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Weighted Exponential Distribution: Approximate Bayes Estimations with Fuzzy Data

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Abstract

This paper aims to focus on Bayes estimations for the shape and scale parameters along with the reliability function of weighted exponential distribution with fuzzy data. Bayes estimations have been obtained using symmetric and asymmetric loss functions. Lindley's approximation method has been used for the integrals that cannot be solved in closed form. At the end, some comparison for Bayes estimations are studied through a Monte Carlo simulation study. [DOI: 10.22401/ANJS.00.1.24]

Keywords: Weighted Exponential Distribution; Bayes Estimation; Lindley's Approximation; Squared Error Loss Function; Linex Loss Function; Fuzzy Data.

1. Introduction

Based on the idea of Azzalini [3] who introduced the shape parameter for several symmetric distributions, Gupta and Kundu in (2009) [4] presented a shape parameter to an exponential distribution which belongs to asymmetric distributions and thus got a new class of distribution named weighted exponential distribution that provides, in several cases, better fit than Weibull, gamma or generalized exponential distributions.

The probability density function (pdf) of weighted exponential (from now, we will symbolize it by WE) distribution is given by [4]:

$$f_{WE}(x;\alpha,\lambda) = \frac{\alpha+1}{\alpha}\,\lambda\, e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right);\ \alpha,\lambda>0 \quad \text{...........(1)}$$

for $x>0$ and zero otherwise, where α is the shape parameter and λ is the scale parameter.

The corresponding cumulative distribution function of WE distribution is given by [2]:

$$F_{WE}(x;\alpha,\lambda) = 1-\frac{1}{\alpha}\,e^{-\lambda x}\left(\alpha+1-e^{-\alpha\lambda x}\right) \quad \text{............(2)}$$

The reliability function of WE distribution at time (t), can be obtained by:

$$R_{WE}(t;\alpha,\lambda) = 1-F(t;\alpha,\lambda) = \frac{1}{\alpha}\,e^{-\lambda t}\left(\alpha+1-e^{-\alpha\lambda t}\right);\ t>0 \quad \text{...................................(3)}$$

The hazard function at time (t), is given by [7]:

$$h_{WE}(t;\alpha,\lambda) = \frac{f_{WE}(t;\alpha,\lambda)}{R_{WE}(t;\alpha,\lambda)} = \frac{(\alpha+1)\,\lambda\,(1-e^{-\alpha\lambda t})}{\alpha+1-e^{-\alpha\lambda t}};\ t>0 \quad \text{...........................(4)}$$

Recently, the WE distribution has received many attentions in the statistical literature. Most of this literature focused on estimation the unknown parameters according to the non-Bayes methods, for example, method of maximum likelihood, method of moments, or according to the method that need to preliminary information for estimating after relying on prior distribution (i.e., Bayes estimation) based on real (precise) and censored data. However, in real situations all observations or data are not precise numbers but more or less non-precise, also called fuzzy. So, this paper deals with the fuzzy data in order to estimate the unknown parameters and reliability function of WE distribution according to Bayes estimation method.

2. Bayes Estimations with Fuzzy Data

In this section we considered Bayes estimation theory to estimate the shape and scale parameters along with the reliability function of WE distribution when the available data are shown in the form of fuzzy data.

Definition [9]:

A fuzzy set $\tilde{A}$ in χ is a set of ordered pairs: $\tilde{A}=\{(x,\mathcal{M}_{\tilde{A}}(x))|x\in\chi\}$, where χ is a collection of objects with universal element x, A is a subset of a set χ and $\mathcal{M}_{\tilde{A}}(x)$ is the membership function of x in $\widetilde{A}$ which maps χ to the membership space M, $\mathcal{M}_{\widetilde{A}}(x) : \chi \to [0,1]$. (When M contains only two points 0 and 1, $\widetilde{A}$ is non-fuzzy (crisp) and $\mathcal{M}_{\widetilde{A}}(x)$ is identical to the characteristic function of a non-fuzzy set).

In general, we can define fuzzy set as a set included elements have a membership varying degrees in the set. Classical (crisp) sets allow only full membership or no membership at all, while fuzzy sets allow partial membership.

Now, assume that $\underline{x} = (x_1, x_2, x_3, \dots, x_n)$ be a random sample of size (n) drawn from a certain population and it has pdf given by equation (1) and assume that $\underline{x}$ is not observed exactly (precisely) but only partial information is available in the form of a fuzzy subset $\underline{\tilde{x}}$ with the membership function $\mathcal{M}_{\tilde{x}_i}(x)$. The observed-data likelihood function, $L(\alpha, \lambda|\underline{\tilde{x}})$, can be obtained according to Zadeh's definition of the probability of a fuzzy event through using the expression, $\mathrm{P}(\widetilde{\mathrm{A}}) = \int_{\mathbb{R}^n} \mathcal{M}_{\widetilde{\mathrm{A}}}(\mathrm{x})\, \mathrm{dP}\ ; x \in \mathbb{R}^n$, as,

$$L(\alpha, \lambda|\underline{\tilde{x}}) = \left(\frac{\alpha+1}{\alpha}\right)^n \lambda^n \prod_{i=1}^{n} \int e^{-\lambda x} \left(1 - e^{-\alpha\lambda x}\right) \mathcal{M}_{\tilde{x}_i}(x)\, dx \quad \dots\dots (5)$$

and then natural log of equation (5) can be obtained, as,

$$\tilde{\ell}_{WE} = \ln L(\alpha, \lambda|\underline{\tilde{x}}) = n \ln(\alpha + 1) - n \ln \alpha + n \ln \lambda + \sum_{i=1}^{n} \ln \int e^{-\lambda x} \left(1 - e^{-\alpha\lambda x}\right) \mathcal{M}_{\tilde{x}_i}(x)\, dx \quad \dots\dots (6)$$

In order to obtain Bayes estimation, relative to squared error and linear-exponential (Linex) as symmetric and asymmetric loss functions respectively, assume that the prior distributions of α and λ are taken to be independent Gamma (a, b) and Gamma (c, d) respectively with pdfs,

$$P(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha}; \alpha > 0, a, b > 0 \quad \dots\dots (7)$$

$$P(\lambda) = \frac{d^c}{\Gamma(c)} \lambda^{c-1} e^{-d\lambda}; \lambda > 0, c, d > 0 \quad \dots\dots (8)$$

