

## C-Coretractable and Strongly C-Coretractable Modules

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### Abstract

Let  $R$  be a ring with identity and  $M$  be an  $R$ -module with unity. In this paper we introduce the notion of  $C$ -coretractable modules. Some basic properties of this class of modules are investigated and some relationships between these modules and other related concepts are introduced. Also, we give the notion of strongly  $C$ -coretractable and study its comparison with  $C$ -coretractable.

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### Introduction

Throughout this paper all rings have identities and all modules are unital right  $R$ -modules. A module  $M$  is called coretractable if for each a proper submodule  $N$  of  $M$ , there exists a nonzero  $R$ -homomorphism  $f: M/N \rightarrow M$  [4], and an  $R$ -module  $M$  is called strongly coretractable module if for each proper submodule  $N$  of  $M$ , there exists a nonzero  $R$ -homomorphism  $f: M/N \rightarrow M$  such that  $\text{Im}f + N = M$  [10]. It is clear that every strongly coretractable module is coretractable but it is not conversely, in [11], [12], [13] and [14] have more information of these concepts. In this paper, we introduce the notion of  $C$ -coretractable module where an  $R$ -module  $M$  is called  $C$ -coretractable module if for each proper closed submodule  $N$  of  $M$ , there exists a nonzero homomorphism  $f \in \text{Hom}_R(M/N, M)$ . This work consists of two sections, in section one we study some basic properties of  $C$ -coretractable modules. A characterization of these modules are given where we prove that a direct sum of two  $C$ -coretractable modules is also  $C$ -coretractable (Theorem 1.10).

Also we prove that the isomorphic image of  $C$ -coretractable is again  $C$ -coretractable, but a submodule of  $C$ -coretractable module may be not  $C$ -coretractable and the same thing with respect to direct summand and quotient, see Remark (1.2(11)), also we study this concept with other related modules as multiplication, self-generator and Noetherian

comultiplication. In section two, we introduce and study strongly  $C$ -coretractable modules where an  $R$ -module  $M$  is called strongly  $C$ -coretractable module if for each proper closed submodule  $N$  of  $M$ , there exists a nonzero homomorphism  $f \in \text{Hom}_R(M/N, M)$  such that  $\text{Im}f + N = M$  and clear that every strongly  $C$ -coretractable is  $C$ -coretractable. Many relationships between strongly  $C$ -coretractable modules and other concepts are presented. Finally, in section three the concept of mono- $c$ -coretractable where an  $R$ -module  $M$  is called mono- $C$ -coretractable if for each proper closed submodule of  $M$ , there exists  $f \in \text{End}_R(M)$ ,  $f \neq 0$  and  $N = \ker f$ .

### 1. C-Coretractable Modules

In this section, we introduce a generalization of coretractable namely  $C$ -coretractable and we investigate many properties of these concepts.

#### *Definition (1.1):*

An  $R$ -module  $M$  is called  $C$ -coretractable module if for each proper closed submodule  $N$  of  $M$ , there exists a nonzero homomorphism  $f \in \text{Hom}_R(M/N, M)$ . A ring  $R$  is called  $C$ -coretractable if  $R$  is  $C$ -coretractable  $R$ -module.

**Examples and Remarks (1.2):**

(1) An R-module M is C-coretractable if for each proper closed submodule N of M, there exists  $g \in \text{End}_R(M)$ ,  $g \neq 0$  and  $g(N) = 0$ .

**Proof:** It is clear by [11].

(2) It is clear that every coretractable R-module is C-coretractable. But the converse is not true in general. For example, Q as Z-module is C-coretractable, since it has only two closed submodules (0) and Q. But Q is not coretractable module. Also the Z-module Z is C-coretractable, but it is not coretractable.

(3) Every extending R-module is C-coretractable. Where an M is called extending if every submodule of M is essential in a direct summand. Equivalently, M is extending module if each closed submodule is direct summand [8].

**Proof:** Let N be a proper closed submodule of M. Since M is extending module, then  $N < \oplus M$ , hence  $M = N \oplus K$  for some  $K \leq M$  and so  $M/N \cong K$ , thus there exists an isomorphism  $g: M/N \rightarrow K$ . Define  $f = i \circ g$  and  $f \neq 0$  where i is the inclusion mapping. Therefore M is a C-coretractable module.

