

## Approximate Solutions of the Generalized Two-Dimensional Fractional Partial Integro-Differential Equations

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### Abstract

In this article, the existence and uniqueness theorem of the solution of the generalized two dimensional 'fractional' partial integro-differential equations (2DFPIDEs) have been proved. Then, the sequence of approximate solutions of the generalized 2DFPIDEs employing the variational iteration method (VIM) had been derived and proved and then we prove its convergence to the exact solution. Finally, illustrative examples are simulated using computer software Mathcad 15 and then a comparison between the approximate results and 'the exact solution are given', which discover its competence. [DOI: [10.22401/ANJS.22.2.08](https://doi.org/10.22401/ANJS.22.2.08)]

Keywords: Fractional order (FO), Fractional partial integro-differential equation (FPIDE), variation iteration method (VIM).

### 1. Introduction:

Fractional order partial differential equations FOPDE, is a popularization of 'classical integer' 'order partial differential equations' are 'increasingly' used to pattern problems in 'fluid flow', 'finance; and other areas' of 'applications. Fractional; derivatives provide 'an excellent' writing for the characterization of memory' and inherited properties' of various problems and operations.

Podlubny I. in 1999 [20] pointed that 'Half-order' 'derivatives and integrals' proved to be more salutary for the formulation' of certain' electrochemical' problems than the classical' models. Fractional integration and differentiation operators are also used for extensions of the diffusion and wave equations, [20].

Also, a great deal of effort has been expended over the last 10 years or so in attempting to find robust and stable numerical and analytical methods for solving FPDE of physical interest, such numerical' and 'analytical' methods including finite difference method [9,21,8], Adomian decomposition method (ADM) [11–12,1,17], VIM [3,13,18,14], and homotopy perturbation method (HPM) [3,15,19]. The VIM and the ADM 'have been' 'extensively' used' to solve' FPDE, since they provide immediate and visible symbolic terms of analytic solutions, as

well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

Many 'authors' have studied the numerical solution of various types of FIDEs and FPIDEs. For examples, the application of a backward Euler method to solve FPIDEs by Serna in (1988), Marcos in (1990) used FDM for solving nonlinear singular FPIDEs, (Saadatmandi & Dehghan, in (2011) used the Legendre collocation method to solve FIDEs, Zhu and Fan in (2013) studied nonlinear fractional-order Volterra 'integro-differential' equations (IDEs) by employing second Chebyshev wavelet' operational' matrix 'of fractional 'integration' the main characteristic of this approach is that it reduced the IDEs into a nonlinear 'system' of 'algebraic' equations'. Mahdy and Shwayyea in (2016) solved FPIDE by using two numerical methods such as least squares method and shifted Laguerre polynomials pseudo-spectral method, Wang and Zhu in (2016) used second Chebyshev wavelets to solve FPIDE a weakly singular kernel [7].

In this paper, we will study and solve numerically the 2DFPIDEs using the VIM, this study includes the statement and the proof of the iteration formula used to solve 2DFPIDEs and then its convergence to the exact solution.

**2. Basic Concepts**

In this department, we will give some basic definitions and properties of FO derivatives' and integrals related to the present work.

**Definition (1), [10]:**

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a function,  $\alpha$  a positive real number,  $n$  the integer satisfying  $n - 1 < \alpha \leq n$ , and  $\Gamma$  is the Euler gamma function. Then, the left' and the right Riemann-Liouville (R-L) fractional integrals' of order  $\alpha$  are defined by:

$${}_a I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \tau)^{\alpha-1} f(\tau) d\tau$$

And

$${}_b I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} f(\tau) d\tau$$

**Definition (2), [10]:**

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a function,  $\alpha$  a positive real number,  $n$  the integer satisfying  $n - 1 < \alpha \leq n$ , the Caputo' fractional derivative' of order  $\alpha$  is defined as follows:

$${}_0^C D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau, & m - 1 < \alpha \leq m \\ \frac{d^m}{dx^m} f(x), & m = \alpha \end{cases}$$

**Definition (3), [2]:**

A normed linear space is a vector space  $V$ , over  $\mathbb{R}$  and function  $\| \cdot \|: V \rightarrow \mathbb{R}$  which satisfies:

- 1-  $\|v\| > 0 \forall v \in V$ .
- 2-  $\|v\| = 0 \iff v = 0$ .
- 3-  $\|\alpha v\| = |\alpha| \|v\| \forall v \in V, \alpha \in \mathbb{R}$ .
- 4-  $\|v + w\| \leq \|v\| + \|w\| \forall v, w \in V$ .

