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Generalized Jordan Triple (σ, τ)-Higher Homomorphisms on Prime Rings

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Abstract

Herstein proved that any Jordan homomorphism onto a prime ring of characteristic of R different from 2 and 3 is either a homomorphism or an anti-homomorphism. In this paper the concept of Generalized Jordan triple (σ,τ)–Higher Homomorphisms (GJT(σ,τ)-HH) where σ and τ are two commuting homomorphisms are introduced as follows:

A family of additive mappings $F=(f_i)_{i\in\mathbb{N}}$ of R into R' is said to be a Generalized Triple (σ,τ)-Higher Homomorphism (GT(σ,τ)-HH) if there exist a triple (σ,τ)-higher homomorphism (T (σ,τ) – HH) $\theta=(\phi_i)_{i\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$ and for all $a,b\in R$, we have:

$$f_n(aba)=\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

and θ is said to be the relating triple (σ,τ)-HH.

We will primarily extend the result of Herstein on it. It should be proved that every GJT(σ,τ)-HH of ring R into prime ring R' is either GT(σ,τ)-HH or triple (σ,τ) higher anti-homomorphism (T(σ,τ)-HAH).

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1. Introduction

Jordan homomorphisms of associative rings and algebras play a significant role in various mathematical areas, in particular of ring theory. Throughout this paper R will denote an associative ring with center $Z(R)$. A ring R is said to be a ring with an involution if there exists a mapping $*: R\to R$ such that for every $a,b\in R$, $a^{**}=a$, $(a+b)^*=b^*+a^*$ and $(ab)^*=b^*a^*$. R is called prime if $aRb=(0)$ implies $a=0$ or $b=0$ with $a,b\in R$, and it is called semiprime if $aRa=(0)$ with $a\in R$ implies $a=0$. A ring R is said to be 2-torsion free, if $2a=0$, with $a\in R$, implies $a=0$.

An additive mapping θ of a ring R into a 2-torsion free ring R` is said to be a homomorphism (respectively anti-homomorphism) if $\theta(ab)=\theta(a)\theta(b)$ (respectively $\theta(ab)=\theta(b)\theta(a)$), therefore θ is said to be a Jordan homomorphism if $\theta(ab+ba)=\theta(a)\,\theta(b)+\theta(b)(a)$ and is called a JTH if $\theta(aba)=\theta(a)\theta(b)\theta(a)$ for all $a,b\in R$ (See [1,2,3,4]). It is clear that every homomorphism (anti-homomorphism) is a Jordan homomorphism and every Jordan homomorphism is a JTH but the converse in general is not true (see example 2, in [4]).

In the recent paper [2] Herstein had proved, Jordan homomorphism onto a prime ring of characteristic of R different from 2 and 3 is either a homomorphism or an anti-homomorphism. In [4] Jacobson & Rickart, proved that any Jordan homomorphism of an arbitrary ring into an integral domain is either a homomorphism or an anti-homomorphism. In [1] Bresar studied a JTH of a ring R onto 2 torsion free semiprime ring R`, he had proved that every JTH of a ring onto a prime ring of characteristic not 2 is either a homomorphism or an anti-homomorphism.

Generalized homomorphisms have been primarily defined by Majeed & Shaheen [5] as follows: An additive mapping F of a ring R into ring R' is said to be a generalized homomorphism (resp. GJH) if there exists a homomorphism (Jordan homomorphism) θ, such that:

$F(ab)=F(a)\theta(b)(F(ab+ba)$
$\qquad=F(a)\,\theta(b)+F(b)\theta(a))$, for all $a,b\in R$

where θ is called the relating homomorphism (resp. Jordan homomorphism), they have proved that every GJH onto the prime ring of characteristic not 2 is either a homomorphism or an anti-homomorphism.

