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Essentially Small Quasi-Dedekind Module Relative to a Module

Mukdad Qaess Hussein [1], Marrwa Abdallah Salih [2] and Nagham Ali Hussen [3]
[1] Department of Computer Science, College of Education for Pure Science, Diyala University, Diyala-Iraq.
[2] Department of Mathematics, College of Education for Pure Science, Tikrit University, Tikrit-Iraq.
[3] College of Engineering, University of Information Technology and Communications, Baghdad-Iraq.

Abstract

Let R be a ring with identity and let M be a unitary left module over R. In this paper, we study direct summand (direct sum) of essentially small quasi-Dedekind module (essentially small quasi-Dedekind modules). Also, give the definition of essentially small quasi-Dedekind relative to a module with some examples. We give some of their basic properties and some examples that illustrate these properties. [DOI: 10.22401/ANJS.00.1.23]

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Introduction

This paper study the direct summand of essentially small quasi-Dedekind module and the direct sum of essentially small quasi-Dedekind modules need not be essentially small quasi-Dedekind. We give the definition of essentially small quasi-Dedekind module relative to a module.

A submodule A of an R-module M is called small in M ($A \ll M$) if whenever a submodule B of M with M = A + B implies B = M, [1].

An R-submodule N of an R-module M is called essentially small ($N \ll_e M$), if for every nonzero small submodule K of M, $K \cap N \neq \{0\}$. Equivalently, for each $0 \neq x \in M$, there exists $0 \neq r \in R$ such that $0 \neq rx \in N$.

An R-module M is called essentially small quasi-Dedekind if Hom(M/N, M) = {0} for all $N \ll_e M$.

A ring R is essentially small quasi-Dedekind if R is an essentially small quasi-Dedekind R-module.

A submodule N of an R-module M is called small invertible if $N^{-1}N = M$, where $N^{-1} = \{x \in R_T : xN \ll M\}$ and R_T is the localization of R at T in the usual sense, T = {s ∈ S: sm = 0 for some m ∈ M, then m = 0}, where S is the set of all nonzero divisors of R.

An R-module M is called small quasi-Dedekind, if every nonzero R submodule N of M is small quasi-invertible; that is Hom(M/N, M) = {0}, for all $\{0\} \neq N \ll M$.

A ring R is small quasi-Dedekind if R is a small quasi-Dedekind R-module.

The property of essentially small quasi-Dedekind module is inherited by direct summand.

Proposition 1:

A direct summand of an essentially small quasi-Dedekind module is an essentially small quasi-Dedekind module.

Proof:

Let M be an essentially small quasi-Dedekind R-module and let $N \leq^{\oplus} M$, then $M = N \oplus K$, for some submodule $K \leq M$. Let $f \in End_R(N)$, $f \neq 0$, to prove that Ker f $\ll_e$ N. Consider the following: $M \xrightarrow{\rho} N \xrightarrow{f} N \xrightarrow{i} M$, where ρ is the natural projection, and i is the inclusion mapping. Hence $h = iof\rho \in End_R(M)$ and $h \neq 0$, so Ker h $\ll_e$ M and since $Kerf \subseteq Kerh$ then Kerf $\ll_e$ M. Now assume that Kerf $\ll_e$ N, we shall show this implies Ker h $\ll_e$ M and so we get a contradiction.

Let x + y be any nonzero element of M, where $x \in N, y \in K$. If $x \neq 0$ and (y = 0 or $y \neq 0$), then since Kerf $\ll_e$ N, there exists $0 \neq r \in R$ such that $0 \neq rx \in Kerf$. Hence $rx + ry \neq 0$, because if rx+ry = 0, then rx = –ry ∈ N∩K = {0} which is a contradiction. Also h(rx + ry) = 0; that is $0 \neq r(x+y) \in Ker\,h$. If x = 0 and $y \neq 0$, then $x + y = y \neq 0$ and 1.y = y, h(y) = iof ρ (0+y) =

iof(0) = f(0) = 0; that is $0 \neq 1(x+y) = y \in Ker\, h$. Therefore Ker h $\ll_e$ M which is a contradiction. Thus our assumption is false and hence Ker f $\ll_e$ N; that is N is an essentially small quasi-Dedekind R-module.

The following example shows the direct sum of essentially small quasi-Dedekind modules is not necessarily essentially small quasi-Dedekind module.

Example 2:

It is known that Z and Z_2 are essentially small quasi-Dedekind as Z-modules. But $Z \oplus Z_2$ is not an essentially small quasi-Dedekind Z-module.

Let M and N be R-modules. We say that M is an essentially small quasi-Dedekind (K-nonsingular) relative to N if, for all $f \in Hom(M,N)$, $f \neq 0$, implies Kerf $\ll_e$ M.