(4) C-coretractable module need not be extending, for example: consider the Z-module  $M = Z_8 \oplus Z_2$  is not extending Z-module since  $N = \langle (\bar{2}, \bar{1}) \rangle$  is closed, but it is not a direct summand of M. Note that all proper closed submodules of M are:  $N_1 = \langle (\bar{0}, \bar{0}) \rangle$ ,  $N_2 = \langle (\bar{1}, \bar{0}) \rangle$ ,  $N_3 = \langle (\bar{0}, \bar{1}) \rangle$ ,  $N_4 = \langle (\bar{1}, \bar{1}) \rangle$ ,  $N_5 = \langle (\bar{4}, \bar{1}) \rangle$  and  $N_6 = \langle (\bar{2}, \bar{1}) \rangle$ . But each of  $N_1, N_2, N_3, N_4$  and  $N_5$  are direct summand submodules of M, since  $N_1 \oplus M = N_2 \oplus N_3 = N_4 \oplus N_5 = M$ . Hence there exists  $f_i: M/N_i \rightarrow M$ ,  $f_i \neq 0$  and  $i = 1, 2, 3, 4, 5$ . Now, take  $N = \langle (\bar{2}, \bar{1}) \rangle = \{ (\bar{0}, \bar{0}), (\bar{2}, \bar{1}), (\bar{4}, \bar{0}), (\bar{6}, \bar{1}) \}$ ,  $M/N = \{ (\bar{0}, \bar{0}) + N, (\bar{1}, \bar{0}) + N, (\bar{2}, \bar{0}) + N, (\bar{3}, \bar{0}) + N \}$ . Define  $f: M/N \rightarrow M$  by  $f((\bar{a}, \bar{0}) + N) = (2\bar{a}, \bar{0})$  for all  $((\bar{a}, \bar{0}) + N) \in M/N$ , then it is clear that f is a nonzero Z-homomorphism. Thus M is C-coretractable.

(5) It is known that every quasi-injective (or  $\pi$ -injective) module is extending module see [6, Proposition(7.2), P.55]. Hence every quasi-injective (or  $\pi$ -injective) module is C-coretractable by part (3). Where "An R-module M is called  $\pi$ -injective if  $f(M) \subseteq M$  for each idempotent f of  $E(M)$ " [6].

(6) The quasi-injective hull of any R-module M is C-coretractable module.

**Proof:** It follows directly by part (5).

(7) Every semi simple module M is a coretractable and hence M is a C-coretractable.

(8) Every uniform module is a C-coretractable.

**Proof:** Since M is a uniform module, hence M is extending by [3, Proposition (1.12), P.10] and hence M is a C-coretractable.

(9) It is obvious that every simple closed module is C-coretractable, but not conversely. Where a module M is called simple closed if M has only two closed submodules which are (0) and M [2]. However the Z-module  $M = Z_8 \oplus Z_2$  is C-coretractable module by part (4) and it is not simple closed.

(10) The isomorphic image of C-coretractable R-module is also C-coretractable.

**Proof:** Let  $M \cong M'$  and M is C-coretractable module, there exists an isomorphism  $f: M \rightarrow M'$ . Let W be a proper closed submodule of M'. Then  $N = f^{-1}(W)$  is proper closed submodule of M. Since M is C-coretractable module. then there exists  $g \in \text{End}_R(M)$ ,  $g \neq 0$  and  $g(N) = 0$ . Now, the mapping  $f \circ g \circ f^{-1} \in \text{End}_R(M')$ . Moreover,  $f \circ g \circ f^{-1}(M') = f \circ g(f^{-1}(M')) = f \circ g(M) = f(g(M))$ . But  $g(M) \neq 0$  and f is an isomorphism, so that  $f(g(M)) \neq 0$ . Thus  $f \circ g \circ f^{-1}(M') \neq 0$ . Also  $f \circ g \circ f^{-1}(W) = f \circ g(f^{-1}(W)) = f \circ g(N) = f(g(N)) = f(0) = 0$ . Therefore M' is C-coretractable module.

(11) C-coretractability is not preserved by taking submodules, factor modules and direct summands, since for any R-module M and a cogenerator R-module C,  $C \oplus M$  is a cogenerator by [4], and so  $C \oplus M$  is a coretractable module. Thus  $C \oplus M$  is C-coretractable, but M need not be C-coretractable. Where an R-module M is called cogenerator if for every nonzero homomorphism  $f: M_1 \rightarrow M_2$  where  $M_1$  and  $M_2$  are R-modules, there exists  $g: M_2 \rightarrow M$  such that  $g \circ f \neq 0$ .

**Proposition (1.3):** Let N be a direct summand submodule of a C-coretractable R-module M. If N is a fully invariant submodule of M. Then N is C-coretractable.