In [6] Faraj had introduced the concept generalized Higher Homomorphism (GHH) as follows; A family of additive mappings $F=(f_i)_{i\in\mathbb{N}}$ of R into R' is called a GHH (respectively GJHH) if there exists aHH $\theta=(\phi_i)_{i\in\mathbb{N}}$, such that $f_n(ab)=\sum_{i=1}^{n} f_i(a)\,\phi_i(b)$ (respectively $f_n(ab+ba)=\sum_{i=1}^{n} f_i(a)\,\phi_i(b)+f_i(b)\,\phi_i(a)$, for all $n\in\mathbb{N}$, $a,b\in R$, where θ is said to be the relating HH (respectively JHH. If R' is 2-torsion-free, then the definition of GJHH is equivalent to the following; $f_n(a^2)=\sum_{i=1}^{n} f_i(a)\,\phi_i(a)$, he had extended the result of Herstien and proved that every GJHH on to prime ring of characteristic not 2 is either a homomorphism or an anti-homomorphism.

Following [7] Salih & Jarallah, they have introduced the concept of GJ(σ,τ)-HH of R into R' as follows: A family of additive mappings $F=(f_i)_{i\in N}$ of R into R' and σ,τ are two homomorphisms of R such that $\sigma\tau=\tau\sigma$, is said to be a GJ(σ,τ)-HH if there exist a Jordan (σ,τ)-HH $\theta=(\phi_i)_{i\in\mathbb{N}}$ from R into R', such that for each $n\in\mathbb{N}$ and for all $a,b\in R$,

$$f_n(ab+ba)=\sum_{i=1}^{n} f_i(\sigma^i(a)\phi_i(\tau^i(b))+\sum_{i=1}^{n} f_i(\sigma^i(b)\phi_i\left(\tau^i(a)\right)$$

where θ is said to be the relating Jordan (σ,τ)-HH.

In the research [8] the authors have presented the concept of GTHH (resp. GJTHH) as follows: A family of additive mappings $F=(f_i)_{i\in\mathbb{N}}$ of R into R' is said to be a GTHH (respectively GJHH) if there exist a family of additive mappings $\theta=(\phi_i)_{i\in\mathbb{N}}$ of R into R', such that $f_n(abc)=\sum_{i=1}^{n} f_i(a)\,\phi_i(b)\phi_i(c)$ (respectively $f_n(aba)=\sum_{i=1}^{n} f_i(a)\,\phi_i(b)\phi_i(a)$, for each $n\in\mathbb{N}$ and for all $a,b\in R$. They have given some results about them.

The purpose of this paper is to extend the above concepts to GT (σ,τ)-HH and GJT (σ,τ)-HAH. We will study the relation between these definitions and prove some results about it, depending on the results in [9].

2. Preliminaries

First, we will give some definitions and Lemmas.

Definition 2.1 [9]. A family of additive mappings $\theta=(\phi_i)_{i\in\mathbb{N}}$ of R into R' is said to be a triple (σ,τ)-HH if for each $n\in\mathbb{N}$ and for all $a,b\in R$, we have:

$$\phi_n(abc)=\sum_{i=1}^{n}\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right)$$

and θ is said to be a Jordan triple (σ,τ)-HH if for each $n\in\mathbb{N}$ and for all $a,b\in R$, we have:

$$\phi_n(aba)=\sum_{i=1}^{n}\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

Lemma 2.2 [9, Lemma 3.1]. If $\theta=(\phi_i)_{i\in\mathbb{N}}$ is a JT (σ,τ)-HH of R into R', then for each $n\in\mathbb{N}$ and for all $a,b,c,r\in R$,

$$A_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(a,b,c))+B_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))A_n(\tau^n(a,b,c))=0$$

Where:

$$A_n(a,b,c)=\phi_n(abc)-\sum_{i=1}^{n}\phi_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right)$$

$$B_n(a,b,c)=\phi_n(abc)-\sum_{i=1}^{n}\phi_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

Note that if $A_n(a,b,c)=0$, then ϕ is a T (σ,τ)-HH and if $B_n(a,b,c)=0$, then ϕ is a T (σ,τ)-HAH.

Lemma 2.3 [9, Proposition 3.5]. Let $\theta=(\phi_i)_{i\in\mathbb{N}}$ be a Jordan triple higher (σ,τ)-homomorphism from prime ring R into prime ring R', then θ is higher (σ,τ)-homomorphism.

