Special Issus: 1st Scientific International Conference, College of Science, Al-Nahrain University, 21-22/11/2017, Part I, pp.166-170

When Compact Sets are α-Closed

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Abstract

This paper is devoted to introduce new concepts so called K(αc)-spaces several various theorems about these concepts are provided in addition, further properties are studied such as the relationships between those concepts and other types of K(αc)-spaces are investigated.
[DOI: 10.22401/ANJS.00.1.22]

Keywords: α-open set, α-compact, $\alpha - lindelof$.

1-Introduction

It is known that there is no relationship between compact sets and closed sets, so this motivates the author [1] to introduce a concept of Kc-spaces; these are the spaces in which every compact subset is closed, and we know that there is no relationship between compact sets and semi closed sets so in 2015 the researcher [6] studied another type of Kc-space namely K(sc)-space which is every compact are semi-closed sets. The researcher [6] has been studied by another type of Kc-space namely K(θc)-space which is every compact space are θ-closed sets and this space (K(θc)-space) will be weaker than Kc-space. Also the same researcher [6] introduced another type of Kc-space namely θK(θc)-space which is weaker than Kc-space. After studying all the above of these space, we studied the relationship compact and α-closed and we found that there is no relationship between compact and α-closed, as we explained the examples during the course of the research, so we studied another type of Kc-space namely K(αc)-space which is every compact sets are α-closed. The aim of this paper is to continuous study Kc-spaces.

The basic definitions that needed in this work are recalled. In this work, a subset A of a topological space (X,τ) is called $\alpha -$open if $A \subset int\, cl\, int(A)$ and $\alpha -$closed set $if\ cl\, int\, cl(A) \subset A$ [2], as the following example $(R,\tau_{Excluded})$ be an $\alpha -$open, where $\tau_{Excluded} = \{U \subseteq X, x_0 \notin U, \text{for some } x_0 \in X\} \cup \{X\}$. Every open is $\alpha -$open, but the convers may be not true[3],as the following example X={1,2,3}, $\tau = \{\emptyset, X, \{1\}\}$, let $A = \{1,2\}$ therefore $A^0 = \{1,2\}^0 = \{1\}$ and hence $\overline{A^0}$=$\overline{\{1\}}$=X⇒$\overline{A^0}^0$=$X^0 = X$. This implies to $\{1,2\} \subseteq \overline{\{1,2\}^0}^0$ is $\alpha -$open but $\{1,2\}$is not open, a subset A of a topological space X is an open if and only if there exists an open set U such that $U \subset A \subset int\, cl(A)$[4], let U be an open set a space X, then every subset B of $X, U \cap \overline{B} \subseteq \overline{U \cap B}$ [5].

2-On K(αc) −space

Since we provided the fact that there is no relationship between compact and closed ($\alpha -$closed) sets, as in the following example shows:

(1)- In (R,τ_{ind}) the subset Q in R is compact, but not $\alpha -$closed, in the fact the non empty subsets of R is compact, but not $\alpha -$closed.

(2)- In (R,τ_{coc}) the set R is $\alpha -$closed, which is not compact.

(3)- In (R,τ_D) the set R is closed ($\alpha -$closed), but not compact.

Now we introduce the following concept:

Definition (1):

A space X is called k(α c)-space if every compact subset of X is α- closed.

Example (1):

(X,τ_D), where τ_D be an discrete space is K(αc) −space.

Definition (2):

A nonempty set X with a topological space τ is said to be $\alpha -$compact, if every cover of X with $\alpha -$open sets has finite sub cover.

Definition (3):

A nonempty set X with a topological space τ is said to be $\alpha-$Lindel of, if every cover of X with $\alpha-$open sets has a countable sub cover.

Example (2):

(R,$\tau_{Excluded}$) is $\alpha-$compact and $\alpha-$ lindelof.

Remark (1):

(i) Every $\alpha-$compact (α-Lindelof) is compact (Lindelof).

(ii) Every $\alpha-$compact is $\alpha-$Lindelof, but the convers may be not true.