The joint prior distribution, say $P(\alpha, \lambda)$, of unknown parameters can be written as:

$$P(\alpha, \lambda) = P(\alpha)\, P(\lambda) = \frac{b^a d^c}{\Gamma(a)\Gamma(c)} \alpha^{a-1} \lambda^{c-1} e^{-(b\alpha + d\lambda)} \quad \dots\dots (9)$$

The joint posterior density function of α and λ given fuzzy data, say $\Pi(\alpha, \lambda|\underline{\tilde{x}})$, can be obtained by:

$$\Pi(\alpha, \lambda|\underline{\tilde{x}}) = \frac{L(\alpha,\lambda|\underline{\tilde{x}})\, P(\alpha,\lambda)}{\int_\lambda \int_\alpha L(\alpha,\lambda|\underline{\tilde{x}})\, P(\alpha,\lambda)\, d\alpha\, d\lambda} \quad \dots\dots (10)$$

The formula of squared error loss function for θ is [1]:

$$L(\hat{\theta}_S, \theta) = (\hat{\theta}_S - \theta)^2$$

where $\hat{\theta}_S$ is an estimation of θ based on squared error loss function. Bayes estimator of θ based on this loss function is obtained as:

$$\hat{\theta}_S = E_\Pi(\theta|\underline{\tilde{x}}) \quad \dots\dots (11)$$

where the expectation is taken with respect to the posterior distribution of θ.

The formula of Linex loss function for θ is, [5]

$$L(\hat{\theta}_L, \theta) = d\left[e^{z(\hat{\theta}_L - \theta)} - z(\hat{\theta}_L - \theta) - 1\right];\ z \neq 0\,, d > 0$$

where $\hat{\theta}_L$ is an estimation of θ based on Linex loss function and Bayes estimator of θ based on this loss function is obtained as:

$$\hat{\theta}_L = -\frac{1}{z} \ln\left[E_\Pi\left(e^{-z\theta} \mid \underline{\tilde{x}}\right)\right] \quad \dots\dots (12)$$

where the expectation is taken with respect to the posterior distribution of $e^{-z\theta}$.

Now, according to equations (11) and (12), Bayes estimation of any function of the parameters, say $u(\alpha, \lambda)$, based on squared error and Linex loss functions can be written respectively as:

$$\hat{u}_S(\alpha, \lambda) = E\left[u(\alpha, \lambda) \mid \underline{\tilde{x}}\right] = \frac{\int_0^\infty \int_0^\infty u(\alpha,\lambda)\, L(\alpha,\lambda|\underline{\tilde{x}})\, P(\alpha,\lambda) d\alpha\, d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\underline{\tilde{x}})\, P(\alpha,\lambda) d\alpha\, d\lambda} \quad \dots (13)$$

$$\hat{u}_L(\alpha, \lambda) = -\frac{1}{z} \ln\left[E\left(u(\alpha, \lambda)|\underline{\tilde{x}}\right)\right] = -\frac{1}{z} \ln\left[\frac{\int_0^\infty \int_0^\infty u(\alpha,\lambda)\, L(\alpha,\lambda|\underline{\tilde{x}})\, P(\alpha,\lambda) d\alpha\, d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\underline{\tilde{x}})\, P(\alpha,\lambda) d\alpha\, d\lambda}\right] \quad \dots\dots (14)$$

Note that, the above ratio of two integrals cannot be simplified into a closed form. Therefore, we consider Lindley's approximation. Lindley (1980) [6] developed an approximate procedure for assessment the ratio of two integrals. Consider $I(\underline{\tilde{x}})$ defined as:

$$I(\underline{\tilde{x}}) = E\left[u(\alpha, \lambda) \mid \underline{\tilde{x}}\right] = \frac{\int_0^\infty \int_0^\infty u(\alpha,\lambda) e^{\tilde{\ell}_{WE} + \rho(\alpha,\lambda)} d\alpha\, d\lambda}{\int_0^\infty \int_0^\infty e^{\tilde{\ell}_{WE} + \rho(\alpha,\lambda)} d\alpha\, d\lambda} \quad \dots\dots (15)$$

where, $u(\alpha, \lambda)$ is a function of α and λ only, $\tilde{\ell}_{WE}$ is the natural log-likelihood function, given by equation (6), $\rho(\alpha, \lambda)$ is the natural log-joint prior density function.

Then, for sufficiently large sample size, the ratio of two integrals $I(\underline{\tilde{x}})$ can be approximated as, (see [10]).

$$I(\underline{\tilde{x}}) = u(\hat{\alpha},\hat{\lambda}) + \frac{1}{2}\left[(\hat{u}_{\alpha\alpha} + 2\hat{u}_{\alpha}\hat{\rho}_{\alpha})\hat{\sigma}_{\alpha\alpha} + (\hat{u}_{\alpha\lambda} + 2\hat{u}_{\alpha}\hat{\rho}_{\lambda})\hat{\sigma}_{\alpha\lambda} + (\hat{u}_{\lambda\alpha} + 2\hat{u}_{\lambda}\hat{\rho}_{\alpha})\hat{\sigma}_{\lambda\alpha} + (\hat{u}_{\lambda\lambda} + 2\hat{u}_{\lambda}\hat{\rho}_{\lambda})\hat{\sigma}_{\lambda\lambda}\right] + \frac{1}{2}\left[(\hat{u}_{\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha})\left(\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{\ell}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha} + \hat{\ell}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda} + \hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right) + (\hat{u}_{\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{u}_{\alpha}\hat{\sigma}_{\lambda\alpha})\left(\hat{\ell}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{\ell}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha} + \hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{\ell}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\right)\right] \quad \ldots\ldots\ldots\ldots (16)$$

where, $\hat{\alpha}$ and $\hat{\lambda}$ are the MLE's of α and λ respectively. The MLE's of α and λ can be obtained as solutions of the first partial derivative for equation (6) with respect to that parameters, equations (17) and (18). It is clearly there is no closed-form solution to that equations, therefore, Newton–Raphson iterative techniques can be used to obtain the solution.

σ_{ij} is the $(i,j)^{th}$ elements of matrix $\left[\frac{-\partial^2\tilde{\ell}_{WE}}{\partial\alpha\partial\lambda}\right]^{-1}$ where sub-scripts (i,j) refers to α,λ , respectively.

$\hat{u}_{\alpha}$ and $\hat{u}_{\lambda}$ are the first derivative of the function $u(\alpha,\lambda)$ with respect to α and λ respectively evaluated at $\hat{\alpha}$ and $\hat{\lambda}$.

