.78E+02
20	1.30E+52	9.71E+67	2.09E+02	4.48E+14	7.70E+03
25	1.39E+65	9.63E+84	1.41E-00	2.06E+18	6.07E+04
30	1.49E+78	9.562e+101	9.50E+05	9.50E+21	4.78E-05
35	1.39E+91	9.4909e+118	6.3991e+005	4.3707e-025	3.7649e+006
40	1.8176e+117	9.4204e+135	4.3117e+002	2.0114e+029	2.9658e+007

We can conclude from Table(1) that some of the upper bounds may fail to decay along with e^{At} ,where the behavior of e^{At} depends upon the sign of $\alpha(A)$ according to following relation:

$$\lim_{t\to\infty} e^{At} = 0 \Leftrightarrow \alpha(A) < 0$$

hence, the Jordan bound (16) decay precisely when e^{At} decays while the other bound grows regardless of the sign of $\alpha(A)$,the example also shows when A has an ill-conditioned eigensystem (i.e. $\kappa(X)$ large) then the decay of Jordan bound will not become slow. In general, the effectiveness , to the rounding error of one bound relative to the other depends on the matrix A and time t. However, when A is normal, then $\alpha(A)=\mu(A)$ in (10), $m=\kappa(X)=1$ in (16), and, thus we have

$$\left\|e^{At}\right\| \le e^{\alpha(A)t},$$

also since we have $\left\|e^{At}\right\| \ge \left|e^{\lambda t}\right|$ for all $\lambda \in \lambda(A)$,then $\left\|e^{At}\right\| \ge e^{\alpha(A)t}$ imply that if

$$A^*A = AA^* \; then \; \left\|e^{At}\right\| = e^{\alpha(A)t}.$$

When A is not normal, it is possible for $\left\|e^{At}\right\|$ to grow initially even though $\alpha(A)$ is negative. In this case the factor $\left(m\,\kappa(X) \max_{0\le j\le m-1} \frac{t^j}{j!} \right)$ in (16), will necessary, accommodate the "hump" in the graph of $\left\|e^{At}\right\|$ and thus whether or not $\left\|e^{At}\right\|$ grows will depends upon the sign of $\mu(A)$:

$$\sup_{t\ge 0} \left\|e^{At}\right\| = 1 \Leftrightarrow \mu(A) \le 0$$

The above result follows from (10) and the fact that $\mu(A)$ is the derivative of $f(t)=\left\|e^{At}\right\|$ at t = 0,

Perturbation bounds. In this section we will use the results given in the previous section to (8). To simplify this process it is convenient to use the following lemma which is given in [6],[7],[4] and[5].

lemma(1): If $M(t)$ is monotone increasing function on $[0,\infty)$ and $\left\|e^{At}\right\| \le M(t)e^{\beta t}$ for all $t \ge 0$, then

$$\phi(t) = \frac{\left\|e^{(A+E)t} - e^{At}\right\|}{\left\|e^{At}\right\|} \le \|E\| t M(t)^2 e^{[\beta-\alpha(A)+\|E\|M(t)]t}$$

Using the above lemma with following theorem we can show that $\phi(t)$ has the form

Theorem (1):

$$\phi(t) = t\|E\|_2 \, e^{(\|A\| - \alpha(A) - \|E\|)t} \quad \ldots(17)$$

Proof. By virtue of (9) we can apply Lemma(1) with $\mu(t)=1$ and $\beta = \|A\|$

Theorem (2):

$$\phi(t) \le t\|E\| e^{(\|\mu(A) - \alpha(A) + E\|)t} \quad \ldots\ldots(18)$$

Proof. By using (10) we can set $\mu(t)=1$ and $\beta=\mu(A)$ and invoke Lemma1.

Theorem(3) :If the Jordan decomposition of A is given by (12) and (13), then

$$\phi(t) \le t\|E\| M_J(t)^2 e^{M_J(t)\|E\|t} \quad \ldots\ldots\ldots\ldots(19)$$

where $M_J(t) = m\kappa(X) \max_{0\le j\le m-1} \frac{t^j}{j!}$.

Proof. $M_J(t)$ is a monotone increasing function and from (16), $\left\|e^{At}\right\| \le M_J(t)e^{\alpha(A)t}$ we can apply Lemma(1) with $M(t)=M_J(t)$ and $\beta=\mu(A)$.