**Proof:** Since N is a direct summand of M, so there exists a submodule W of M such that

$N \oplus W = M$ . Let  $K$  be a proper closed submodule of  $N$ , then  $K \oplus W$  is a closed submodule in  $N \oplus W = M$ . Since  $M$  is a C-coretractable module, so there exists  $f \in \text{End}_R(M)$ ,  $f \neq 0$  and  $f(K \oplus W) = 0$ . Suppose that  $g$  is a restriction map from  $N$  into  $M$ ,  $g \neq 0$ . Also  $N$  is fully invariant direct submodule. Then  $N$  is stable module [1, Lemma(2.1.6)]. So  $g(N) \subseteq N$ . Therefore  $g \in \text{End}_R(N)$ ,  $g \neq 0$ .  $g(K) = f|_N(K) = 0$ . Thus  $N$  is a C-coretractable.

**Corollary (1.4):** Let  $N$  be a direct summand submodule of duo  $R$ -module  $M$ . If  $M$  is a C-coretractable. Then  $N$  is also C-coretractable.

**Proof:** Since every submodule of duo  $R$ -module is fully invariant. Therefore the result follows directly by Proposition (1.3).

**Proposition (1.5):** Let  $N$  be a direct summand submodule of a C-coretractable module  $M$ . If  $N$  cogenerates  $M$ . Then  $N$  is a C-coretractable.

**Proof:** Suppose  $N$  cogenerates  $M$ , so there exists  $g \in \text{Hom}_R(M, N)$ ,  $g \neq 0$ . Let  $K$  be a closed submodule of  $N$ . Since  $N$  is direct summand of  $M$ , then  $N \oplus W = M$  for some a submodule  $W$  of  $M$ . So  $K \oplus W$  is closed submodule in  $N \oplus W = M$ . Then there exists  $f \in \text{End}_R(M)$ ,  $f \neq 0$ ,  $f(K \oplus W) = 0$  (Since  $M$  is C-coretractable module  $M$ ), hence  $g \circ f \neq 0$ , Let  $h$  be a restriction of  $g \circ f$  on  $N$ , so  $h \in \text{End}_R(N)$  and  $h(K) = g(f(K)) = 0$ . Therefore,  $N$  is a C-coretractable.

**Proposition(1.6):** Let  $M$  be an  $R$ -module. If  $M$  is quasi-Dedekind module. Then  $M$  is a C-coretractable module if and only if  $M$  is a simple closed module.

**Proof:** ( $\Rightarrow$ ) Let  $N$  be a proper closed submodule of  $M$ . Since  $M$  is C-coretractable module, then there exists  $f \in \text{End}_R(M)$ ,  $f \neq 0$  and  $f(N) = 0$ ;  $N \subseteq \ker f$ . As  $M$  is a quasi-Dedekind module,  $f$  is monomorphism, hence  $\ker f = 0$ . Thus  $N = 0$  and hence  $M$  is simple closed module.

( $\Leftarrow$ ) It follows directly by Examples and Remarks (1.2(8)).

Note that simple closed module need not be quasi-Dedekind, for example  $Z_4$  as  $Z$ -module is simple closed, but not quasi-Dedekind. Recall that an  $R$ -module  $M$  is

called C-Rickart if  $\ker f$  is closed submodule of  $M$  for all  $f \in \text{End}_R(M)$ .

**Theorem (1.7):** Let  $M$  be a C-Rickart module, then  $M$  is a coretractable module if and only if for all proper submodule  $K$  of  $M$ , there exists a proper closed submodule  $W$  of  $M$  such that  $K \subseteq W$  and  $M$  is C-coretractable module.

**Proof:** ( $\Rightarrow$ ) It is clear that  $M$  is C-coretractable module. Now, let  $K$  be a proper submodule of  $M$ . Since  $M$  is coretractable module, then there exists a nonzero  $R$ -homomorphism  $f: M \rightarrow M$ ,  $f(K) = 0$ ; that is  $K \subseteq \ker f$ . But  $M$  is a C-Rickart module, so  $\ker f$  is a closed submodule of  $M$ . As  $f \neq 0$ ,  $\ker f \neq M$  and hence  $\ker f$  is a proper closed submodule such that  $\ker f \subset M$ .

( $\Leftarrow$ ) Let  $K$  be a proper submodule of  $M$ . By hypothesis there exists a proper closed submodule  $W$  of  $M$  such that  $K \subseteq W$ . Since  $M$  is C-coretractable module, there exists a nonzero hence  $f \in \text{End}_R(M)$  such that  $f(W) = 0$ . This implies that  $f(K) = 0$ , and so  $M$  is a coretractable module.