Lemma 2.4 [9, Theorem 3.4]. Every JT (σ,τ)-HH of ring R into prime ring R' is either triple (σ,τ)-HH or triple (σ,τ)-HAH.

Lemma 2.5 [1]. Let R be a 2-torsion free semiprime ring. If $x,y\in R$ such that $xry+yrx=0$, then $xry=yrx=0$, for all $r\in R$.

Definition 2.6 [7]. A family of additive mappings $F=(f_i)_{i\in\mathbb{N}}$ of R into R' is said to be a GJT (σ,τ)-HH if there exist a Jordan triple (σ,τ)-HH $\theta=(\phi_i)_{i\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$ and for all $a,b\in R$, we have:

$$f_n(aba)=\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

and θ is called the relating Jordan triple (σ,τ)-HH.

Lemma 2.7. Let $F=(f_i)_{i\in\mathbb{N}}$ be a GJT (σ,τ)-HH of R into2-torsion free ring R' associated with JT (σ,τ)-HH $\theta=(\phi_i)_{i\in\mathbb{N}}$. Then for each $n\in\mathbb{N}$ and for all $a,b,c\in R$,

$$f_n(abc+cba)=\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right)+f_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

Proof. The same proof of Lemma 2.6 in [7]. ■

Now, we will introduce the definition of GT (σ,τ)-HH as follows.

Definition 2.8. A family of additive mappings $F=(f_i)_{i\in\mathbb{N}}$ of R into R' is said to be a GT (σ,τ)-HH if there exist a triple (σ,τ)-HH $\theta=(\phi_i)_{i\in\mathbb{N}}$, such that for each $n\in\mathbb{N}$ and for all $a,b\in R$, we have:

$$f_n(abc)=\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

and θ is said to be the relating triple (σ,τ)-HH.

It is clear that every Generalized Higher Homomorphism is a Generalized Jordan Higher Homomorphism, but the converse need not be true in general. Following example shows

It is clear that every GT(σ,τ)-HH is a GJT(σ,τ)-HH, but the converse is not true in general. In [8], the authors presented an example of a ring that is JHH but not HH, we will extend it to GT (σ,τ)- HH as follows:

Example 2.9. Suppose that S is a ring with non-trivial involution *, $R=S\oplus S\oplus S$, $a\in S$ such that $a\in Z(S)$ and $s_i a s_j=0$, for all $s_i,s_j\in R$, for all i and j. Let $F=(f_i)_{i\in\mathbb{N}}$ be a family of mappings of R into itself defined by for each $n\in N$ and $(s,t,s)\in R$:

$$f_n((s,t,s))=\begin{cases}\left(-(2-n)a\sigma^i(s),(n-1)\sigma^i\tau^{n-i}(t^*),-(2-n)a\sigma^i(s)\right) & for\ n=1,2,\\ 0 & n\geq 3.\end{cases}$$

In [9], there is a JT (σ,τ)- HH $\theta=(\phi_i)_{i\in\mathbb{N}}$ is defined by:

$$\phi_n(s,t,s)=\begin{cases}\left((2-n)a\sigma^i(s),(n-1)\sigma^i\tau^{n-i}(t^*),(2-n)a\sigma^i(s)\right), & for\ n=1,2\\ 0 & n\geq 3.\end{cases}$$

Therefore, it is clear that F is a GJT (σ,τ)-HH but not a GT (σ,τ)- HH.

Remark 2.10. Let $F=(f_i)_{i\in\mathbb{N}}$ be a GT (σ,τ)-HH from R into R' and $\theta=(\phi_i)_{i\in\mathbb{N}}$ be a related T (σ,τ)-HH. Then for each $n\in\mathbb{N}$ and for all $a,b\in R$, we will write

$$\delta_n(a,b,c)=f_n(abc)-\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right)$$

$$\gamma_n(a,b,c)=f_n(abc)-\sum_{i=1}^{n} f_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

For the purpose of this paper, we can list elementary properties about above

1. $\delta_n(a,b,c)+\delta_n(c,b,a)=0$.
2. $\gamma_n(a,b,c)+\gamma_n(c,b,a)=0$.