Examples (3):

(a) In (Z,τ_D) the set Z is $\alpha-$Lindelof (Lindelof) but not $\alpha-$compact (compact).

In fact every infinite countable set with discrete topology satisfy the above results.

(b) (R,τ_{coc}) is $\alpha-$Lindelof, but not $\alpha-$compact

Proposition (1):

Every $\alpha-$closed set in $\alpha-$compact ($\alpha-$Lindelof) is $\alpha-$compact ($\alpha-$Lindelof).

Proof:

Let (X,τ) be an $\alpha-$compact space and Y is $\alpha-$closed in X, to prove that Y is $\alpha-$compact, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an $\alpha-$open cover $for\ Y$ (means that) Y$\subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$, where U_α is α-open in X $\forall\alpha\in\Lambda$, since X = Y$\cup Y^c \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \cup Y^c$, also Y is $\alpha-$closed in X $\Rightarrow Y^c is\ \alpha-$open in X, so $\bigcup_{\alpha\in\Lambda}\{U_\alpha\} \cup Y^c$ is $\alpha-$open cover for X, which is $\alpha-$compact, then there exists $\alpha_1, \dots, \alpha_n \in \Lambda$ such that:

$$X=\bigcup_{i=1}^{n} U_{\alpha_i} \cup Y^c \Rightarrow Y \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

which implies that $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover from $\{U_\alpha\}_{\alpha\in\Lambda}$ of Y, therefore Y is α-compact. ■

Definition (4), [4]:

A space X is said to αT_1-space if every two distinct points x, y in X, there exists two $\alpha-$open sets $U, V,$ such that $x \in U,$ but $x \notin V$ and $y \in V$ but $y \notin U$.

Example (4):

(R,τ_{cof}) is αT_1-space.

Definition (5), [4]:

A space X is said to αT_2-space if every two distinct points x, y in X, there exists two $\alpha-$open sets U, V, such that $x \in U$, but $x \notin V$ and $y \in V$ but $y \notin U$ and $U \cap V = \emptyset$.

Example (5):

(R, τ_D) is $\alpha T_2 - space$.

Theorem (1):

Every $\alpha-$compact set of αT_2-space is $\alpha-$closed .

Proof:

Let A be an $\alpha-$compact set in a topological space X, to prove that A is $\alpha-$closed, let p$\in A^c$ since X αT_2-space, so $\forall q \in A,$ there exists $U, V \in \tau^\alpha$(=the set of all $\alpha-$open sets) with $q \in U, p \in V$ such that $U \cap V = \emptyset,$ now the collection $\{U(q): q \in A\}$ is an $\alpha-$open cover of A, which is $\alpha-$compact, then there exist finite sub cover of A, that is $A \subset \bigcup_{i=1}^{n} U(q_i), let\ V^* = \bigcap_{i=1}^{n} V_{q_i}(p) \Rightarrow U^* = \bigcup_{i=1}^{n} U(q_i), then\ V^*$ is an $\alpha-$open set containing p (finite intersection of $\alpha-$open sets), we claim that $U^* \cap V^* = \emptyset, let\ x \in U^*$ there fore $x \in U(q_i)$ and suppose $x \notin V^*$ and hence $A \cap V^* = \emptyset, V^* \subseteq A^c \Longrightarrow A^c$ is $\alpha-$open in X. This implies to $A\ is\ \alpha - closed$. ■

Now we introduce weak form of K(αc)$-$space which was previously submitted.

Definition (6):

A space X is called αK(αc)$-$spaceif every $\alpha-$compact subset of X is $\alpha-$closed .

Example (6):

(a) (R,τ_{coc}) is $\alpha K(\alpha c)-$space, $K(\alpha c)-$space and $Kc-$space, but not an $Lc-$space.

(b) (R, τ_D) is $\alpha K(\alpha c)-$space and satisfy all other concepts in above.