$\hat{u}_{\alpha\alpha}$ is the second derivative of the function $u(\alpha,\lambda)$ with respect to α evaluated at $\hat{\alpha}$ and $\hat{\lambda}$. Other expressions can be inferred exactly in similar style.

$$\hat{\rho}_{\alpha} = \frac{\partial \ln P(\alpha,\lambda)}{\partial\alpha}\bigg|_{\substack{\alpha=\hat{\alpha}\\ \lambda=\hat{\lambda}}} = \frac{a-1}{\hat{\alpha}} - b$$

$$\hat{\rho}_{\lambda} = \frac{\partial \ln P(\alpha,\lambda)}{\partial\lambda}\bigg|_{\substack{\alpha=\hat{\alpha}\\ \lambda=\hat{\lambda}}} = \frac{c-1}{\hat{\lambda}} - d$$

$$\frac{\partial\tilde{\ell}_{WE}}{\partial\alpha} = \frac{-n}{\alpha(\alpha+1)} + \sum_{i=1}^{n}\frac{\int \lambda\, x\, e^{-\lambda x(\alpha+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx} = 0 \quad \ldots (17)$$

$$\frac{\partial\tilde{\ell}_{WE}}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\frac{\int x e^{-\lambda x}\left((\alpha+1)e^{-\alpha\lambda x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx} = 0 \quad \ldots\ldots\ldots (18)$$

$$\frac{\partial^2\tilde{\ell}_{WE}}{\partial\alpha^2} = \frac{n(2\alpha+1)}{\alpha^2(\alpha+1)^2} - \sum_{i=1}^{n}\left[\frac{\int \lambda^2\, x^2\, e^{-\lambda x(\alpha+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx} + \left(\frac{\int \lambda\, x\, e^{-\lambda x(\alpha+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}\right)^2\right] \quad \ldots\ldots (19)$$

$$\frac{\partial^2\tilde{\ell}_{WE}}{\partial\lambda^2} = \frac{-n}{\lambda^2} + \sum_{i=1}^{n}\left[\frac{\int x^2\, e^{-\lambda x}\left(1-(\alpha+1)^2\, e^{-\alpha\lambda x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx} - \left(\frac{\int x\, e^{-\lambda x}\left((\alpha+1)e^{-\alpha\lambda x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}\right)^2\right] \quad (20)$$

$$\frac{\partial^2\tilde{\ell}_{WE}}{\partial\alpha\partial\lambda} = \frac{\partial^2\tilde{\ell}_{WE}}{\partial\lambda\partial\alpha} = \sum_{i=1}^{n}\left[\frac{\int x e^{-\lambda x(\alpha+1)}\left(1-\lambda x(\alpha+1)\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx} - \frac{\int \lambda x\, e^{-\lambda x(\alpha+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx \int x\, e^{-\lambda x}\left((\alpha+1)e^{-\alpha\lambda x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\left(\int e^{-\lambda x}\left(1-e^{-\alpha\lambda x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx\right)^2}\right] \quad \ldots\ldots\ldots (21)$$

$$\hat{\ell}_{\alpha\alpha\lambda} = \frac{\partial^3\tilde{\ell}_{WE}}{\partial\alpha\partial\alpha\partial\lambda}\bigg|_{\substack{\alpha=\hat{\alpha}\\ \lambda=\hat{\lambda}}} = \hat{\ell}_{\alpha\lambda\alpha} = \frac{\partial^3\tilde{\ell}_{WE}}{\partial\alpha\partial\lambda\partial\alpha}\bigg|_{\substack{\alpha=\hat{\alpha}\\ \lambda=\hat{\lambda}}} = \hat{\ell}_{\lambda\alpha\alpha} = \frac{\partial^3\tilde{\ell}_{WE}}{\partial\lambda\partial\alpha\partial\alpha}\bigg|_{\substack{\alpha=\hat{\alpha}\\ \lambda=\hat{\lambda}}}$$

$$= \sum_{i=1}^{n}\frac{\int \hat{\lambda}\, x^2\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\left(\hat{\lambda}(\hat{\alpha}+1)x-2\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx} + \sum_{i=1}^{n}\frac{\int \hat{\lambda}^2 x^2\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx \int x\, e^{-\hat{\lambda}x}\left((\hat{\alpha}+1)e^{-\hat{\alpha}\hat{\lambda}x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\left(\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx\right)^2} + 2\sum_{i=1}^{n}\frac{\int \hat{\lambda}x\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx \int x\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\left(\hat{\lambda}(\hat{\alpha}+1)x-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\left(\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx\right)^2} + 2\sum_{i=1}^{n}\frac{\left(\int \hat{\lambda}\, x\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx\right)^2 \int x\, e^{-\hat{\lambda}x}\left((\hat{\alpha}+1)e^{-\hat{\alpha}\hat{\lambda}x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\left(\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx\right)^3} \quad \ldots\ldots\ldots (22)$$

$$\hat{\ell}_{\lambda\alpha\lambda} = \frac{\partial^3\tilde{\ell}_{WE}}{\partial\lambda\partial\alpha\partial\lambda}\bigg|_{\substack{\alpha=\hat{\alpha}\\ \lambda=\hat{\lambda}}} = \hat{\ell}_{\alpha\lambda\lambda} = \frac{\partial^3\tilde{\ell}_{WE}}{\partial\alpha\partial\lambda\partial\lambda}\bigg|_{\substack{\alpha=\hat{\alpha}\\ \lambda=\hat{\lambda}}}$$

$$= \sum_{i=1}^{n}\frac{\int(\hat{\alpha}+1)\, x^2\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\left(\hat{\lambda}x(\hat{\alpha}+1)-2\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx} - \sum_{i=1}^{n}\frac{\int x^2\, e^{-\hat{\lambda}x}\left(1-(\hat{\alpha}+1)^2 e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx \int \hat{\lambda}x\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\left(\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx\right)^2} - 2\sum_{i=1}^{n}\frac{\int x\, e^{-\hat{\lambda}x}\left((\hat{\alpha}+1)e^{-\hat{\alpha}\hat{\lambda}x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}\left[\frac{\int x\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\left(1-\hat{\lambda}(\hat{\alpha}+1)x\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx} - \frac{\int x\, e^{-\hat{\lambda}x}\left((\hat{\alpha}+1)e^{-\hat{\alpha}\hat{\lambda}x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx \int \hat{\lambda}x\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\left(\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx\right)^2}\right] \quad \ldots\ldots\ldots (23)$$