Note that if $\delta_n(a,b,c)=0$, then F is a GT(σ,τ)-HH and if $\gamma_n(a,b,c)=0$, then F is a GT (σ,τ)- HAH.

Lemma 2.11. If $F=(f_i)_{i\in\mathbb{N}}$ is a GT (σ,τ)-HH from a ring R into a ring R' and $\theta=(\phi_i)_{i\in\mathbb{N}}$ be a related JT (σ,τ)-HH, then for all $a,b\in R$ and $n\in\mathbb{N}$

i. $\delta_n(a+b,c,d)=\delta_n(a,c,d)+\delta_n(b,c,d)$,

ii. $\delta_n(a,b+c,d)=\delta_n(a,b,d)+\delta_n(a,c,d)$,

iii. $\delta_n(a,b,c+d)=\delta_n(a,b,c)+\delta_n(a,b,d)$.

Proof. i.

$$\delta_n(a+b,c,d)=f_n((a+b)cd)-\sum_{i=1}^{n} f_i\left(\sigma^i(a+b)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)$$
$$=f_n(acd+bcd)-\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)-\sum_{i=1}^{n} f_i\left(\sigma^i(b)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)$$

Since f_n is an additive mapping for each n, then:

$$\delta_n(a+b,c,d)=f_n(acd)-\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)+f_n(bcd)-\sum_{i=1}^{n} f_i\left(\sigma^i(b)\right)\phi_i\left(\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\tau^i(d)\right)$$
$$=\delta_n(a,c,d)+\delta_n(b,c,d)$$

By the same way we can prove ii and iii. ■

3. Main Results

Lemma 3.1. If $F=(f_i)_{i\in\mathbb{N}}$ is a GJT (σ,τ)-HH of R into R' and $\theta=(\phi_i)_{i\in\mathbb{N}}$ be a related JT (σ,τ)-HH, then for each $n\in\mathbb{N}$ and for all $a,b,c,r\in R$,

$$\delta_n\left(\sigma^n(a\,b,c)\right)\phi_n\left(\sigma^n(r)\right)B_n\left(\tau^n(a\,b,c)\right)+\gamma_n\left(\sigma^n(abc)\right)\phi_n\left(\sigma^n(r)\right)A_n\left(\tau^n(a,b,c)\right)=0$$

Proof. Assume that F is a GJT (σ,τ)-HH and take $a,b,c,r\in R$.

By induction on $n\in\mathbb{N}$. If $n=1$; define $w=abcrcba+cbarabc$, then we get the require result.

We can assume that

$$\delta_m\left(\sigma^m(a,b\,,c)\right)\phi_m\left(\sigma^m(r)\right)B_m\left(\tau^m(a,b\,,c)\right)+\gamma_m\left(\sigma^m(a,b,c)\right)\phi_m\left(\sigma^m(r)\right)A_m\left(\tau^m(a,b,c)\right)=0$$

is true for all $a,b,c,r\in R,n\in\mathbb{N}$ and $m<n$.

Now, we have:

$$f_n(w)=f_n(a(bcrcb)a+c(barab)c)$$
$$=\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(bcrcb)\right)\ \phi_i\left(\tau^i(a)\right)+\sum_{i=1}^{n} f_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(barab)\right)\phi_i\left(\tau^i(c)\right)$$
$$=\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\left(\sum_{j=1}^{i}\phi_j(\sigma^j\sigma^i\tau^{n-i}(b))\phi_j\left(\sigma^j\tau^{i-j}\sigma^i\tau^{n-i}(crc)\right)\phi_j\left(\tau^j\sigma^i\tau^{n-i}(b)\right)\right)\phi_i\left(\tau^i(a)\right)+\sum_{i=1}^{n} f_i(\sigma^i c)\left(\sum_{j=1}^{i}\phi_j(\sigma^j\sigma^i\tau^{n-i}(b))\phi_j\left(\sigma^j\tau^{i-j}\sigma^i\tau^{n-i}(ara)\right)\phi_j\left(\tau^j\sigma^i\tau^{n-i}(b)\right)\right)\phi_i\left(\tau^i(c)\right)$$
$$=\sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\left(\sum_{j=1}^{i}\phi_j\left(\sigma^j\sigma^i\tau^{n-i}(b)\right)\left(\sum_{k=1}^{j}\phi_k(\sigma^k\sigma^j\tau^{n-j}\sigma^i\tau^{n-i}(c))\phi_k\left(\sigma^k\tau^{j-k}\sigma^j\tau^{n-j}\sigma^i\tau^{n-i}(r)\right)\phi_k\left(\tau^k\sigma^j\tau^{n-j}\sigma^i\tau^{n-i}(c)\right)\right)\phi_j\left(\tau^j\sigma^i\tau^{n-i}(b)\right)\right)\phi_i\left(\tau^i(a)\right)+\sum_{i=1}^{n} f_i\left(\sigma^i(c)\right)\left(\sum_{j=1}^{i}\phi_j\left(\sigma^j\sigma^i\tau^{n-i}(b)\right)\left(\sum_{k=1}^{j}\phi_k(\sigma^k\sigma^j\tau^{n-j}\sigma^i\tau^{n-i}(a))\ \phi_k\left(\sigma^k\tau^{j-k}\sigma^j\tau^{n-j}\sigma^i\tau^{n-i}(r)\right)\phi_k\left(\tau^k\sigma^j\tau^{n-j}\sigma^i\tau^{n-i}(a)\right)\right)\phi_j\left(\tau^j\sigma^i\tau^{n-i}(b)\right)\right)\phi_i\left(\tau^i(c)\right)$$
$$=\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i\left(\sigma^i\sigma^i\tau^{n-i}(c)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i\tau^{n-i}(r)\right)\phi_i(\sigma^i\tau^{n-i}(c)))\phi_i\left(\sigma^i\tau^n(b)\right)\phi_i\left(\tau^i(a)\right)+\sum_{i=1}^{n} f_i(\sigma^i(c))\,\phi_i(\sigma^i\tau^{n-i}(b))\phi_i\left(\sigma^i\sigma^i\tau^{n-i}(a)\right)\phi_i\left(\sigma^i\tau^{n-i}\sigma^i\tau^{n-i}(r)\right)\phi_i(\sigma^i\tau^{n-i}a))\phi_i\left(\sigma^i\tau^n(b)\right)\phi_i\left(\tau^i(c)\right)$$
$$=\sum_{i=1}^{n}\left(\sum_{j=1}^{i} f_i\left(\sigma^i(a)\right)\phi_j(\sigma^j\sigma^i\tau^{n-i}(b))\phi_j\left(\tau^n(c)\right)\right)\phi_i(\sigma^j\sigma^i\tau^{n-i}(r))\ \left(\sum_{j=1}^{i}\left(\phi_j(\tau^j\sigma^j\tau^{n-j}(c)))\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(a)\right)\right)\right)+\sum_{i=1}^{n}\left(\sum_{j=1}^{i} f_i\left(\sigma^i(c)\right)\phi_j(\sigma^j\sigma^i\tau^{n-i}(b))\phi_j\left(\tau^n(a)\right)\right)\phi_i(\sigma^j\sigma^i\tau^{n-i}(r))\sum_{j=1}^{i}\left(\phi_j(\tau^j\sigma^j\tau^{n-j}(a)))\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(c)\right)\right) \quad (1)$$

On the other hand, $f_n(w)=f_n\left((abc)r(cba)+(cba)r(abc)\right)$. Thus by Lemma 2.7, we get:

$$f_n(w)=\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(cba)\right)+f_i\left(\sigma^i(cba)\right)\phi_i\left(\sigma^i\tau^{n-i}(r))\right)\phi_i\left(\tau^i(abc)\right)$$