Proposition:

A subset M of a topological space X is an $\alpha-$closed if and only if there exists an closed set F such that $\overline{F^O} \subseteq M \subseteq F$.

Proof:

Suppose that M is $\alpha-$closed to prove that $\overline{F^O} \subseteq M \subseteq F$, where F is closed

Since:

cl int cl (M)$\subset M$ ...(1)

and

M$\subset cl(M)$ ... (2)

then by (1) and (2), we get that:

Cl int cl (M)$\subset M \subset cl(M)$, Put cl(M) $= F$ so $cl\ int\ (F) \subset M \subset F$

Conversely: now to prove that M is $\alpha -$closed, whenever there exists an closed sets F, such that $\overline{F^O} \subseteq M \subseteq F$, since $cl\ int\ (F) \subset M \subset F$, therefore $cl\ cl\ int\ (F) \subset cl(M) \subset cl(F) = F$

And hence $cl\ int\ (F) \subset cl(M) \subset F$

$\Rightarrow int\ \ cl\ int\ (F) \subset int\ cl(M) \subset int\ (F)$

$\Rightarrow cl\ int\ \ cl\ int\ (F) \subset cl\ int\ cl(M) \subset$ $cl\ int\ (F)$ (Since $cl\ int\ (F) \subseteq$ M$\subseteq F$)

Therefore $cl\ int\ cl\ (M) \subset M$

This implies to M is $\alpha -$closed. ■

Lemma (1):

let (X, τ) and (Y, τ') be two topological spaces, let $f: X \to Y$ be a homeomorphism from a space X in to space Y if F is $\alpha -$closed in X , then $f(F)$ is $\alpha -$closed in Y.

Proof:

Let F be an $\alpha -$closed in X that is; $\overline{\overline{M^0}} \subseteq F \subseteq M,$ to prove that $f(F) is\ \alpha -$closed in Y

Since $\overline{\overline{M^0}} \subseteq F \subseteq M,$ therefore $\overline{f(M)^0} = f(\overline{\overline{M^0}}) \subseteq f(F) \subseteq f(M),$ "($\overline{f(M)^0} = f(\overline{\overline{M^0}})$", since M is closed set in X and f is closed function (since f is homeomorphism), so f(M) is closed set in Y, since $M^0 \subseteq M$, so $f(M^0) \subseteq f(M)$, But $(f(M))^0 = f(M^0) \subseteq f(M)$($f$ is a homeomorphism), also:

$\overline{(f(M))^0} = \overline{\overline{f(M^0)}}$ ... (2)

$\overline{f(M^o)} = f(\overline{M^0})$ ... (3)

Whenever f is home & M^0 is a subset of X, then by (2) and (3) we get that

$\overline{(f(M))^0} = f(\overline{M^0})$ ... (4)

And by (1) and (4) we get that$\overline{(f(M))^0} \subseteq f(F) \subseteq f(M)$, so $f\ (F)$ is an $\alpha -$closed in Y.

Definition (7), [5]:

Let $f: X \to Y$ is a function of a space, then:

(i) f is called a continuous function if $f^{-1}(A)$ is an open set in X for every open set A in Y.

(ii) f is called an open function if f(A) is an open set in Y, for every open set A in X.

Definition (8), [4]:

Let f: X→ Y be a function of a space, then f is called $\alpha -$continuous function, if $f^{-1}(A)$is an $\alpha -$open set in X for every open set A in Y.

Definitions (9):

Let f: X→ $Y\ be$ a function of a space, then:

(i) f is called an α-open function if f (A) is an α-open in Y, for every open set A in X.

(ii) f is called an $\alpha^* -$open function, if $f(A)$ is an open in $Y,$ for every $\alpha -$ open set A in X.

(iii) f is called an $\alpha^{**} -$open function, if $f(A)$ is an $\alpha -$ open in $Y,$ for every

(iv) α −open set A in X.

(v) f is called an α-closed function if f(A) is an α-closed in Y, for every closed set A in X.

(vi) f is called an $\alpha^* -$closed function, if $f(A)$ is an closed in $Y,$ for every α −closed set A in X.