$$= \frac{\partial^3\tilde{\ell}_{WE}}{\partial\alpha^3}\bigg|_{\substack{\alpha=\hat{\alpha}\\ \lambda=\hat{\lambda}}} = \frac{2n\left[\hat{\alpha}(\hat{\alpha}+1)-(2\hat{\alpha}+1)^2\right]}{\hat{\alpha}^3(\hat{\alpha}+1)^3} + \sum_{i=1}^{n}\frac{\int \hat{\lambda}^3\, x^3\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx} + 3\sum_{i=1}^{n}\frac{\int \hat{\lambda}^2 x^2\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx \int \hat{\lambda}\, x\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\,\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\left(\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx\right)^2} +$$

$$2\sum_{i=1}^{n}\left(\frac{\int \hat{\lambda}\, x\, e^{-\hat{\lambda}x(\hat{\alpha}+1)}\ \mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}\right)^3 \quad \text{.........} (24)$$

$$\hat{\ell}_{\lambda\lambda\lambda}=\frac{\partial^3 \tilde{\ell}_{WE}}{\partial\lambda^3}\Big|_{\substack{\alpha=\hat{\alpha}\\ \lambda=\hat{\lambda}}}$$

$$=\frac{2n}{\hat{\lambda}^3}+\sum_{i=1}^{n}\frac{\int x^3 e^{-\hat{\lambda}x}\left(\hat{\alpha}(\hat{\alpha}+1)^2 e^{-\hat{\alpha}\hat{\lambda}x}+(\hat{\alpha}+1)^2 e^{-\hat{\alpha}\hat{\lambda}x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}$$

$$-3\sum_{i=1}^{n}\frac{\int x^2 e^{-\hat{\lambda}x}\left(1-(\hat{\alpha}+1)^2 e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx \int x\, e^{-\hat{\lambda}x}\left((\hat{\alpha}+1)\, e^{-\hat{\alpha}\hat{\lambda}x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\left(\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx\right)^2}$$

$$+2\sum_{i=1}^{n}\left(\frac{\int x\, e^{-\hat{\lambda}x}\left((\hat{\alpha}+1)e^{-\hat{\alpha}\hat{\lambda}x}-1\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}{\int e^{-\hat{\lambda}x}\left(1-e^{-\hat{\alpha}\hat{\lambda}x}\right)\mathcal{M}_{\tilde{x}_i}(x)\,dx}\right)^3 \quad \text{...........} (25)$$

Now, relative to squared error loss function, the approximate Bayes estimations would be as follows:

For the parameter α: Assume that $u(\alpha,\lambda)=\alpha$ and then,

$u_\alpha=1,\ u_{\alpha\alpha}=u_\lambda=u_{\lambda\lambda}=u_{\alpha\lambda}=u_{\lambda\alpha}=0.$

$$\hat{\alpha}_S=E(\alpha|\underline{\tilde{x}})=\hat{\alpha}+\hat{\rho}_\alpha\hat{\sigma}_{\alpha\alpha}+\hat{\rho}_\lambda\hat{\sigma}_{\alpha\lambda}+\frac{1}{2}\left[\hat{\sigma}_{\alpha\alpha}\left(\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)+\hat{\sigma}_{\lambda\alpha}\left(\hat{\ell}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\right)\right] \quad \text{..............................} (26)$$

- For the parameter λ :Assume that $u(\alpha,\lambda)=\lambda$ and then,

$u_\lambda=1, u_{\lambda\lambda}=u_\alpha=u_{\alpha\alpha}=u_{\alpha\lambda}=u_{\lambda\alpha}=0.$

$$\hat{\lambda}_S=E(\lambda|\underline{\tilde{x}})=\hat{\lambda}+\hat{\rho}_\lambda\hat{\sigma}_{\lambda\lambda}+\hat{\rho}_\alpha\hat{\sigma}_{\lambda\alpha}+\frac{1}{2}\left[\hat{\sigma}_{\lambda\lambda}\left(\hat{\ell}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\right)+\hat{\sigma}_{\alpha\lambda}\left(\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)\right] \quad \text{..............................} (27)$$

- For the reliability function: Assume that $u(\alpha,\lambda)=R(t)=\frac{1}{\alpha}\ e^{-\lambda t}\left(\alpha+1-\ e^{-\alpha\lambda t}\right)$ and then:

$u_\alpha=\frac{1}{\alpha}e^{-\lambda t}\left(\lambda t e^{-\alpha\lambda t}-\frac{1}{\alpha}+\frac{1}{\alpha}e^{-\alpha\lambda t}\right)$; $u_\lambda=te^{-\lambda t}\left(e^{-\alpha\lambda t}-\frac{1}{\alpha}+\frac{1}{\alpha}e^{-\alpha\lambda t}-1\right)$

$u_{\alpha\alpha}=\frac{2}{\alpha^3}e^{-\lambda t}\left(1-e^{-\alpha\lambda t}\right)-\frac{\lambda t}{\alpha}e^{-\lambda t(\alpha+1)}\left(\lambda t+\frac{2}{\alpha}\right)$;

$u_{\lambda\lambda}=t^2e^{-\lambda t}\left(1+\frac{1}{\alpha}\right)-t^2e^{-\lambda t(\alpha+1)}\left(\alpha+\frac{1}{\alpha}+2\right)$;

$u_{\alpha\lambda}=u_{\lambda\alpha}=\frac{t}{\alpha^2}e^{-\lambda t}\left(1-e^{-\alpha\lambda t}\right)-\lambda t^2e^{-\lambda t(\alpha+1)}\left(1+\frac{1}{\alpha}\right).$

$$\hat{R}_S(t)=E\left(R(t)|\underline{\tilde{x}}\right)=\frac{1}{\hat{\alpha}}\ e^{-\hat{\lambda}t}\left(\hat{\alpha}+1-\ e^{-\hat{\alpha}\hat{\lambda}t}\right)+\frac{1}{2}\left[(\hat{u}_{\lambda\lambda}+2\hat{u}_\lambda\hat{\rho}_\lambda)\hat{\sigma}_{\lambda\lambda}+(\hat{u}_{\alpha\lambda}+2\hat{u}_\alpha\hat{\rho}_\lambda)\hat{\sigma}_{\alpha\lambda}+(\hat{u}_{\lambda\alpha}+2\hat{u}_\lambda\hat{\rho}_\alpha)\hat{\sigma}_{\lambda\alpha}+(\hat{u}_{\alpha\alpha}+2\hat{u}_\alpha\hat{\rho}_\alpha)\hat{\sigma}_{\alpha\alpha}+(\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda}+\hat{u}_\alpha\hat{\sigma}_{\lambda\alpha})\left(\hat{\ell}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\right)+(\hat{u}_\lambda\hat{\sigma}_{\alpha\lambda}+\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha})\left(\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)\right] \quad \text{..............................} (28)$$