**Definition (4), [2]:**

Let  $f: A \rightarrow \mathbb{R}^n$  be a continuously differentiable function over an interval  $[a, b]$ , where  $A \subset \mathbb{R}^n$  is said to satisfy Liptschitz condition if there exists a constant  $L > 0$  (dependent on both function and the interval), such that:  $\|f(a) - f(b)\| \leq L \|a - b\|$ , for every pair of points  $a, b \in A$ .

**2.1 Properties of Fractional Derivatives and Integration:**

For  $\alpha > 0$ , the following property is satisfied [9].

1. If  $m - 1 < \alpha \leq m, m \in \mathbb{N}$  and  $u$  is any function, then:

$${}_a I_x^\alpha {}^C D_x^\alpha u(x) = u(x) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{x^k}{k!}, x > 0.$$

$$\text{and } {}_0^C D_x^\alpha {}_a I_x^\alpha u(x) = u(x).$$

$$2. {}_a I_x^\alpha {}_0 I_x^\beta u(x) = {}_0 I_x^\beta {}_a I_x^\alpha u(x) = {}_a I_x^{\alpha+\beta} u(x).$$

**3. Problem Statement**

The statement of the considered problem of this paper is to solve the following 2DFPIDEs of fractional order  $0 < \alpha \leq 1$  using the VIM,

$${}_0^C D_t^\alpha u(x, t) = g(x, t) + {}_a I_x^\beta {}_0 I_t^\gamma k(x, t, u(x, t)) \dots\dots\dots (1)$$

where  $0 < \alpha \leq 1, (x, t) \in \Omega$  with initial conditions:

$$u(x, 0) = u_0(x), x \in [a, b].$$

where  $k$  is the kernel function,  $g$  is given function and  $u$  is the unknown real valued function for  $a \leq x \leq b, 0 \leq t \leq T, \beta, \gamma \in \mathbb{R}^+$ . Also  ${}_0^C D_t^\alpha$  denotes the Caputo fractional derivative of order  $\alpha$  and  ${}_a I_x^\beta, {}_0 I_t^\gamma$  denotes the R-L fractional order integral operators of order  $\beta$  and  $\gamma$ , respectively and the domain of definition is given by:

$$\Omega = \{(x, t), a \leq x \leq b, 0 \leq t \leq T\}$$

**4. Existence' and Uniqueness' of Solution:**

One of the most important tasks in this article is to find the approximate solution of 2DFPIDEs, and hence we need to investigate first the existence and uniqueness theorem for the analytical solutions of the 2DFPIDEs. Due to this, several authors have paid more attention in the past to provide these theorems, for example, the fixed-point principles based on Banach fixed point theorem or the contraction mapping principle were used by Nemytskii in (1933) to establish various existence and uniqueness theorems for integral equations and in this work for 2DFPIDEs. In this department, we shall state and prove the existence' and uniqueness' theorem of the solution of the generalized 2DFPIDEs of an arbitrary fractional order derivative.

**Theorem 1:**

Consider the generalized 2DFPIDEs given by equation (1) over the region  $\Omega = \{(x, t): a \leq x \leq b, 0 \leq t \leq T\}$  and suppose

that the kernel  $k$  satisfies Lipschitz condition with respect to  $u$  and constant  $L$ , such that  $L < \frac{\Gamma(w+1)\Gamma(\beta+1)}{T^w(b-a)^\beta}$ . Then equation (1) has a unique solution.

**Proof:**

Apply  ${}_0I_t^\alpha, 0 < \alpha \leq 1$ , on the both sides of equation (1), and then using property (1) in section 2.1

$${}_0I_t^\alpha {}^C D_t^\alpha u(x, t) = {}_0I_t^\alpha g(x, t) + {}_0I_t^\alpha {}_a I_x^\beta {}_0I_t^\gamma k(x, t, u(x, t))$$

and hence:

$$u(x, t) - u_0(x) = {}_0I_t^\alpha g(x, t) + {}_0I_t^w {}_a I_x^\beta k(x, t, u(x, t)), w = \alpha + \gamma \dots\dots\dots (2)$$

Therefore:

$$u(x, t) = u_0(x) + {}_0I_t^\alpha g(x, t) + {}_0I_t^w {}_a I_x^\beta k(x, t, u(x, t)) \dots\dots\dots (3)$$