(vii) f is called an $\alpha^{**} -$closed function, if $f(A)$ is an $\alpha -$ closed in $Y,$ for every α −closed set A in X.

(viii) f is called α^*-continuous function if $f^{-1}(A)$ is an open set in X for every $\alpha -$ open set A in Y.

(ix) f is α**-continuous function if $f^{-1}(A)$ is an α-open set in X for every $\alpha -$ open set A in Y.

Definition (10):

A space X is called (αK)c−space, if every $\alpha -$compact subset of X is closed.

Example (7):

The discrete space (X,τ_D) satisfy the above definition.

Theorem (2):

let f: X→ Y be bijective $\alpha^* -$open and closed function, then Y is (αK) c-space, whenever X is (αK)c-space.

***Proof*:**

Let A be $\alpha -$compact in Y, to prove that $f^{-1}(A)$ is $\alpha -$compact in X, let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an $\alpha -$open cover to $f^{-1}(A)$ (means that) $f^{-1}(A) = \cup_{\alpha \in \Lambda}(U_\alpha)$

Therefore $ff^{-1}(A) = f(\cup_{\alpha \in \Lambda}\ U_\alpha) = \cup_{\alpha \in \Lambda} f(U_\alpha)$ and hence $A \subseteq \cup_{\alpha \in \Lambda} f(U_\alpha),$ since f(U_α) is $\alpha -$open in $Y\ \forall \alpha \in \Lambda$ therefore but A is $\alpha -$compact, so:

$A \subseteq \bigcup_{i=1}^{n} f(U_{\alpha_i})$
which implies to:
$f^{-1}(A) \subseteq f^{-1} \bigcup_{i=1}^{n} f(U_{\alpha_i})$
$= \bigcup_{i=1}^{n} ff^{-1}(U_{\alpha_i})$
$= \bigcup_{i=1}^{n} U_{\alpha_i}$ (since f is one to one)
Therefore:
$f^{-1}(A) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$
and so $f^{-1}(A)$
is α −compact in X, which is $(\alpha K)c$ −space, so $f^{-1}(A)$ is closed $\Rightarrow f(f^{-1}(A)) = A$ is closed in Y, which implies X is (αk)c-space ■

Proposition (2):

Let $f: X \rightarrow Y$ be bijective open and α^*-closed (α^{**} −closed) function, then Y is $K(\alpha c)$-space, whenever X is K(αc)-space.

Proof:

Let A be compact in Y, to prove that $f^{-1}(A)$ is compact in X, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover to $f^{-1}(A)$ (means that) $f^{-1}(A)$ $=\bigcup_{\alpha\in\Lambda}(U_\alpha) \Longrightarrow ff^{-1}(A) = f(\bigcup_{\alpha\in\Lambda} U_\alpha) = \bigcup_{\alpha\in\Lambda} f(U_\alpha) \Rightarrow A \subseteq \bigcup_{\alpha\in\Lambda} f(U_\alpha)$, since f($U_\alpha$) is open in Y $\forall \alpha \in \Lambda$ ⇒but A *is* compact, so $A \subseteq \bigcup_{i=1}^{n} f(U_{\alpha_i}) \Longrightarrow f^{-1}(A) \subseteq$
$f^{-1} \bigcup_{i=1}^{n} f(U_{\alpha_i}) = \bigcup_{i=1}^{n} ff^{-1}(U_{\alpha_i}) =$
$\bigcup_{i=1}^{n} U_{\alpha_i}$ (since f is one to one)
$\Longrightarrow f^{-1}(A) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$
so $f^{-1}(A)$ is compact in X, which is $K(\alpha c)$ −space, so $f^{-1}(A)$ is $\alpha -$ closed
$\Rightarrow f(f^{-1}(A)) = A$ is closed in Y, so A is closed, which implies it is α −closed in Y, which A is compact
Therefore, it is K(αc) –space. ■

Proposition (3):

let f: X→ Y be bijective α^* −open and α^{**} − closed function, then Y is $\alpha K(\alpha c)$-space, whenever X is αK(αc)-space.