Recall that an  $R$ -module  $M$  is called purely extending if every submodule is essential in pure submodule. Equivalently,  $M$  is purely extending if and only if every closed submodule is pure in  $M$  [3].

**Proposition (1.8):** Let  $M$  be a purely extending  $R$ -module. Then the following assertion hold:

- (1) If  $M$  is Noetherian projective, then  $M$  is a C-coretractable module.
- (2) If  $M$  finitely generated flat module over Noetherian ring, then  $M$  is a C-coretractable module.

**Proof:** As  $M$  is a purely extending module, then every closed submodule is pure by [3, Theorem(2.2), P.39].

- (1) As  $M$  is Noetherian projective module, hence every pure submodule is a direct summand of  $M$  by [7, Proposition (2.11), P.63], and hence  $M$  is extending. and so  $M$  is C-coretractable module.
- (2) Since  $M$  is finitely generated flat module over Noetherian ring, then every pure submodule is a direct summand of  $M$  by [7, Proposition (2.10), P.62], and hence  $M$  is extending. Thus  $M$  is C-coretractable.

**Corollary (1.9):** If  $R$  is Noetherian purely extending ring, then  $R$  is a  $C$ -coretractable.

**Proof:** Since  $R$  is projective, then the result follows directly by Proposition (1.8).

Recall that "An  $R$ -module  $M$  has the closed intersection property (briefly CIP) if the intersection of any two closed submodules is again closed" [2].

**Theorem (1.10):** Let  $\{M_\alpha : \alpha \in I\}$  be a family of  $C$ -coretractable  $R$ -module. If for any  $\alpha, \beta \in I$ ,  $M_\alpha$  is  $M_\beta$ -injective and  $M = \bigoplus_{\alpha \in I} M_\alpha$  such that  $M$  has CIP. Then  $M$  is  $C$ -coretractable module. In particular, if  $M$  is a quasi-injective and satisfy CIP, then  $\bigoplus_{\alpha \in I} M_\alpha$  (where  $M = M_\alpha$  for all  $\alpha \in I$ ) is  $C$ -coretractable.

**Proof:** Let  $K$  be a proper closed submodule of  $M$ , then there exists  $\beta \in I$  such that  $M_\beta \not\subseteq K$ . Since  $M$  satisfies CIP and  $M_\beta$  is closed in  $M$ , so  $K \cap M_\beta$  is proper closed in  $M$ . But  $K \cap M_\beta$  is a proper submodule in  $M_\beta$  and  $M_\beta$  is  $C$ -coretractable module. Thus there exists a nonzero homomorphism  $f: M_\beta / (K \cap M_\beta) \rightarrow M_\beta$ . Now, the natural map  $g: M_\beta / (K \cap M_\beta) \rightarrow M/K$  (Which is defined by  $g(x + (K \cap M_\beta)) = x + K$  for all  $x \in M_\beta \subset M$ ) is a monomorphism. As  $M_\beta$  is  $M_\alpha$ -injective for any  $\alpha \in I$  by hypothesis,  $M_\beta$  is  $M/K$ -injective by [5, Proposition (16.13)], so there exists  $h: M/K \rightarrow M_\beta$  such that  $h \circ g = f$ , hence  $0 \neq i \circ h \in \text{Hom}_R(M/K, M)$ , where  $i: M_\beta \rightarrow M$  is the natural inclusion. Thus  $M$  is  $C$ -coretractable.

Now by applying Theorem (1.10) we give the following examples

- (1) Let  $M = \bigoplus_{\alpha \in I} Q_\alpha$  (where  $Q_\alpha = Q$  for all  $\alpha \in I$ ). As  $Q$  is quasi-injective and satisfy CIP, then  $M$  is  $C$ -coretractable.
- (2) Let  $M = \bigoplus_{\alpha \in I} M_\alpha$  (where  $M_\alpha = Z_{p^\infty}$  for all  $\alpha \in I$ ). As  $Z_{p^\infty}$  is quasi-injective and satisfy CIP, then  $M$  is  $C$ -coretractable.

**Theorem (1.11):** Let  $M = \bigoplus_{\alpha \in I} M_\alpha$ , where  $M_\alpha$  is an  $R$ -modules for all  $\alpha \in I$  such that every closed submodule of  $M$  is fully invariant submodule. If  $M_\alpha$  is  $C$ -coretractable module for all  $\alpha \in I$ , then  $M$  is a  $C$ -coretractable module.