Now, let  $u_1(x, t), u_2(x, t) \in C_t^m([a, b] \times [0, T])$  be any two functions, which satisfies equation (1), and hence satisfy equation (3);

$$u_1(x, t) = u_0(x) + {}_0I_t^\alpha g(x, t) + \frac{1}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{w-1} (x-y)^{\beta-1} k(y, s, u_1(y, s)) dy ds = Tu_1 \dots\dots\dots (4)$$

and

$$u_2(x, t) = u_0 + {}_0I_t^\alpha g(x, t) + \frac{1}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{w-1} (x-y)^{\beta-1} k(y, s, u_2(y, s)) dy ds = Tu_2 \dots\dots\dots (5)$$

Now, subtracting equation (5) from equation (4) and carrying the supremum norm, get

$$\|Tu_1 - Tu_2\| = \left\| \frac{1}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{w-1} (x-y)^{\beta-1} - [k(y, s, u_1(y, s)) - k(y, s, u_2(y, s))] dy ds \right\| \dots\dots\dots (6)$$

$$\leq \frac{1}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{w-1} (x-y)^{\beta-1} \|k(y, s, u_1(y, s)) - k(y, s, u_2(y, s))\| dy ds$$

because  $k$  satisfies Lipschitz condition, then:

$$\|Tu_1 - Tu_2\| \leq \frac{L}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{w-1}$$

$$(x-y)^{\beta-1} \|u_1(y, s) - u_2(y, s)\| dy ds \dots\dots (7)$$

$$= \frac{L}{\Gamma(w)\Gamma(\beta)} \|u_1(y, s) - u_2(y, s)\| \frac{(x-a)^{\beta} t^w}{\beta w} \dots\dots (8)$$

By using the definition of the gamma function, equation (8) will be:

$$\|Tu_1 - Tu_2\| \leq \frac{L(x-a)^\beta t^w}{\Gamma(w+1)\Gamma(\beta+1)} \|u_1(y, s) - u_2(y, s)\| \dots\dots\dots (9)$$

Therefore, upon taking the supremum value of  $x$  and  $t$  over  $\Omega$ , we have:

$$\|Tu_1 - Tu_2\| \leq \frac{L(b-a)^\beta T^w}{\Gamma(w+1)\Gamma(\beta+1)} \|u_1(y, s) - u_2(y, s)\| \quad a \leq x \leq b, 0 \leq t \leq T$$

Since  $L \leq \frac{\Gamma(w+1)\Gamma(\beta+1)}{T^w(b-a)^\beta}$ , then  $\frac{L(b-a)^\beta T^w}{\Gamma(w+1)\Gamma(\beta+1)} < 1$

which implies that the operator  $T$  is a contractive mapping and therefore by using Banach fixed point theorem it has a unique fixed point.

Hence, equation (1) has a unique solution. ■

To explain the applicability of the above theorem, consider the following example:

**Example 1:**

Consider the linear generalized 2DFPIDEs

$${}^C D_t^{0.5} u(x, t) = g(x, t) + {}_a I_x^{0.5} {}_0 I_t^{0.75} [(xt)u(x, t)]$$

,  $\alpha = 0.5, \beta = 0.5, \gamma = 0.75 \dots\dots\dots (10)$

where  $(x, t) \in [0, 1] \times [0, 1]$

Now the kernel  $k$  satisfies Lipschitz constant condition since it is differentiable with respect to  $u$  with constant

$$L = \sup_{(x,t) \in [0,1] \times [0,1]} \left| \frac{\partial k}{\partial u} \right| = \sup_{(x,t) \in [0,1] \times [0,1]} |xt| = |1 * 1| = 1$$

Now, from theorem (1),  $w = \alpha + \gamma = 0.5 + 0.75 = 1.25, T = 1, a = 0$  and  $b = 1$  and also:

$$\frac{\Gamma(w+1)\Gamma(\beta+1)}{\Gamma^w(b-a)^\beta} = \frac{\Gamma(2.25)\Gamma(1.25)}{1^{1.25} 1^{0.5}} = 1.004$$

It is clear that  $1 < 1.004$ , i.e.,  $L < \frac{\Gamma(w+1)\Gamma(\beta+1)}{T^w(b-1)^\beta}$ , which implies that by theorem

(1) the generalized 2DFPIDEs (10) has a unique solution. ■

**5. The VIM for Solving 2DFPIDEs:**

To illustrate the main aspects of the VIM, we consider the following general nonlinear ordinary differential equation in operator form

$$L(u(x)) + N(u(x)) = g(x), x \in [a, b]$$

“where  $L$  is a linear operator and  $N$  is a nonlinear operator respectively,  $g$  is a nonhomogeneous term. According to VIM, we can construct the correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_{x_0}^x \lambda(x, s) \{L(u_n(s)) + N(\tilde{u}_n(s)) - g(s)\} ds$$

where  $\lambda$  is a Lagrange multiplier [4,5,6,16] which can be identified optimally via variational theory,  $u_n$  is the  $n^{th}$  approximate solution, and  $\tilde{u}_n$  is considered as a restricted variation, i.e.,  $\delta\tilde{u}_n = 0$ . After identification of Lagrange multiplier, the successive approximations  $u_{n+1}(x), n \geq 0$ , of the solution  $u$  can be readily obtained. Consequently, the exact solution will be of the form:

$$u(x) = \lim_{n \rightarrow \infty} u_n(x).$$

Now, the variational iteration formulation to evaluate the approximate solution of 2DFPIDEs given by equation (1) will be derived which is due to the derivation first the general Lagrange multiplier.