Now by using (2.12), the following result can be obtained as a corollary to any of above Theorems (2) or(3) ,

$$\text{If } A^*A = AA^* \; then \; \phi(t) < t\|E\| e^{\|E\|t}.$$

The upper bounds appearing in (17)-(19) are just an upper bounds. They are not necessarily accurate measures of the sensitivity of e^{At}. Since these bounds are smaller when A is normal, and such result suggest that there is a connection between the normality of A and the sensitivity of e^{At} to a rounding error ,we will shed some light on this connection in the next section.

The condition number of the exponential matrix.

In [8] Rice gave a general theory of the condition number, and thus using this theory to measure the sensitivity, at any point $A \in X$, of a map F from a metric space X to a metric space Y. In this section we will investigate the sensitivity of e^{At} to rounding error, by applying Rice's definitions to $F(t)=e^{At}$.

Before we begin, it is instructive to look at the idea of conditioning in a more familiar setting, let $\kappa(A)=\|A\|\|A^{-1}\|$ be the "condition number of a matrix with respect to inversion". The sensitivity of A^{-1} to rounding error can be justify from the following inequality

$$\frac{\|(A+E)^{-1}-A^{-1}\|_2}{\|A^{-1}\|_2}\le\frac{\|E\|_2}{\|A\|_2}\frac{\kappa(A)}{1-\|E\|_2\|A^{-1}\|_2}$$

$(\|E\|_2\|A^{-1}\|_2<1)$..............................(20)

It is always possible to choose the perturbation matrix E such that the above upper bound is attained. Thus, if $\kappa(A)$ is large, it is possible that a small change in the element of A will induce a relatively large change in the element of A^{-1}, that is we claim $\kappa(A)$ measures the sensitivity of the map $(A\mapsto A^{-1})$.

We now formulate a relevant condition number consistent with Rice's theory.

Definition. The condition number of the exponential matrix A at time t is defined by

$$\nu(A,t)=\lim_{\delta\to 0}\nu_\delta(A,t),$$

where

$$\nu_\delta(A,t)=\sup_{\|E\|\le\delta\|A\|}\frac{\|e^{(A+E)t}-e^{At}\|}{\delta\|e^{At}\|}.$$

Geometrically $r=\delta\|e^{At}\|\nu_\delta(A,t)$ is the radius of the smallest sphere in $C^{n\times n}$ which is centered at e^{At} and encloses the image of the set $\{B\ \|A-B\|\le\delta\|A\|\}$ under the mapping $B\mapsto e^{At}$, when small relative changes in the element of A produce relatively large alterations in the element of e^{At} and then $\nu_\delta(A,t)$ correspondingly large.

Our investigation of the condition number $\nu(A,t)$ begins with the following theorem.

Theorem(4) :If $D(e^{At})$ denotes the Frechet derivative of $F(t)=e^{At}$ then

$$\nu(A,t)=\frac{\|D(e^{At})\|_2\|A\|_2}{\|e^{At}\|_2},$$

where

$$\|D(e^{At})\|_2=\sup_{\|E\|\le 1}\left\|\int_0^t e^{A(t-s)}Ee^{As}ds\right\|_2.$$

Proof. As we noted in the introduction,

$\|e^{(A+E)t}-e^{At}\|=D(e^{At})E+o(\|E\|)$ where

$D(e^{At})E=\int_0^t e^{A(t-s)}Ee^{As}ds$, thus,

$$\nu(A,t)=\lim_{\delta\to 0}\sup_{\|E\|\le\delta\|A\|}\frac{\|D(e^{At})E+o(\|E\|)\|}{\delta\|e^{At}\|}$$

$$=\lim_{\delta\to 0}\sup_{\|E\|\le\|A\|}\left[\frac{\|D(e^{At})E\|}{\delta\|A\|}\frac{\|A\|}{\|e^{At}\|}+\frac{o(\delta\|A\|)}{\delta\|e^{At}\|}\right]$$

$$=\sup_{\|E\|\le 1}\|D(e^{At})E\|\frac{\|A\|_2}{\|e^{At}\|_2}=\|D(e^{At})E\|\frac{\|A\|}{\|e^{At}\|}.$$

Corollary(1) : $\nu(A,t)\ge t\|A\|$ *for all* $t\ge 0$.

Proof.

$$\nu(A,t)\ge\left\|\int_0^t e^{A(t-s)}Ie^{As}ds\right\|_2\frac{\|A\|_2}{\|e^{At}\|_2}=t\|A\|_2.$$

Corollary(2) :If A is normal then $\nu(A,t)=t\|A\|_2$.