Proof:

Let A be α −compact in Y, to prove that $f^{-1}(A)$ is α −compact in X, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an α −open cover *to* $f^{-1}(A)$ (means that) $f^{-1}(A)$ $=\bigcup_{\alpha\in\Lambda}(U_\alpha) \Longrightarrow ff^{-1}(A) = f(\bigcup_{\alpha\in\Lambda} U_\alpha) = \bigcup_{\alpha\in\Lambda} f(U_\alpha) \Rightarrow A \subseteq \bigcup_{\alpha\in\Lambda} f(U_\alpha)$, since f($U_\alpha$) is α −open in Y $\forall \alpha \in \Lambda \Rightarrow$ *but A is* α −compact, so:
$A \subseteq \bigcup_{i=1}^{n} f(U_{\alpha_i})$
Hence:
$f^{-1}(A) \subseteq f^{-1} \bigcup_{i=1}^{n} f(U_{\alpha_i})$
$= \bigcup_{i=1}^{n} ff^{-1}(U_{\alpha_i})$
$= \bigcup_{i=1}^{n} U_{\alpha_i}$ (since f is one to one)
which implies to $f^{-1}(A) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$
so $f^{-1}(A)$ is α −compact in X, which is $\alpha K(\alpha c)$ −space, so $f^{-1}(A)$ is α- closed, so $f(f^{-1}(A)) = A$ is $\alpha -$ closed in Y, which A is α −compact, then X is αK(αc)-space ■

Proposition (4):

The α*-continuous image and serjective of α −compact (α −Lindelof) set is α −compact (α −Lindelof).

Proof:

Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an α −open cover for Y (means that) $Y = \bigcup_{\alpha\in\Lambda} U_\alpha$, but f is continuous, so $f^{-1}(U_\alpha)$ is α −open in X $\forall \alpha \in \Lambda$
$f^{-1}(Y) = f^{-1}(\bigcup_{\alpha\in\Lambda} U_\alpha) =$
$\bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha)$, also $f^{-1}(Y) = X$
$\Longrightarrow X = \bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha) \Rightarrow \{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ is an α −open cover for X, which is α −compact ⟹there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$, such that X=$\bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i}) =$
$f^{-1}(\bigcup_{i=1}^{n}(U_{\alpha_i}))\ therefore\ f(X) =$
$f(f^{-1} \bigcup_{i=1}^{n}(U_{\alpha_i})) = \bigcup_{i=1}^{n}(U_{\alpha_i})$ (since f is on to) and hence $f(X) = Y \Rightarrow Y = \bigcup_{i=1}^{n} U_{\alpha_i}$ which implies $Y\ is\ \alpha$ −compact. ■

Proposition (5), [4]:

Every finite space is α −compact.

Proof:

Let (X,τ) be a finite topological space say X= $\{a_1, \ldots, a_n\}$, to prove that X is α −compact, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an α −open cover to X (means that)
$X = \bigcup_{\alpha\in\Lambda} U_\alpha$, since $a_i \in X = \bigcup_{\alpha\in\Lambda} U_\alpha, X = \bigcup_{i=1}^{n}\{a_i\} = \bigcup_{i=1}^{n} U_{\alpha_i} \rightarrow X\ is\ \alpha -$ compact. ■

Proposition(6):

The property of being a K(αc) −space is a topological property.

Proof:

Let $f: X \rightarrow Y\ b$e a homeomorphism function from an $K(\alpha c)$ −space X in to a space Y to show that Y is also $K(\alpha c)$ −space.

Let A be a compact subset of Y, but $f^{-1}(A)$ is also compact of X, which is $K(\alpha c)$ −space, then $f^{-1}(A)$ is α −closed in X, since f is onto, then $A = f(f^{-1}(A))$, then

A is $\alpha-$closed of $Y,$ therefore Y is an $K(\alpha c)-$space.

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