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$= \sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\left(\sum_{i=1}^{n}\left(\phi_j\left(\sigma^j\tau^i(a)\right)\right)\phi_j\left(\sigma^j\tau^{j-i}\tau^i(b)\right)\phi_j\left(\tau^j\tau^i(c)\right)+\phi_j\left(\sigma^j\tau^i(c)\right)\right.$
$\left.\phi_j\left(\sigma^j\tau^{j-i}\tau^i(b)\right)\phi_j\left(\tau^j\tau^i(a)\right)-\phi_i\left(\tau^i(abc)\right)\right)+\sum_{i=1}^{n}\left(\sum_{j=1}^{i}\left(f_j\left(\sigma^j\sigma^i(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(c)\right)+\right.\right.$
$\left.\left.f_j\left(\sigma^j\sigma^i(c)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(a)\right)-f_i\left(\sigma^i(abc)\right)\right)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$

$= \sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\sum_{j=1}^{i}\phi_j\left(\sigma^j\tau^i(a)\right)\phi_j\left(\sigma^j\tau^j(b)\right)\phi_j\left(\tau^j\tau^i(c)\right)+\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)$
$\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\sum_{j=1}^{i}\phi_j\left(\sigma^j\tau^j(c)\right)\phi_j\left(\sigma^j\tau^j(b)\right)\phi_j\left(\tau^j\tau^i(a)\right)-\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)+$
$\sum_{i=1}^{n}\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)+\sum_{i=1}^{n}\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(c)\right)$
$\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)-\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$

$= -\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\left(\phi_i\left(\tau^i(abc)\right)-\sum_{j=1}^{i}\phi_j\left(\sigma^j\tau^i(a)\right)\phi_j\left(\sigma^j\tau^j(b)\right)\phi_j\left(\tau^j\tau^i(c)\right)\right)-$
$\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\left(\phi_i\left(\tau^i(abc)\right)-\sum_{j=1}^{i}\phi_j\left(\sigma^j\tau^i(c)\right)\phi_j\left(\sigma^j\tau^j(b)\right)\phi_j\left(\tau^j\tau^i(a)\right)\right)-+$
$\sum_{i=1}^{n}\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)+$
$\sum_{i=1}^{n}\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(c)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)$

$= -\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i\left(\tau^i(a,b,c)\right)-\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i\left(\tau^i(abc)\right)+$
$\sum_{i=1}^{n}\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right)+$
$\sum_{i=1}^{n}\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(c)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\phi_i\left(\tau^i(abc)\right) \quad (2)$

From equations (1) and (2), we get:

$0 = -\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i\left(\tau^i(a,b,c)\right)-\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i\left(\tau^i(abc)\right)+$
$\sum_{i=1}^{n}\left(\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(c)\right)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\left(\phi_i\left(\tau^i(abc)\right)-\sum_{j=1}^{i}\phi_j(\tau^j\sigma^j\tau^{n-j}(c)))\right.$
$\left.\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(a)\right)\right)+\sum_{i=1}^{n}\left(\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(c)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(a)\right)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)\left(\phi_i\left(\tau^i(abc)\right)-\right.$
$\left.\sum_{j=1}^{i}\phi_j(\tau^j\sigma^j\tau^{n-j}(a)))\phi_j\left(\tau^j(b)\right)\phi_j\left(\tau^j(c)\right)\right)$

$0 = -\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i\left(\tau^i(a,b,c)\right)-\sum_{i=1}^{n} f_i\left(\sigma^i(abc)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i\left(\tau^i(abc)\right)+$
$\sum_{i=1}^{n}\left(\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(c)\right)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i\left(\tau^i(a,b,c)\right)+$
$\sum_{i=1}^{n}\left(\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(c)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(a)\right)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i\left(\tau^i(a,b,c)\right)$

$0 = -\sum_{i=1}^{n}\left(f_i\left(\sigma^i(abc)\right)-\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(a)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(c)\right)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i\left(\tau^i(a,b,c)\right)-$
$\sum_{i=1}^{n}\left(f_i\left(\sigma^i(abc)\right)-\sum_{j=1}^{i} f_j\left(\sigma^j\sigma^i(c)\right)\phi_j\left(\sigma^j\tau^{i-j}\sigma^i(b)\right)\phi_j\left(\tau^j\sigma^i(a)\right)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i\left(\tau^i(a,b,c)\right)$