Proof. In view of the previous corollary, all we must show is that when A is normal, then $\nu(A,t)\le t\|A\|$ but this result follows by taking the norm for the expression $\nu(A,t)$ and then using (2.12) we have

$$\nu(A,t)\le\int_0^t\|e^{A(t-s)}\|_2\|e^{As}\|_2ds\frac{\|A\|_2}{\|e^{At}\|_2}$$

$$=\int_0^t e^{\alpha A(t-s)}e^{\alpha(A)s}ds\frac{\|A\|_2}{e^{\alpha(A)t}}=t\|A\|.$$

For the above corollary we can conclude that, if the matrix A is normal, then, the condition number $\nu(A,t)$ is small as possible. It is rather more difficult to identify the case when A for which $\nu(A,t)>>t\|A\|$. This is in contrast to the matrix inversion problem where the ill-conditioned matrices can be characterized through the singular value decomposition. We can conclude this section a result analogous to (20), and to do so we need to introduce the following functions:

$$\hat{v}(A,t) = \max_{0 \le s \le t} v(A,s),$$

$$\varphi(A,t) = \int_0^t \left\| e^{-As} \right\| \left\| e^{As} \right\| ds \;,$$

the function $\hat{v}(A,t)$ can be regarded as an exponential condition number of a matrix A over the interval [0,t].We also remark that $\varphi(A,t)$ is monotone increasing function and thus $\varphi(A,t) \ge t$. (Incidentally, $\varphi(A,t) = t$ only if $A = \lambda I + S$ where $S^{*} = S$.)

Theorem(5) : if $\varphi(A,t)\|E\|_2 \le 1$,then

$$\frac{\left\| e^{(A+E)t} - e^{At} \right\|_2}{\left\| e^{At} \right\|_2} \le \frac{\|E\|_2}{\|A\|_2} \frac{\hat{v}(A,t)}{1 - \|E\|_2 \varphi(A,t)} \;.$$

Proof: see[7].

Our numerical shows that when A is normal we may replace $\hat{v}(A,t)$ with $v(A,t)$ in (21) and, again we observe that the upper bound , dose not increase . Finally the bounds discussed in §2 are tested with the following problem:

consider the heat equation in one dimensional space

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

with boundary conditions $u(0,t) = u(2,t) = 0$, $t > 0$,

and initial conditions $u(x,0) = 1$,for $0 \le x \le 2$,

Let $h = 0.2$,the dimension of matrices B and C, given in (4) will be an 9×9, the bounds (9), (16), (10), and (11) illustrated on in Table (2) below for a selected values of t.

Table (2)

t	$\left\| e^{At} \right\|$	Power (a)	Jordan (b)	Log Norm (c)	Log Norm(d)
0	3	1	1.4142	1	7.7485
5	8.19E+42	2.81E+41	164.43	4.48E-06	9.38E+06
10	4.75E+25	7.88E+82	19119	2.03E-11	1.13E+13
15	2.75E+38	2.71E+124	2.22E+06	8.96E-17	1.37E+19
20	1.59E+51	6.21E+165	2.58E+08	4.01E-22	1.66E+25
25	9.23E+63	1.74E+207	3.01E+10	1.79E-27	2.01E+31
30	5.35E–76	4.89E+248	3.45E–12	8.03E-33	2.41E+37
35	1.10E+89	1.37E–290	4.05E+14	3.59E-38	2.05E+43
40	1.79E+102	Inf	4.72E+16	1.61E-43	3.56E+49

the symbol Inf refer to a number greater than 1.7977e+308 which the computer cannot holed it , again we can conclude that the Jordan bound decay when e^{At} decays while the other bound are not.

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الخلاصة

في هذا البحث قمنا باختبار كيفية تأثر المصفوفة الأسية (e^{At}) لتغير الطيف في المصفوفة A الناتجة من حل مسائلة القيمة الابتدائية بأسلوب(IIOC) وهو نظام الفروقات المنتهية ذو الرتبة العالية المتراص ، لقد توصلنا إلى كون المصفوفة الأسية (e^{At}) اقل تأثرا عندما تكون A مصفوفة متعامد مستخدمين (log norms) و (Jordan factorizations) وخلال مراحل صياغة العدد الشرطي الأسي، حاولنا تفسير العلاقة الغير صريحة ما بين هذا الشرط و بين النظام الذاتي للمصفوفة A .