**Proof:** Let  $N$  be a proper closed submodule of  $M$ . Since  $N$  is fully invariant submodule of  $M$ ,  $N = \bigoplus_{\alpha \in I} N_\alpha \cap M_\alpha$ . Set  $N \cap M_\alpha = N_\alpha$  for all  $\alpha \in I$ . Then  $N = \bigoplus_{\alpha \in I} N_\alpha$ . Now  $N_\alpha \leq^\oplus N$ , so  $N_\alpha$  is closed in  $N$ , but  $N$  is closed in  $M$ . Also, as  $N$

is proper submodule of  $M$ , there exists at least one  $\alpha_i, N_{\alpha_i}$  is proper submodule of  $M_{\alpha_i}$ . But  $M_{\alpha_i}$  is  $C$ -coretractable, so there exists  $f_{\alpha_i}: M_{\alpha_i}/N_{\alpha_i} \rightarrow M_{\alpha_i}$  and  $f_{\alpha_i} \neq 0$ . As  $M/N \cong \bigoplus_{\alpha \in I} (M_\alpha/N_\alpha)$ . Define  $h: M/N \rightarrow M_{\alpha_i}$  by  $h(m+N) = f_{\alpha_i}(m_{\alpha_i} + N_{\alpha_i})$  for any  $m = \bigoplus_{\alpha \in I} m_\alpha \in M$ . Then  $h \neq 0$  and  $g = i \circ h: M/N \rightarrow M, g \neq 0$ .

**Theorem (1.12):** Let  $M_1$  and  $M_2$  be  $R$ -modules,  $M = M_1 \oplus M_2$ , such that  $\text{ann}(M_1) + \text{ann}(M_2) = R$ . If  $M_1$  and  $M_2$  are  $C$ -coretractable module, then  $M$  is a  $C$ -coretractable module.

**Proof:** Let  $N$  be a proper closed submodule of  $M$ . Since  $\text{ann}(M_1) + \text{ann}(M_2) = R$ , then  $N = N_1 \oplus N_2$  for some submodules  $N_1$  and  $N_2$  of  $M_1$  and  $M_2$  respectively. Hence by a similar argument of [10, Theorem (2.8)],  $M$  is  $C$ -coretractable module.

Recall that an  $R$ -module  $M$  is called quasi-Dedekind if every proper nonzero submodule  $N$  of  $M$  is quasi-invertible where a submodule  $N$  of  $M$  is called quasi-invertible if  $\text{Hom}_R(M/N, M) = 0$  [15]. Also every nonsingular coretractable  $R$ -module is semi simple see [4]. However this result cannot be generalized for  $C$ -coretractable module. For example,  $Z$  as  $Z$ -module is nonsingular  $C$ -coretractable  $R$ -module, but it is not semisimple. However we have the following:

**Remark (1.13):** If  $M$  is nonsingular and quasi-Dedekind module, then  $M$  is simple closed and hence  $M$  is a  $C$ -coretractable module.

**Proof:** Suppose that there exists a nonzero proper closed submodule  $N$  of  $M$ . Hence  $N$  is quasi-invertible submodule since  $M$  is quasi-Dedekind, then by [15, Proposition (3.13), P.19], then  $N$  is an essential submodule of  $M$  which is a contradiction. Thus  $M$  is a simple closed module.

We need recall that an  $R$ -module  $M$  is called multiplication if for each submodule  $N$  of  $M$ , there exists a right ideal in  $R$  such that  $MI = N$  [6].

**Proposition (1.14):** Let  $M$  be a multiplication  $R$ -module. Then  $M$  is a  $C$ -coretractable module.

**Proof:** Let  $N$  be a proper closed submodule of  $M$ . Then if  $N=0$ , then there exists  $f:M/N \rightarrow M$  and  $f \neq 0$ . If  $N \neq 0$ , then  $N$  is not essential since  $N$  is closed submodule of  $M$  and so  $N$  is not a rational submodule of  $M$ ; that is  $M$  is not rational extension of  $N$ . Hence  $N$  is not quasi-invertible submodule of  $M$  by [15, Proposition (3.9), P.18], so that  $\text{Hom}(M/N, M) \neq 0$ . Therefore  $M$  is a C-coretractable module.

The converse of Proposition (1.14) is not true in general, since for example  $Q$  as  $Z$ -module is C-coretractable module, but it is not multiplication module.

It is known that every cyclic module over commutative ring  $R$  is a multiplication module. Hence we have the following directly:

**Corollary (1.15):** Let  $M$  be a cyclic module over commutative ring  $R$ , then  $M$  is a C-coretractable module.

**Corollary (1.16):** If  $R$  is a commutative ring with unity, then  $R$  is a C-coretractable.

Recall that an  $R$ -module  $M$  is called self-generator if for every submodule  $N$  of  $M$ ,  $N = \sum_f f(M)$ , where  $f \in \text{Hom}(M, N)$  [5, P.241].