Relative to Linex loss function, approximate Bayes estimations would be as follows:

- For the parameter α: Assume that $u(\alpha,\lambda)=e^{-z\alpha}$ and then,

$u_\alpha=-z\,e^{-z\alpha},\ u_{\alpha\alpha}=z^2\,e^{-z\alpha},\ \ u_\lambda=u_{\lambda\lambda}=u_{\alpha\lambda}=u_{\lambda\alpha}=0.$

$$\hat{\alpha}_L=-\frac{1}{z}\ln\left[E\left(e^{-z\alpha}|\underline{\tilde{x}}\right)\right]$$

where,

$$E\left(e^{-z\alpha}|\underline{\tilde{x}}\right)=e^{-z\hat{\alpha}}+\hat{\rho}_\alpha\hat{\sigma}_{\alpha\alpha}+\hat{\rho}_\lambda\hat{\sigma}_{\alpha\lambda}+\frac{1}{2}\left[\hat{\sigma}_{\alpha\alpha}\left(\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)+\hat{\sigma}_{\lambda\alpha}\left(\hat{\ell}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\right)\right] \quad \text{..............................} (29)$$

- For the parameter λ: Assume that $u(\alpha,\lambda)=e^{-z\lambda}$ and then,

$u_\lambda=-z\,e^{-z\lambda},\ u_{\lambda\lambda}=z^2\,e^{-z\lambda}$, $u_\alpha=u_{\alpha\alpha}=u_{\alpha\lambda}=u_{\lambda\alpha}=0.$

$$\hat{\lambda}_L=-\frac{1}{z}\ln\left[E\left(e^{-z\lambda}|\underline{\tilde{x}}\right)\right]$$

where:

$$E\left(e^{-z\lambda}|\underline{\tilde{x}}\right)=e^{-z\hat{\lambda}}+\hat{\rho}_\lambda\hat{\sigma}_{\lambda\lambda}+\hat{\rho}_\alpha\hat{\sigma}_{\lambda\alpha}+\frac{1}{2}\left[\hat{\sigma}_{\lambda\lambda}\left(\hat{\ell}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\right)+\hat{\sigma}_{\alpha\lambda}\left(\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\ell}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{\ell}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)\right] \quad \text{..............................} (30)$$

- For the reliability function: Assume that $u(\alpha,\lambda)=e^{-zR(t)}=e^{-\frac{z}{\alpha}e^{-\lambda t}\left(\alpha+1-\ e^{-\alpha\lambda t}\right)}$ and then,

$u_\alpha=\frac{z}{\alpha}\ e^{-\lambda t-\frac{z}{\alpha}e^{-\lambda t}\left(\alpha+1-e^{-\alpha\lambda t}\right)}\left(\frac{1}{\alpha}-\frac{1}{\alpha}e^{-\alpha\lambda t}-\lambda te^{-\alpha\lambda t}\right)$

$u_\lambda=zte^{-\lambda t-\frac{z}{\alpha}e^{-\lambda t}\left(\alpha+1-e^{-\alpha\lambda t}\right)}\left(1+\frac{1}{\alpha}-\frac{1}{\alpha}e^{-\alpha\lambda t}-e^{-\alpha\lambda t}\right)$

$u_{\alpha\alpha}=e^{-\frac{z}{\alpha}e^{-\lambda t}\left(\alpha+1-e^{-\alpha\lambda t}\right)}\left[\frac{z}{\alpha}e^{-\lambda t}\left(\lambda^2t^2e^{-\alpha\lambda t}-\frac{2}{\alpha^2}+\frac{2\lambda t}{\alpha}e^{-\alpha\lambda t}+\frac{2}{\alpha^2}e^{-\alpha\lambda t}\right)+\left(\frac{z}{\alpha}e^{-\lambda t}\left(\frac{1}{\alpha}-\frac{1}{\alpha}e^{-\alpha\lambda t}-\lambda te^{-\alpha\lambda t}\right)\right)^2\right]$

$$u_{\lambda\lambda} = e^{-\frac{z}{\alpha}e^{-\lambda t}(\alpha+1-e^{-\alpha\lambda t})}\left[zte^{-\lambda t}\left(\alpha te^{-\alpha\lambda t} + 2te^{-\alpha\lambda t} - t - \frac{t}{\alpha} + \frac{t}{\alpha}e^{-\alpha\lambda t}\right) + \left(zte^{-\lambda t}\left(1 + \frac{1}{\alpha} - \frac{1}{\alpha}e^{-\alpha\lambda t} - e^{-\alpha\lambda t}\right)\right)^2\right]$$

$$u_{\alpha\lambda} = u_{\lambda\alpha} = zte^{-\lambda t-\frac{z}{\alpha}e^{-\lambda t}(\alpha+1-e^{-\alpha\lambda t})}\left[\frac{z}{\alpha}\, e^{-\lambda t}\left(\frac{1}{\alpha} - \frac{1}{\alpha}e^{-\alpha\lambda t} - \lambda te^{-\alpha\lambda t}\right)\left(1 + \frac{1}{\alpha} - \frac{1}{\alpha}e^{-\alpha\lambda t} - e^{-\alpha\lambda t}\right) + \lambda te^{-\alpha\lambda t}\left(1 + \frac{1}{\alpha}\right) - \frac{1}{\alpha^2}\left(1 - e^{-\alpha\lambda t}\right)\right]$$

$$\hat{R}_L(t) = -\frac{1}{z}\ln\left[E\left(e^{-zR(t)}|\underline{\tilde{x}}\right)\right]$$

where:

$$E\left(e^{-zR(t)}|\underline{\tilde{x}}\right) = e^{-\frac{z}{\hat{\alpha}}e^{-\hat{\lambda}t}\left(\hat{\alpha}+1-e^{-\hat{\alpha}\hat{\lambda}t}\right)} + \frac{1}{2}\left[(\hat{u}_{\lambda\lambda} + 2\hat{u}_\lambda\hat{\rho}_\lambda)\hat{\sigma}_{\lambda\lambda} + (\hat{u}_{\alpha\lambda} + 2\hat{u}_\alpha\hat{\rho}_\lambda)\hat{\sigma}_{\alpha\lambda} + (\hat{u}_{\lambda\alpha} + 2\hat{u}_\lambda\hat{\rho}_\alpha)\hat{\sigma}_{\lambda\alpha} + (\hat{u}_{\alpha\alpha} + 2\hat{u}_\alpha\hat{\rho}_\alpha)\hat{\sigma}_{\alpha\alpha} + (\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda} + \hat{u}_\alpha\hat{\sigma}_{\lambda\alpha})\left(\hat{\ell}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{\ell}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha} + \hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{\ell}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}\right) + (\hat{u}_\lambda\hat{\sigma}_{\alpha\lambda} + \hat{u}_\alpha\hat{\sigma}_{\alpha\alpha})\left(\hat{\ell}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{\ell}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha} + \hat{\ell}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda} + \hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}\right)\right] \quad \text{.......} (31)$$

3. Simulation Study and Results

The simulation program has been written by using MATLAB (R2010b) computer program. The simulation study consist the following steps:

- Choose the sample size (n): $n = 25, 50, 75$ and 100 . Set the default (true) values for the shape parameter of WE distribution (α): $\alpha = 0.5, 1$ and 1.5 . Further that, without the loss of generality, choose the default value for the scale parameter (λ) as $\lambda = 1$. Choose the values of hyper-parameters associated with gamma prior distributions to be $a = b = c = d = 0.0001$ and $a = b = c = d = 2$ in order to deal with a non-informative and informative priors respectively. Choose four times (t) to assess the estimating reliability function: $t = 1,2,3,4$. Choose the values of Linex loss function constant (z): $z = \mp 0.8$. Choose the number of sample replicated (L): $L = 100$.
- Generate a random sample, say $\underline{x}$, of size n distributed as WE distribution. Since the explicit form of the inverse function of the WE distribution cannot be obtained, the random samples are generated as the summation of two independent random variables distributed as exponential distribution with parameters (λ) and $(\lambda(\alpha + 1))$.

Encode the simulated data according to the following fuzzy information system (FIS) Fig.(1), where each observation in sample will be fuzzy based on an suitable selected membership function.

$\mathcal{M}_{\tilde{x}_1}(x) = \begin{cases} 1 & ; x \le 0.05 , \\ \frac{0.25 - x}{0.2} & ;\ 0.05 \le x \le 0.25 , \\ 0 & ;\ \ otherwise\ , \end{cases}$	$\mathcal{M}_{\tilde{x}_2}(x) = \begin{cases} \frac{x - 0.05}{0.2} & ; 0.05 \le x \le 0.25 , \\ \frac{0.5 - x}{0.25} & ; 0.25 \le x \le 0.5 , \\ 0 & ;\ otherwise\ , \end{cases}$
$\mathcal{M}_{\tilde{x}_3}(x) = \begin{cases} \frac{x - 0.25}{0.25} & ;\ 0.25 \le x \le 0.5 , \\ \frac{0.75 - x}{0.25} & ;\ 0.5 \le x \le 0.75 , \\ 0 & ;\ otherwise\ , \end{cases}$	$\mathcal{M}_{\tilde{x}_4}(x) = \begin{cases} \frac{x - 0.5}{0.25} & ; 0.5 \le x \le 0.75 , \\ \frac{1 - x}{0.25} & ; 0.75 \le x \le 1 , \\ 0 & ;\ otherwise\ , \end{cases}$
$\mathcal{M}_{\tilde{x}_5}(x) = \begin{cases} \frac{x - 0.75}{0.25} & ; 0.75 \le x \le 1 , \\ \frac{1.5 - x}{0.5} & ;\ 1 \le x \le 1.5 , \\ 0 & ;\ otherwise\ , \end{cases}$	$\mathcal{M}_{\tilde{x}_6}(x) = \begin{cases} \frac{x - 1}{0.5} & ;\ 1 \le x \le 1.5 , \\ \frac{2 - x}{0.5} & ;\ 1.5 \le x \le 2 , \\ 0 & ;\ otherwise\ , \end{cases}$
$\mathcal{M}_{\tilde{x}_7}(x) = \begin{cases} \frac{x - 1.5}{0.5} & ; 1.5 \le x \le 2 , \\ 3 - x & ;\ 2 \le x \le 3 , \\ 0 & ;\ otherwise\ , \end{cases}$	$\mathcal{M}_{\tilde{x}_8}(x) = \begin{cases} x - 2 & ; 2 \le x \le 3 , \\ 1 & ;\ x \ge 3 , \\ 0 & ;\ otherwise. \end{cases}$

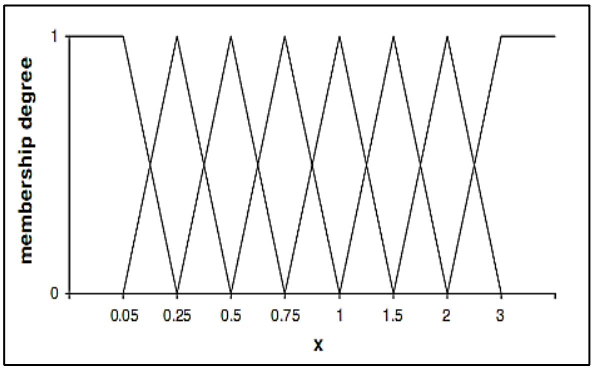

Fig.(1) FIS used to Encode the Simulated Data [8].

› Compute the estimators of the unknown shape and scale parameters along with the reliability function of WE distribution. The initial values required for iterative proceeding algorithms have been chosen to be the moment estimators [2] as:

$\hat{\alpha}_{MO} = \frac{2M-3+\sqrt{2M-3}}{2-M}$ and $\hat{\lambda}_{MO} = \frac{\hat{\alpha}_{MO}+2}{(\hat{\alpha}_{MO}+1)\bar{x}}$

Provided that, $M = \frac{m_2}{m_1^2}$; $\frac{3}{2} < M < 2$; $m_1 = E(X)$ and $m_2 = E(X^2)$.

The iterative process stops when the absolute difference between two successive iterations become less than $\varepsilon = 0.0001$.