**Theorem 2:**

Consider the 2DFPIDEs given by equation (1) and let  $u_n \in C_t^m([a, b] \times [0, T])$  be the approximate solution then the sequence of approximate solutions using the VIM are approximated by:

$$u_{n+1}(x, t) = u_n(x, t) - {}_0I_t^\alpha [{}_0^C D_t^\alpha u_n(x, s) - g(x, s) - {}_aI_x^\beta {}_0I_t^\gamma k(y, s, u(y, s))] \dots\dots\dots (11)$$

for  $n = 0, 1, \dots$

**Proof:**

Consider equation (1), which has the form

$${}_0^C D_t^\alpha u(x, t) = g(x, t) + {}_aI_x^\beta {}_0I_t^\gamma k(x, t, u(x, t))$$

using the VIM, the correction functional will be as follow:

$$u_{n+1}(x, t) = u_n(x, t) + {}_0I_t^\alpha [\lambda(x, s) \{ {}_0^C D_t^\alpha u_n(x, s) - g(x, s) - {}_aI_x^\beta {}_0I_t^\gamma k(y, s, \tilde{u}_n(y, s)) \}] \dots\dots\dots (12)$$

We will approximate the fractional derivative  ${}_0^C D_t^\alpha$  by the first derivative and hence

$$u_{n+1}(x, t) = u_n(x, t) + {}_0I_t^\alpha [\lambda(x, s) \{ D_t u_n(x, s) - g(x, s) - {}_aI_x^\beta {}_0I_t^\gamma k(y, s, \tilde{u}_n(y, s)) \}] \dots\dots\dots (13)$$

and by taking the first variation on both sides of equation (13) with respect to  $u_n$  with the assumptions that  $\delta u_n(x, 0) = 0$  and  $\delta g(x, s) = 0$ .

The following equation is obtained

$$\delta u_{n+1}(x, t) = \delta u_n(x, s) + \int_0^t D_t \lambda(x, s) \delta u_n(x, s) ds \dots\dots\dots (14)$$

Therefore, upon using integration by parts with respect to  $s$ , we get:

$$\delta u_{n+1}(x, s) = \delta u_n(x, s) + \lambda(x, s) \delta u_n(x, s) |_{s=t} - \int_0^t \delta u_n(x, s) \lambda'(x, s) ds$$

Therefore:

$$\delta u_{n+1}(x, t) = (1 + \lambda) \delta u_n(x, s) |_{s=t} - \int_0^t \delta u_n(x, s) \lambda'(x, s) ds \dots\dots\dots (15)$$

and as a result, using variational theory the following necessary condition is obtained for an arbitrary  $\delta u_n$

$$\lambda'(x, s) = 0 \dots\dots\dots (16)$$

with initial condition

$$1 + \lambda(x, s) |_{s=t} = 0 \dots\dots\dots (17)$$

Solving equation (16) with initial condition (17), the following Lagrange multiplier is obtained

$$\lambda(x, s) = -1$$

Therefore, by substituting the value of  $\lambda$  back into the correction functional (12), the following variational iteration formula is obtained:

$$u_{n+1}(x, t) = u_n(x, t) - {}_0I_t^\alpha [{}_0^c D_t^\alpha u_n(x, s) - g(x, s) - {}_a I_x^\beta {}_0 I_t^\gamma k(y, s, u_n(y, s))] \quad \blacksquare$$

**6. Convergence of the VIM for Solving 2DFPIDE:**

The VIM derived in theorem (2) may be used to solve a large number of problems involving approximation, which will converge rapidly to the exact solution, easily and accurately. For linear problems, the exact solution may be obtained by only one iteration, this is due to the fact that the Lagrange multiplier can be exactly identified [7].

In this section, the following convergence theorem for the approximate solution obtained using the constructed VIM (11) will be stated and proved.