**Proposition (1.17):** Let  $M$  be a self-generator duo  $R$ -module. Then  $M$  is a C-coretractable module.

**Proof:** Let  $N$  be a nonzero proper closed submodule of  $M$ . Hence  $N$  is not essential submodule of and hence  $N$  is not rational submodule of  $M$ . Thus  $M$  is not rational extension of  $N$ . Then by [15, Proposition (3.14), P.20],  $N$  is not quasi-invertible; that is  $\text{Hom}(M/N, M) \neq 0$ . Thus  $M$  is a C-coretractable module.

**Proposition (1.18):** Let  $M$  be a Noetherian comultiplication  $R$ -module. Then  $M$  is an extending and hence it is C-coretractable module.

**Proof:** Since  $M$  is a Noetherian comultiplication  $R$ -module, then  $M$  is an Artinian quasi-injective, but  $M$  is quasi-injective implies that  $M$  is extending module. Thus  $M$  is a C-coretractable module.

## 2. Strongly C-Coretractable Modules

In this section, we introduce and study the concept of strongly C-coretractable module. Many important properties are given.

**Definition (2.1):** An  $R$ -module  $M$  is called strongly C-coretractable module if for each proper closed submodule  $N$  of  $M$ , there exists a nonzero homomorphism  $f \in \text{Hom}_R(M/N, M)$  and  $\text{Im} f + N = M$ . Equivalently,  $M$  is a strongly C-coretractable  $R$ -module if for each proper closed submodule  $N$  of  $M$ , there exists  $g \in \text{End}_R(M)$ ,  $g \neq 0$ ,  $g(N) = 0$  and  $\text{Im} f + N = M$ . A ring  $R$  is called strongly C-coretractable if  $R$  is strongly C-coretractable  $R$ -module.

### Examples and Remarks (2.2):

(1) Every strongly coretractable module is a strongly C-coretractable module. But the converse is not true in general. For example Consider  $Z_{12}$  as  $Z$ -module the only proper closed submodules of  $Z_{12}$  are  $(\bar{0})$ ,  $(\bar{3})$  and  $(\bar{4})$  (Which are direct summand of  $Z_{12}$ ), hence  $Z_{12}$  is strongly C-coretractable but  $Z_{12}$  is not strongly coretractable.

(2) Every strongly C-coretractable module is a C-coretractable. However we claim the converse is not true in general. But we have no example.

(3) Every semisimple module is a strongly C-coretractable since every semisimple is a strongly coretractable module and hence strongly C-coretractable by part (1).

(4) Every simple closed module is a strongly C-coretractable.

(5) Every extending  $R$ -module is a strongly C-coretractable module. In particular,  $M = Z^{(n)}$  as  $Z$ -module;  $M = Z \oplus Z \oplus Z \dots \oplus Z$  ( $n$ -time).

(6) The quasi-injective hull of any  $R$ -module  $M$  is a strongly C-coretractable.

**Proof:** It is clear by part (5).

(7) Every Noetherian comultiplication  $R$ -module  $M$  is a strongly C-coretractable.

**Proof:** By Proposition (1.18),  $M$  is extending module. Thus  $M$  is strongly C-coretractable module by part (5).

(8) If  $M$  be a faithful multiplication module over extending ring, then  $M$  is a strongly C-coretractable module.

**Proof:** By [3, Proposition (3.5), P.26],  $M$  is an extending  $R$ -module. Hence  $M$  is strongly C-coretractable.

**Proposition (2.3):** Let  $M \cong M'$  where  $M$  is a strongly C-coretractable R-module. Then  $M'$  is a strongly C-coretractable R-module.

**Proof:** Since  $M \cong M'$ , so there exists  $f: M \rightarrow M'$  be an R-isomorphism. Let  $W$  be a proper closed submodule of  $M'$ . Then  $N = f^{-1}(W)$  is proper closed submodule of  $M$ . Since  $M$  is strongly C-coretractable module. So there exists a nonzero R-homomorphism  $h: M/N \rightarrow M$  such that  $h(M/N) + N = M$ , and hence by the same argument proof of [9, Proposition (2.1.4)], we get  $M'$  is a strongly C-coretractable.

**Proposition (2.4):** Let  $M$  be a strongly C-coretractable R-module and  $N$  be a proper closed submodule of  $M$ , then  $M/N$  is strongly C-coretractable too.