› Repeat the above steps 100 times and then compare the different estimators for the shape and scale parameters according to the mean squared error $\text{MSE}(\hat{\alpha})$ and $\text{MSE}(\hat{\lambda})$ and compare the different estimators of reliability function with different times according to the integrated mean squared error $\text{IMSE}(\hat{R}(t))$ as:

$$MSE(\hat{\alpha}) = \frac{\sum_{j=1}^{L}(\hat{\alpha}_j-\alpha)^2}{L} \quad \text{.........................} (32)$$

$$MSE(\hat{\lambda}) = \frac{\sum_{j=1}^{L}(\hat{\lambda}_j-\lambda)^2}{L} \quad \text{.........................} (33)$$

$$IMSE(\hat{R}(t)) = \frac{1}{L}\sum_{j=1}^{L}\left(\frac{1}{n_t}\sum_{i=1}^{n_t}\left(\hat{R}_j(t_i) - R(t_i)\right)^2\right) \quad \text{.....................} (34)$$

where:

$\hat{\alpha}_j, \hat{\lambda}_j$: is the estimate of α and λ respectively at the j^{th} replicate (run).

L: is the number of sample replicated.

n_t : is the number of times chosen to be (4).

$\hat{R}_j(t_i)$: is the estimates of $R(t)$ at the j^{th} replicate (run) and i^{th} time.

The results of simulation study are summarized in tables (1)-(8).

3.1 Simulation Results for Estimating the Parameters

From tables (1)-(6) which present the simulation results for estimated MSE associated with different estimations of the unknown shape and scale parameters of WE distribution with $\alpha = 0.5, 1, 1.5$ and $\lambda = 1$, we have observed:

- The performance of Bayes estimates based on Linex loss function with positive value of its shape parameter (z) is better than that based on negative value for all sample sizes.
- The performance of Bayes estimates with non-informative priors assumption based on Linex loss function with different values of (z) is better than that based on squared error loss function for all sample sizes.
- The performance of Bayes estimates with informative priors assumption based on squared error loss function and that based on Linex loss function with negative value of (z) is almost identical especially with $n \geq 75$.
- The performance of Bayes estimates with informative priors assumption is better than that with non-informative priors assumption for all sample sizes.
- The estimations can be ordered according to their performance as:

Order	1	2	3
Performance	$BL(z = +0.8)$	$BL(z = -0.8)$	BS
	with non-informative priors		
Performance	$BL(z = +0.8)$	BS	$BL(z = -0.8)$
	with informative priors		

3.2 Simulation Results for Estimating the Reliability Function

From tables (7) and (8) which present the simulation results for estimated IMSE associated with different estimations of the reliability function of WE distribution with $\alpha = 0.5, 1, 1.5$ and $\lambda = 1$, we have observed:

- The performance of Bayes estimates based on Linex loss function with positive value of its shape parameter (z) is better than that based on negative value for all sample sizes.
- The performance of Bayes estimates based on Linex loss function with different values of (z) is better than that based on squared error loss function for all sample sizes.
- In general, the performance of Bayes estimates with non-informative priors assumption based on different loss functions is almost identical especially with n ≥ 50.
- The performance of Bayes estimates with informative priors assumption is better than that with non-informative priors assumption for all sample sizes.
- The estimations can be ordered according to their performance as:

<table>
<tr><th>Order</th><th>1</th><th>2</th><th>3</th></tr>
<tr><td rowspan="2">Performance</td><td>$BL(z = +0.8)$</td><td>$BL(z = -0.8)$</td><td>BS</td></tr>
<tr><td colspan="3">with non-informative and informative priors</td></tr>
</table>

4. Conclusions

The most important conclusions according to Monte Carlo simulation study based on fuzzy data, tables (1)…(8), are summarized by:

1. From the estimated mean squared error values associated with different estimates of the unknown shape and scale parameters of weighted exponential distribution with shape parameter $\alpha = 0.5, 1, 1.5$ and scale parameter $\lambda = 1$, we have observed:
   - For all sample sizes, increase the value of the shape parameter, increasing the values of mean squared error associated with Bayes estimates with non-informative and informative gamma priors assumption.
   - With all estimates, the values of mean squared error are decreasing as the sample size increase.
   - For all sample sizes, the performance of Bayes estimates according to Lindley's approximation with informative gamma priors assumption is better than that with non-informative priors.
   - For all sample sizes, the performance of Bayes estimates according to Lindley's approximation with non-informative and informative gamma priors assumption based on linear-exponential (Linex) loss function with positive value of its shape parameter is better than that based on negative value.
   - The performance of Bayes estimates according to Lindley's approximation with informative gamma priors assumption based on squared error loss function and that based on Linex loss function with negative value of its shape parameter is almost identical especially with large sample sizes.
2. From the estimated integrated mean squared error values associated with different estimates of the reliability function of weighted exponential distribution with shape parameter $\alpha = 0.5, 1, 1.5$ and scale parameter $\lambda = 1$, we have observed:
   - For all sample sizes, increase the value of the shape parameter, decreasing the values of integrated mean squared error associated with all estimations.
   - The values of integrated mean squared error associated with all estimations are decreasing as the sample size increase.
   - The performance of Bayes estimates with informative gamma priors assumption is better than that with non-informative priors assumption for all sample sizes. Further, the performance of Bayes estimates based on Linex loss function is better than that based on squared error loss function for all sample sizes.

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Table (1)

MSE Values Associated with Bayes Estimates of the Shape and Scale Parameters of WE Distribution based on Squared Error Loss Function with Non-Informative Priors and Different Sample Sizes.

n	α	λ	$\hat{\alpha}_{BS}$	$\hat{\lambda}_{BS}$
25	0.5	1	0.7513466	0.2055911
	1	1	0.8211137	0.2230011
	1.5	1	1.2935761	0.2739481
50	0.5	1	0.6256030	0.0439936
	1	1	0.7007319	0.1224583
	1.5	1	0.9825211	0.1374294
75	0.5	1	0.5256901	0.0426216
	1	1	0.6903225	0.0603187
	1.5	1	0.8508306	0.0794490
100	0.5	1	0.2490393	0.0119749
	1	1	0.5022281	0.0459603
	1.5	1	0.8117866	0.0599941

Table (2)

MSE Values Associated with Bayes Estimates of the Shape and Scale Parameters of WE Distribution based on Linex Loss Function with Non-Informative Priors, $z = +\,0.8$ and Different Sample Sizes.