**Theorem 3:**

Let  $u, u_n \in C_t^m([a, b] \times [0, T])$  be the exact and approximate solutions of equation (1) and (11), respectively. If  $E_n(x, t) = u_n(x, t) - u(x, t)$  and the kernel  $k$  satisfies Lipschitz condition with constant  $L < \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{T^{\alpha(\beta-\alpha)\beta}}$ , then the sequence of approximate solutions  $\{u_n\}, n = 0, 1, 2, \dots$  converge to the exact solution  $u(x, t)$ .

**Proof:**

The approximate solutions of equation (1) obtained using the VIM is given by:

$$u_{n+1}(x, t) = u_n(x, t) - {}_0I_t^\alpha [{}_0^c D_t^\alpha u_n(x, s) - g(x, s) - {}_a I_x^\beta {}_0 I_t^\gamma k(y, s, u_n(y, s))]$$

Also,  $u$  is the exact solution of equation (1) and hence, it satisfies equation (12), i.e.

$$u(x, t) = u(x, t) - {}_0I_t^\alpha [{}_0^c D_t^\alpha u(x, s) - g(x, s) - {}_a I_x^\beta {}_0 I_t^\gamma k(y, s, u(y, s))]$$

Subtracting  $u$  from  $u_{n+1}$  then:

$$u_{n+1}(x, t) - u(x, t) = u_n(x, t) - u(x, t) - {}_0I_t^\alpha [{}_0^c D_t^\alpha u_n(x, s) - {}_0^c D_t^\alpha u(x, s) - g(x, s) + g(x, s) - {}_a I_x^\beta {}_0 I_t^\gamma (k(y, s, u_n(y, s)) - k(y, s, u(y, s)))] \dots\dots\dots (18)$$

and hence:

$$\begin{aligned} E_{n+1}(x, t) &= E_n(x, t) - {}_0I_t^\alpha [{}_0^c D_t^\alpha E_n(x, s) - {}_a I_x^\beta {}_0 I_t^\gamma (k(y, s, u_n(y, s)) - k(y, s, u(y, s)))] \\ &= E_n(x, t) - E_n(x, t) + E_n(x, 0) \\ &+ {}_0I_t^\alpha [{}_a I_x^\beta {}_0 I_t^\gamma (k(y, s, u_n(y, s)) - k(y, s, u(y, s)))] \end{aligned}$$

and since  $E_n(x, 0) = 0$ , therefore:

$$E_{n+1}(x, t) = {}_a I_x^\beta {}_0 I_t^w [k(y, s, u_n(y, s)) - k(y, s, u(y, s))], w = \alpha + \gamma \dots\dots\dots (19)$$

Since  $k$  satisfies Lipschitz condition and by applying the supremum norm on equation (19) will give

$$\begin{aligned} \|E_{n+1}(x, t)\| &\leq {}_a I_x^\beta {}_0 I_t^w \|k(y, s, u_n) - k(y, s, u)\| \\ &\leq L {}_a I_x^\beta {}_0 I_t^w \|u_n(y, s) - u(y, s)\| \\ &\leq L {}_a I_x^\beta {}_0 I_t^w \|E_n(y, s)\| \dots\dots\dots (20) \end{aligned}$$

Using the definition of R-L fractional integral into equation (20),

$$\|E_{n+1}(x, t)\| \leq \frac{L}{\Gamma(w)\Gamma(\beta)} \int_0^t (t-s)^{w-1} \int_a^x (x-y)^{\beta-1} \|E_n(y, s)\| dy ds \dots\dots\dots (21)$$

$$\leq \frac{L}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (x-y)^{\beta-1} (t-s)^{w-1} \|E_n(y, s)\| dy ds \dots\dots\dots (22)$$

Hence, if  $n = 0$ , then:

$$\begin{aligned} \|E_1(x, t)\| &\leq \frac{L}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (x-y)^{\beta-1} (t-s)^{w-1} \|E_0(y, s)\| dy ds \\ &= \frac{L}{\Gamma(w)\Gamma(\beta)} \frac{(x-a)^\beta t^w}{\beta w} \|E_0(y, s)\| \\ &\leq \frac{L(x-a)^\beta t^w}{\Gamma(w+1)\Gamma(\beta+1)} \|E_0(y, s)\| \dots\dots\dots (23) \end{aligned}$$