**Proof:** Let  $W/N$  be a proper closed submodule of  $M/N$ . Since  $N$  is closed submodule of  $M$ , so  $W$  is closed submodule of  $M$ . But  $M$  is strongly C-coretractable module, hence there exists a nonzero R-homomorphism  $g: M/W \rightarrow M$  and  $g(M/W) + W = M$ . And hence by the same argument of proof of [10, Theorem (2.1.5)] we can get  $M/N$  is also strongly C-coretractable.

**Corollary (2.5):** Let  $N$  be a direct summand submodule of strongly C-coretractable R-module  $M$ , then  $N$  is a strongly C-coretractable module.

**Proof:** Since  $N$  is a direct summand submodule of  $M$ , so there exists submodule  $W$  of  $M$  such that  $N \oplus W = M$ . Thus  $M/W$  is strongly C-coretractable by Proposition (2.4). But  $M/W \cong N$  so that  $N$  is also strongly C-coretractable module by Proposition (2.3).

Recall that "If  $R$  is a principal ideal domain, then any finitely generated torsion free module over  $R$  is extending module" [16, Proposition (1.2.4), P.11], hence we can obtain the following proposition directly.

**Proposition (2.6):** Let  $M$  be a finitely generated torsion free module over principal ideal domain. Then  $M$  is a strongly C-coretractable module.

**Example (2.7):** Each of the  $Z$ -module  $Z \oplus Z$  and  $M_{2 \times 2}(Z)$ , where  $M_{2 \times 2}(Z)$  is the set of all

$2 \times 2$  matrices whose entries in  $Z$  are strongly C-coretractable module.

**Proposition (2.8):** Let  $R$  be a strongly C-coretractable ring and  $M$  be a faithful cyclic R-module, then  $M$  is strongly C-coretractable module.

**Proof:** Let  $R$  be a strongly C-coretractable ring,  $M = mR$  for some  $m \in R$ . Then  $R \cong mR$  since  $M$  is faithful cyclic module. But  $R$  is strongly C-coretractable. So  $M$  is strongly C-coretractable by Proposition (2.3).

### 3. Mono-C-Coretractable Modules

As a generalization of C-coretractable, we present a class of modules in the class of C-coretractable modules. We study and investigate some properties of this concept.

**Definition (3.1):** An R-module  $M$  is called mono-C-coretractable if for each proper closed submodule of  $M$ , there exists  $f \in \text{End}_R(M)$ ,  $f \neq 0$  and  $N = \ker f$ .

#### Examples and Remarks (3.2):

(1) A module  $M$  is mono-C-coretractable if and only if for each proper closed submodule  $N$  of  $M$ , there exists a monomorphism  $f$  from  $M/N$  into  $M$ .

**Proof:** ( $\Rightarrow$ ) Let  $N$  be a proper closed submodule of  $M$ . Since  $M$  is mono-C-coretractable module, so there exists  $f \in \text{End}_R(M)$ ,  $f \neq 0$  and  $N = \ker f$ . Define  $g: M/N \rightarrow M$  by  $g(m+N) = f(m)$  for all  $m \in M$ . It is clear that  $g$  is well-defined homomorphism and  $g \neq 0$  since  $f \neq 0$ .  $\text{Ker } g = \{m+N \in M/N : g(m+N) = 0\} = \{m+N \in M/N : f(m) = 0\} \neq \{m+N \in M/N : m \in \ker f\} = \{m+N \in M/N : m \in N\} = N$ . Thus  $g$  is monomorphism.

( $\Leftarrow$ ) Let  $K$  be a proper closed submodule of  $M$ . By hypothesis there exists  $f: M/N \rightarrow M$ ,  $f$  is monomorphism. Take  $f \circ \pi \in \text{End}(M)$  and  $f \circ \pi(M) = f(\pi(M)) = f(M/N) \neq 0$ . Now,  $\ker(f \circ \pi) = \{m \in M : f \circ \pi(m) = 0\} = \{m \in M : f(m+N) = 0\} = \{m \in M : m \in N\} = N$ . Then  $\ker(f \circ \pi) = N$ .

(2) Every mono-coretractable module is mono-C-coretractable. But the converse is not true in general. For example, the  $Z$ -module  $Z$  is mono-C-coretractable since the only proper closed is  $(0)$  and the only nonzero endomorphism is the identity  $f: Z \rightarrow Z$ ,  $\ker f = 0$

and hence  $N = \ker f$ , but it is not mono-coretractable since it is not coretractable.