n	α	λ	$\hat{\alpha}_{BL}\ (z = +\,0.8)$	$\hat{\lambda}_{BL}\ (z = +\,0.8)$
25	0.5	1	0.7493397	0.1122675
	1	1	0.8072443	0.1352087
	1.5	1	0.9951300	0.2130023
50	0.5	1	0.4015880	0.0407778
	1	1	0.4758065	0.1058622
	1.5	1	0.6456800	0.1141037
75	0.5	1	0.3191192	0.0354204
	1	1	0.3214306	0.0381708
	1.5	1	0.3937044	0.0416263
100	0.5	1	0.1018511	0.0089937
	1	1	0.2629313	0.0320119
	1.5	1	0.3727509	0.0333864

Table (3)

MSE Values Associated with Bayes Estimates of the Shape and Scale Parameters of WE Distribution based on Linex Loss Function with Non-Informative Priors, $z = -\,0.8$ and Different Sample Sizes.

n	α	λ	$\hat{\alpha}_{BL}\ (z = -\,0.8)$	$\hat{\lambda}_{BL}\ (z = -\,0.8)$
25	0.5	1	0.7504430	0.1912071
	1	1	0.8092657	0.2030487
	1.5	1	0.9981224	0.2631765
50	0.5	1	0.6009801	0.0439381
	1	1	0.6991300	0.1215570
	1.5	1	0.9780852	0.1253871
75	0.5	1	0.5110943	0.0425360
	1	1	0.6641505	0.0600129
	1.5	1	0.8312896	0.0782183
100	0.5	1	0.1351335	0.0108225
	1	1	0.4536963	0.0429080
	1.5	1	0.7916566	0.0595802

Table (4)
MSE Values Associated with Bayes Estimates of the Shape and Scale Parameters of WE Distribution based on Squared Error Loss Function with Informative Priors and Different Sample Sizes.

n	α	λ	$\hat{\alpha}_{BS}$	$\hat{\lambda}_{BS}$
25	0.5	1	0.4533799	0.0186437
	1	1	0.4938780	0.0188310
	1.5	1	0.6134889	0.0198231
50	0.5	1	0.3273896	0.0123893
	1	1	0.3658930	0.0155298
	1.5	1	0.3996709	0.0193793
75	0.5	1	0.2824340	0.0099085
	1	1	0.2987288	0.0109832
	1.5	1	0.3025980	0.0190913
100	0.5	1	0.0917286	0.0086008
	1	1	0.1686639	0.0091631
	1.5	1	0.1789889	0.0093009

Table (5)
MSE Values Associated with Bayes Estimates of the Shape and Scale Parameters of WE Distribution based on Linex Loss Function with Informative Priors, $z = +0.8$ and Different Sample Sizes.

n	α	λ	$\hat{\alpha}_{BL}\ (z = +0.8)$	$\hat{\lambda}_{BL}\ (z = +0.8)$
25	0.5	1	0.3387012	0.0108905
	1	1	0.3619889	0.0111837
	1.5	1	0.3928902	0.0168383
50	0.5	1	0.2898795	0.0095859
	1	1	0.2899092	0.0099980
	1.5	1	0.2989183	0.0102761
75	0.5	1	0.1818837	0.0088036
	1	1	0.1978901	0.0092369
	1.5	1	0.2871999	0.0093300
100	0.5	1	0.0913841	0.0069203
	1	1	0.1427086	0.0069823
	1.5	1	0.1454985	0.0071922

Table (6)
MSE Values Associated with Bayes Estimates of the Shape and Scale Parameters of WE Distribution based on Linex Loss Function with Informative Priors, $z = -0.8$ and Different Sample Sizes.

n	α	λ	$\hat{\alpha}_{BL}\ (z = -0.8)$	$\hat{\lambda}_{BL}\ (z = -0.8)$
25	0.5	1	0.4989843	0.0196970
	1	1	0.5260977	0.0218984
	1.5	1	0.6289070	0.0220855
50	0.5	1	0.3819403	0.0130389
	1	1	0.3909911	0.0166884
	1.5	1	0.4078695	0.0200283
75	0.5	1	0.2890858	0.0100795
	1	1	0.2989332	0.0109888
	1.5	1	0.3099570	0.0198655
100	0.5	1	0.0920090	0.0087989
	1	1	0.1689785	0.0091954
	1.5	1	0.1796331	0.0093031

Table (7)
IMSE Values Associated with Bayes Estimates of the Reliability Function of WE Distribution based on Squared Error and Linex Loss Functions with Non-Informative Priors and Different Sample Sizes.

n	α	λ	$\hat{R}_{BS}(t)$	$\hat{R}_{BL}(t)\ (z = -0.8)$	$\hat{R}_{BL}(t)\ (z = +0.8)$
25	0.5	1	0.0095987	0.0095919	0.0089883
	1	1	0.0028115	0.0024998	0.0024440
	1.5	1	0.0023982	0.0023774	0.0023732
50	0.5	1	0.0043492	0.0043448	0.0043441
	1	1	0.0018889	0.0018876	0.0018848
	1.5	1	0.0006438	0.0006435	0.0006433
75	0.5	1	0.0005081	0.0005014	0.0005010
	1	1	0.0003868	0.0003808	0.0003803
	1.5	1	0.0002752	0.0002747	0.0002732
100	0.5	1	0.0003429	0.0003448	0.0003441
	1	1	0.0002359	0.0002351	0.0002349
	1.5	1	0.0001460	0.0001458	0.0001458

Table (8)
IMSE Values Associated with Bayes Estimates of the Reliability Function of WE Distribution based on Squared Error and Linex Loss Functions with Informative Priors and Different Sample Sizes.

n	α	λ	$\hat{R}_{BS}(t)$	$\hat{R}_{BL}(t)\ (z=-0.8)$	$\hat{R}_{BL}(t)\ (z=+0.8)$
25	0.5	1	0.0072386	0.0071706	0.0057094
	1	1	0.0019059	0.0016513	0.0012221
	1.5	1	0.0016223	0.0014031	0.0012096
50	0.5	1	0.0038890	0.0038194	0.0026675
	1	1	0.0011085	0.0010079	0.0009815
	1.5	1	0.0004556	0.0004188	0.0003879
75	0.5	1	0.0002944	0.0002904	0.0002317
	1	1	0.0002669	0.0002266	0.0002141
	1.5	1	0.0002396	0.0002177	0.0001991
100	0.5	1	0.0002418	0.0002309	0.0001957
	1	1	0.0002088	0.0001992	0.0001384
	1.5	1	0.0001086	0.0001073	0.0001067