An *R*-module M is called small uniform, if $M \neq 0$ and every nonzero submodule of *M* is essentially small in *M.*

An R-module M is called semisimple if every submodule of M is direct summand of M [**1**, p.189].

Remarks and Examples 3

1) Let M be an R-module. Then M is an essentially small quasi-Dedekind if and only if M is an essentially small quasi-Dedekind relative to M.
2) Let M be an essentially small quasi-Dedekind R-module. Then M is an essentially small quasi-Dedekind relative to N, for all $N \leq M$.

Proof:

Let $N \leq M$. If N = M, then M is an essentially small quasi-Dedekind relative to N. If N $\lneqq$ M, assume that $f \in Hom(M,N)$, $f \neq 0$. Hence $iof \in End_R(M)$, $iof \neq 0$, where i is the inclusion mapping. Since M is an essentially small quasi-Dedekind R-module, then Ker(iof) $\ll_e$ M. But Kerf = Ker(iof), thus Kerf $\ll_e$ M and so M is an essentially small quasi-Dedekind relative to N.

3) Every small uniform R-module M is not an essentially small quasi-Dedekind relative to N, where N is any R-module.
4) Any semisimple R-module M is an essentially small quasi-Dedekind relative to N, where N is any R-module.
5) Z12 is not essentially small quasi-Dedekind relative to Z6, since there exists $f: Z_{12} \longrightarrow Z_6$ defined by $f(\bar{x}) = 3\bar{x}$ for all $\bar{x} \in Z_{12}$, hence $Kerf = (\bar{2}) \ll_e Z_{12}$.

Theorem 4:

Let $(M_i)_{i \in \Lambda}$ be a family of modules. Then $M = \oplus_{i \in \Lambda} M_i$ is essentially small quasi-Dedekind if and only if M_i is an essentially small quasi-Dedekind relative to M_j, for all $i, j \in \Lambda$.

Proof:

We shall give the details of proof of this theorem for $i \in \Lambda = \{1,2\}$, and the proof for any Λ is similarly.

$\Rightarrow$) Since $M = M_1 \oplus M_2$ is an essentially small quasi-Dedekind R-module, then by Prop1, M_1 and M_2 are essentially small quasi-Dedekind R-modules. So M_1 is an essentially small quasi-Dedekind relative to M_1 and M_2 is an essentially small quasi-Dedekind relative to M_2. Now, to prove that M_1 is an essentially small quasi-Dedekind relative to M_2. Let $f: M_1 \longrightarrow M_2$, f $\neq$ 0. Consider the following: $M \xrightarrow{\rho} M_1 \xrightarrow{f} M_2 \xrightarrow{i} M$, where ρ is the natural projection, and i is the inclusion mapping. Then $h = iof o\rho \in End_R(M)$ and $h \neq 0$, thus Kerh $\ll_e$ M, but $Kerf \subseteq Kerh$ which implies Kerf $\ll_e$ M. Now we have to prove that Kerf $\ll_e$ M_1. Suppose that Kerf $\ll_e$ M_1, then $Kerf \oplus M_2 \ll_e M_1 \oplus M_2 = M$, but we can show that $Kerh = Kerf \oplus M_2$ as follows: Let $x \in Kerf$, $y \in M_2$, h(x + y) = iofoρ(x + y) = iof (x) = f(x) = 0, thus $Kerf \oplus M_2 \subseteq Kerh$, and let $x + y \in Kerh \subseteq M_1 \oplus M_2$, so x ∈ M_1, y ∈ M_2, since h(x + y) = 0 implies (iofo ρ)(x + y) = 0, so iof (x) = 0 then f(x) = 0; that is $x \in Kerf$, thus $Kerh \subseteq Kerf \oplus M_2$. Hence $Kerh = Kerf \oplus M_2 \ll_e M_1 \oplus M_2 = M$, which is a contradiction. Therefore Kerf $\ll_e$ M_1 and hence M_1 is an essentially small quasi-Dedekind relative to M_2.

Similarly, M_2 is an essentially small quasi-Dedekind relative to M_1.

$\Leftarrow$) Let $\psi : M \longrightarrow M$ such that $Ker\psi \ll_e$ M, so $Ker\psi \cap M_1 \ll_e M_1$. Let $\psi|_{M_1} : M_1 \longrightarrow M$ such that $\psi|_{M_1}(x) = \psi(x+0)$, for all $x \in M_1$, then $Ker(\psi|_{M_1}) = Ker\psi \cap M_1$, to see this:

Let $x \in Ker(\psi|_{M_1})$ implies:

$$0 = \psi|_{M_1}(x) = \psi(x+0) = \psi(x)$$

It follows that $x \in Ker\psi \cap M_1$. Also, let $x \in Ker\psi \cap M_1$, so $x \in M_1$ and $0 = \psi(x) = \psi(x+0) = \psi|_{M_1}(x)$, so $x \in Ker(\psi|_{M_1})$. Consider the following: $M_1 \xrightarrow{\psi|_{M_1}} M \xrightarrow{\rho_1} M_1$ and $M_1 \xrightarrow{\psi|_{M_1}} M \xrightarrow{\rho_2} M_2$, where ρ_1, ρ_2 are the natural projections. We claim that $Ker(\rho_1 o\psi|_{M_1}) \cap Ker(\rho_2 o\psi|_{M_1}) \supseteq Ker\psi|_{M_1}$.