(3) Every simple closed module is mono-C-coretractable. But the converse is not true in general. For example, the  $Z$ -module  $Z_6$  is mono-C-coretractable by part (2), but it is not simple closed.

(4) Every mono-C-coretractable module is a C-coretractable, but the converse is not true in general.

(5) Every semisimple module is mono-coretractable and hence it is mono-C-coretractable by part (3). But the converse is not true in general. For example, the  $Z$ -module  $Z_4$  is mono-C-coretractable, but it is not semisimple.

(6) Let  $M$  be an  $R$ -module. If  $M$  is a quasi-Dedekind mono-C-coretractable, then  $M$  is a simple closed.

**Proof:** Since every mono-C-coretractable is C-coretractable, then the result is obtained by Proposition (1.6).

**Proposition (3.3):** Let  $M$  be a Rickart  $R$ -module. Then  $M$  is a mono-C-coretractable if and only if an extending.

**Proof:** ( $\Rightarrow$ ) Let  $N$  be a proper closed submodule of  $M$ . Since  $M$  is mono-C-coretractable, then there exists  $f \in \text{End}_R(M)$ ,  $f \neq 0$  and  $K = \ker f$ , but  $M$  is a Rickart, hence  $\ker f$  is a direct summand of  $M$  for each  $f \in \text{End}_R(M)$  and hence  $M$  is extending module.

( $\Leftarrow$ ) It is clear.

**Proposition (3.4):** Let  $M$  be a strongly Rickart  $R$ -module. Then  $M$  is a mono-C-coretractable if and only if an extending fully stable.

**Proof:** ( $\Rightarrow$ ) Let  $N$  be a proper closed submodule of  $M$ . Since  $M$  is mono-C-coretractable, then there exists  $f \in \text{End}_R(M)$ ,  $f \neq 0$  and  $N = \ker f$ , but  $M$  is a strongly Rickart, hence  $\ker f$  is a stable direct summand of  $M$  for each  $f \in \text{End}_R(M)$  and hence  $N$  is a stable direct summand of  $M$ . Thus  $M$  is extending fully stable module.

( $\Leftarrow$ ) It is clear.

**Proposition (3.5):** Let  $M$  be a mono-C-coretractable fully stable  $R$ -module. Then

every nonzero closed submodule of  $M$  is also mono-C-coretractable module.

**Proof:** Suppose that  $N$  is a nonzero closed submodule of  $M$ . Let  $K$  a proper closed submodule of  $N$ , so  $K$  is closed submodule of  $M$ . But  $M$  is a mono-C-coretractable module, then there exists  $f \in \text{End}_R(M)$ ,  $f \neq 0$ ,  $f(K) = 0$  and  $K = \ker f$ , so if  $f(N) = 0$ , then  $N \subseteq \ker f = K$  so  $N = K$  which is a contradiction. Thus  $f(N) \neq 0$ . Let  $g$  be the restriction map of  $f$  on  $N$ . Since  $M$  is fully stable, so  $g(N) \subseteq N$ . Hence  $g \in \text{End}_R(N)$  and  $g \neq 0$  since  $g(N) = f(N) \neq 0$ . Moreover  $g(K) = f(K) = 0$ , so  $K \subseteq \ker g \subseteq \ker f = K$ . Thus  $K = \ker g$ .

**Proposition(3.6):** Let  $M$  be a quasi-Dedekind  $R$ -module, then  $M$  is a strongly C-coretractable if and only if  $M$  is a mono-C-coretractable.

**Proof:** ( $\Rightarrow$ ) Since  $M$  is a strongly C-coretractable, then  $M$  is C-coretractable and hence it is simple closed, by Proposition(1.6). Thus  $M$  is mono-C-coretractable.

( $\Leftarrow$ ) Let  $M$  be a mono-C-coretractable. As  $M$  is a quasi-Dedekind, so  $M$  is a simple closed and hence  $M$  is strongly C-coretractable.

**Proposition (3.7):** Let  $M$  be a quasi-Dedekind  $R$ -module, then the following statements are equivalent:

- (1)  $M$  is strongly C-coretractable;
- (2)  $M$  is C-coretractable;
- (3)  $M$  is simple closed;
- (4)  $M$  is mono-C-coretractable.

**Proof:** (1) $\Rightarrow$ (2) It follows by Examples and Remarks (3.2(2)), (2) $\Rightarrow$ (3) It follows by Proposition (1.6) since  $M$  is a quasi-Dedekind module, (3) $\Rightarrow$ (4) It follows by Examples and Remarks (3.2 (3)) and (4)  $\Leftrightarrow$  (1) It follows by Proposition (3.6).

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