If  $n = 1$ , then:

$$\begin{aligned} \|E_2(x, t)\| &\leq \frac{L}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (x-y)^{\beta-1} (t-s)^{w-1} \\ &\|E_1(y, s)\| dy ds \\ &\leq \frac{L}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (x-y)^{\beta-1} (t-s)^{w-1} \\ &\frac{L(x-a)^\beta t^w}{\Gamma(w+1)\Gamma(\beta+1)} \|E_0(y, s)\| dy ds \\ &= \frac{L^2}{\Gamma(w)\Gamma(\beta)\Gamma(w+1)\Gamma(\beta+1)} \int_0^t \int_a^x (x-y)^{2\beta-1} \\ &(t-s)^{2w-1} \|E_0(y, s)\| dy ds \\ &= \frac{L^2}{\Gamma(w)\Gamma(\beta)\Gamma(w+1)\Gamma(\beta+1)} \frac{(x-a)^{2\beta} t^{2w}}{2\beta \ 2w} \\ &\|E_0(y, s)\| \\ &\leq \left(\frac{L}{2\Gamma(w+1)\Gamma(\beta+1)}\right)^2 (x-a)^{2\beta} t^{2w} \\ &\|E_0(y, s)\| \dots\dots\dots (24) \end{aligned}$$

If  $n = 2$ , then:

$$\begin{aligned} \|E_3(x, t)\| &\leq \frac{L}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (x-y)^{\beta-1} (t-s)^{w-1} \\ &\|E_2(y, s)\| dy ds \\ &\leq \frac{L}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (x-y)^{\beta-1} (t-s)^{w-1} \\ &\left(\frac{L}{2\Gamma(w+1)\Gamma(\beta+1)}\right)^2 \\ &(x-a)^{2\beta} t^{2w} \|E_0(y, s)\| dy ds \\ &= \frac{L^3}{2^2\Gamma(w)\Gamma(\beta)\Gamma^2(w+1)\Gamma^2(\beta+1)} \\ &\int_0^t \int_a^x (x-y)^{3\beta-1} (t-s)^{3w-1} \|E_0(y, s)\| dy ds \\ &\leq \frac{L^3}{2^2 3^2 \Gamma^3(w+1)\Gamma^3(\beta+1)} (x-a)^{3\beta} t^{3w} \|E_0(y, s)\| \end{aligned}$$

Therefore,

$$\begin{aligned} \|E_{n+1}(x, t)\| &\leq \frac{L^n(x-a)^n \beta t^{nw}}{(2 \times 3 \times \dots \times n)^2 \Gamma^n(w+1)\Gamma^n(\beta+1)} \\ &\|E_0(y, s)\| \\ &= \frac{1}{(n!)^2} \left(\frac{L(b-a)^\beta T^w}{\Gamma(w+1)\Gamma(\beta+1)}\right)^n \|E_0(y, s)\| \end{aligned}$$

now, since  $L < \frac{\Gamma(w+1)\Gamma(\beta+1)}{T^w(b-a)^\beta}$  and hence as  $n \rightarrow \infty$ , then  $\|E_n(x, t)\| \rightarrow 0$ , i.e.,  $u_n(x, t) \rightarrow u(x, t)$  as  $n \rightarrow \infty$ . ■

**7. Illustrative Examples:**

In this section, two examples will be considered and simulated using the VIM (11) the first example for liner case, while the second one for nonlinear.

**Example 2:**

Consider the following linear generalized 2DFPIDE

$${}^c_0D_t^{0.5}u(x, t) = -0.27212t^{2.75}x^{2.5} + 1.12837\sqrt{tx} + {}_aI_x^{0.5}{}_0I_t^{0.75}[(xt)u(x, t)] \dots\dots\dots (25)$$

the initial condition:

$$u(x, 0) = 0 \dots\dots\dots (26)$$

$(x, t) \in [0,1] \times [0,1]$  For comparison purpose, the exact solution is given by  $u(x, t) = x$ . By applying the VIM, and by using the initial approximation as follows:

$$u_0(x, t) = -0.27212t^{2.75} x^{2.5} + 1.12837\sqrt{tx}$$

then, the first second and third approximate solution denoted by  $u_1(x, t), u_2(x, t)$  and  $u_3(x, t)$ , are obtained. The approximate solution are computed for  $x = 0.2, 0.4, 0.6, 0.8$  and  $1, t \in [0,1], \Delta t = 0.2$ . comparison is then made with the exact solution where the values are listed in Figure 1. From the result of Figure 1, the convergence and the accuracy of the result between the exact solution and approximate solution can be seen. It can be observed that the third approximation is in good agreement with exact solution