To prove our assertion: Let $x \in Ker(\psi|_{M_1})$ then $\psi|_{M_1}(x) = 0$, hence:

$$\rho_1 o\psi|_{M_1}(x) = \rho_1(\psi|_{M_1}(x)) = \rho_1(0) = 0$$

$$\rho_2 o\psi|_{M_1}(x) = \rho_2(\psi|_{M_1}(x)) = \rho_2(0) = 0$$

Thus $x \in Ker(\rho_1 o\psi|_{M_1}) \cap Ker(\rho_2 o\psi|_{M_1})$; that is:

$$Ker(\rho_1 o\psi|_{M_1}) \cap Ker(\rho_2 o\psi|_{M_1}) \supseteq Ker\psi|_{M_1}.$$

But $Ker(\psi|_{M_1}) = Ker\psi \cap M_1 \ll_e$ M1, so $Ker(\rho_1 o\psi|_{M_1}) \cap Ker(\rho_2 o\psi|_{M_1}) \ll_e$ M_1 and hence $Ker(\rho_1 o\psi|_{M_1}) \ll_e$ M_1 and $Ker(\rho_2 o\psi|_{M_1}) \ll_e$ M_1. But M_1 is an essentially small quasi-Dedekind relative to M_1 and M_1 is an essentially small quasi-Dedekind relative to M_2, by hypothesis. So that $\rho_1 o\psi|_{M_1} = 0$,

$$\rho_2 o\psi|_{M_1} = 0 \qquad \ldots(1)$$

by a similar procedure, we obtain:

$$\rho_1 o\psi|_{M_2} = 0,\ \rho_2 o\psi|_{M_2} = 0 \qquad \ldots(2)$$

Then by (1) and (2) we conclude $\psi = 0$.

Proposition 5:

Let M be an essentially small quasi-Dedekind (K-nonsingular) module, and let $N \le M$. If N $\ll_e$ $N_i \le^{\oplus} M$, for i = 1,2, then $N_1 = N_2$.

Proof:

Consider the endomorphism $(I-\rho_1)\rho_2$, ρ_i is the natural projections of M onto N_i, i = 1, 2; that is $\rho_1 : M \longrightarrow N_1$, $\rho_2 : M \longrightarrow N_2$. Since $N \subseteq N_1$ and $N \subseteq N_2$, so $\rho_1(n) = n$, $\rho_2(n) = n$ for all $n \in N$. Hence for each $n \in N$

$$\begin{aligned}([I-\rho_1]\rho_2)(n) &= I - \rho_1(\rho_2(n))\\ &= (I-\rho_1)(n)\\ &= I(n) - \rho_1(n) = 0\end{aligned}$$

so:

$$N \subseteq Ker([I-\rho_1]\rho_2) \qquad \ldots(1)$$

Since $N_2 \le^{\oplus} M$, so there exists $K_2 \le M$ such that $N_2 \oplus K_2 = M$, and since for each $k \in K_2$, $([I-\rho_1]\rho_2)(k) = (I-\rho_1)(\rho_2(k)) = (I-\rho_1)(0) = 0$ implies

$$K_2 \subseteq Ker([I-\rho_1]\rho_2) \qquad \ldots(2)$$

Now, from (1) and (2) then $N \oplus K_2 \subseteq Ker([I-\rho_1]\rho_2)$, but N $\ll_e$ N_2, K_2 $\ll_e$ K_2, so $N \oplus K_2 \ll_e N_2 \oplus K_2 = M$. Hence $Ker([I-\rho_1]\rho_2) \ll_e$ M, so $(I-\rho_1)\rho_2 = 0$ (since M is an essentially small quasi-Dedekind module). It follows that $\rho_2 = \rho_1 o \rho_2$. Now, we can prove that $N_2 \subseteq N_1$. Let $x \in N_2$, then $\rho_2(x) = x$. Hence $\rho_1(\rho_2(x)) = \rho_1(x)$, then $\rho_1(x) = \rho_2(x) = x$. Hence $x \in N_1$, thus $N_2 \subseteq N_1$.

Similarly by taking $(I-\rho_2)\rho_1$ and showing it is zero, then we obtain $N_1 \subseteq N_2$. Thus $N_1 = N_2$.

References

[1] F. Kasch, **Modules and rings**, Academic Press., London, 1982.