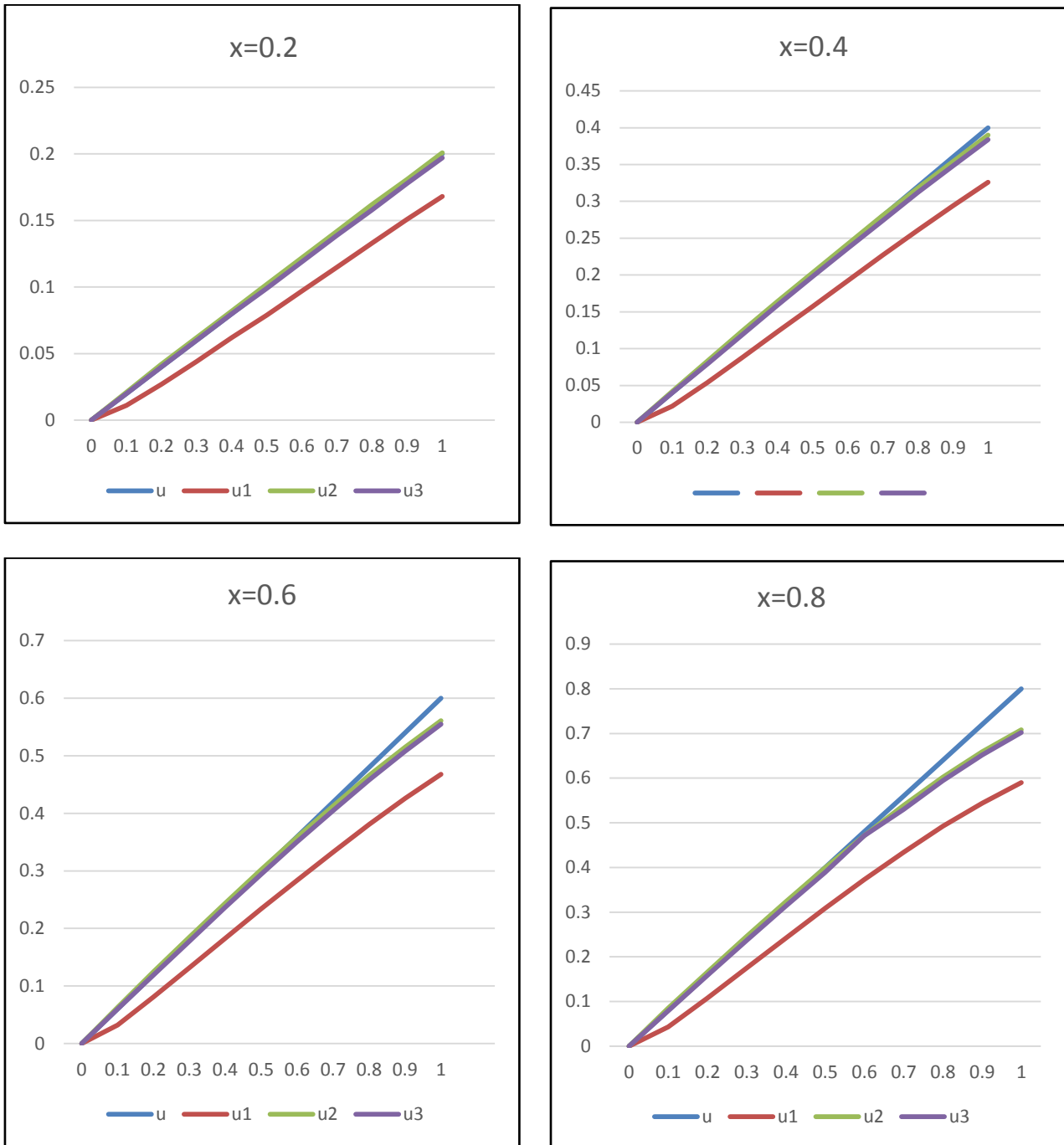


Fig.(1): The exact and the approximate solutions for example 2.

**Example 3:**

Consider the following nonlinear generalized 2DFPIDE

$${}^C_0D_t^{0.5}u(x, t) = -0.16586 t^{3.75}x^{4.5} + 2.25675 t^{1.5}x^2 + {}_aI_x^{0.5} {}_0I_t^{0.75}[(xt) u^2(x, t)] \dots\dots\dots (27)$$

with the initial condition:

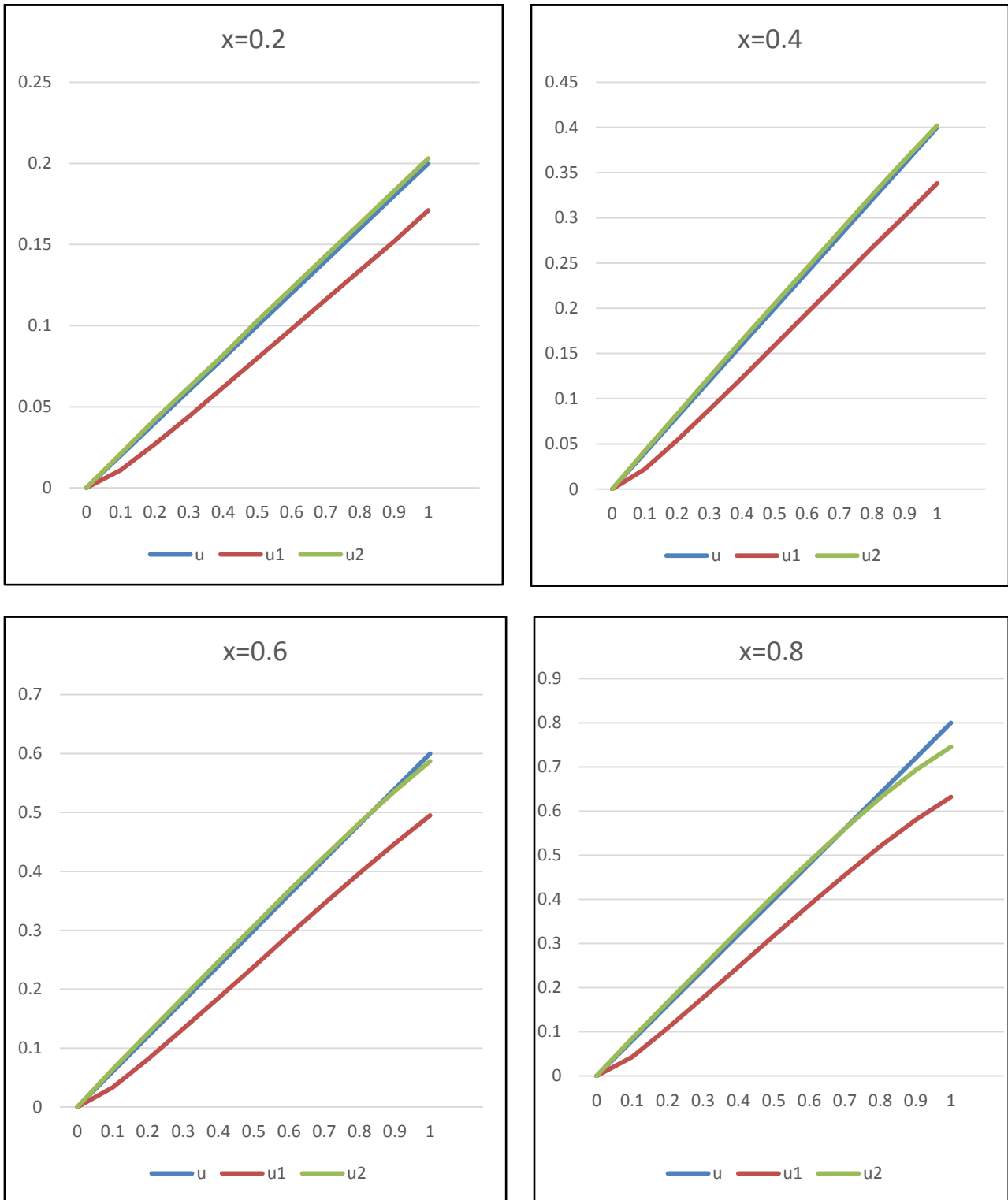
$$u(x, 0) = 0 \dots\dots\dots (28)$$

Where  $(x, t) \in [0,1] \times [0,1]$ . For comparison purpose, the exact solution is

given by  $u(x, t) = xt^2$ . By applying the VIM, and by employing with the initial approximation a follow:

$$u_0(x, t) = -0.16586 t^{3.75}x^{4.5} + 2.25675 t^{1.5}x^2$$

then, for simplicity the first and second approximate solution denoted by  $u_1(x, t)$  and  $u_2(x, t)$  are obtained. The approximate solutions are computed for  $x = 0.2, 0.4, 0.6, 0.8$  and  $1, t \in [0,1]$ ,  $\Delta t = 0.2$ . They are compared is then made with the exact solution and the values are given in Fig.(2).



**Fig.(2): The exact and the approximate solutions for example 3.**

**Conclusions:**

In this paper the existence and uniqueness theorem of the solution of 2DFPIDEs is proposed, in which the proof have been utilized Banach fixed point theorem. The VIM for solving 2DFPIDEs is formulated and the correction functional involved is determined. From there, convergence theorem of the

sequence of approximate solution to the exact solution is provided and proved depending on the error function. The obtained results of the considered of the illustrative examples shows the reliability an applicability of the VIM for solving complicated differential equation.



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