$0 = -\sum_{i=1}^{n}\delta_i\left(\sigma^i(a,b,c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)B_i\left(\tau^i(a,b,c)\right)-\sum_{i=1}^{n}\gamma_i\left(\sigma^i(a,b,c)\right)\phi_i\left(\sigma^i\tau^{n-i}(r)\right)A_i\left(\tau^i(a,b,c)\right)$

Hence, we have:

$\gamma_n\left(\sigma^n(a,b,c)\right)\phi_n\left(\sigma^n(r)\right)A_n\left(\tau^n(a\ b,c)\right)+\delta_n\left(\sigma^n(a,b,c)\right)\phi_n\left(\sigma^n(r)\right)B_n\left(\tau^n(a,b,c)\right)=0.$ ■

Corollary 3.2. Let $F=(f_i)_{i\in\mathbb{N}}$ be a GJT (σ,τ)-HH of R into R' and $\theta=(\phi_i)_{i\in\mathbb{N}}$ be a related JT (σ,τ)-HH, then for each $n\in\mathbb{N}$ and for all $a,b,c,r\in R$

$\delta_n\left(\sigma^n(a,b,c)\right)\phi_n\left(\sigma^n(r)\right)B_n\left(\tau^n(a,b,c)\right)=$
$\gamma_n\left(\sigma^n(a,b,c)\right)\phi_n\left(\sigma^n(r)\right)A_n\left(\tau^n(a,b,c)\right)=0$

Proof. By Lemma 2.5 and Lemma 3.1, we get the result. ■

Theorem 3.3. Let $F=(f_i)_{i\in\mathbb{N}}$ be a GJT (σ,τ)-HH of ring R into prime ring R' and $\theta=(\phi_i)_{i\in\mathbb{N}}$ be a related JT (σ,τ)-HH then for each $n\in\mathbb{N}$ and for all $a,b,c,r,x,y,z\in R$

$\delta_n\left(\sigma^n(a,b,c)\right)\phi_n\left(\sigma^n(r)\right)B_n\left(\tau^n(x,y,z)\right)=0$

Proof. Replace $a+x$ by a in Corollary 3.2, we get:

$\delta_n\left(\sigma^n(a+x,b,c)\right)\phi_n\left(\sigma^n(r)\right)B_n\left(\tau^n(a+x,b,c)\right)=0$

Hence:

$\delta_n\left(\sigma^n(a,b,c)\right)\phi_n\left(\sigma^n(r)\right)B_n\left(\tau^n(a,b,c)\right)+$
$\delta_n\left(\sigma^n(a,b,c)\right)\phi_n\left(\sigma^n(r)\right)B_n\left(\tau^n(x,b,c)\right)+$

$\delta_n(\sigma^n(x,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(a,b,c)) +$
$\delta_n(\sigma^n(x,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b,c)) = 0$

By Corollary 3.2, we obtain:

$$\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b,c)) + \delta_n(\sigma^n(x,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(a,b,c)) = 0$$

Therefore, we get:

$$0 = \delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b,c)) \phi_n(\sigma^n(r))\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b,c)) = -\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c))\phi_n(\sigma^n(r)) \delta_n(\sigma^n(x,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(a,b,c))$$

Since R' is prime, we obtain:

$$\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b,c)) = 0 \quad (3)$$

Replacing b+y for b in equation (3), we get:

$$\delta_n(\sigma^n(a,b+y,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b+y,c)) = 0$$

Hence:

$$\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b,c)) + \delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c)) + \delta_n(\sigma^n(a,y,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b,c)) + \delta_n(\sigma^n(a,y,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c)) = 0$$

We can use equation (3), we get:

$$\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c)) + \delta_n(\sigma^n(a,y,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b,c)) = 0$$

Therefore, we get:

$$0 = \delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c)) \phi_n(\sigma^n(r))\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c)) = -\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c))\phi_n(\sigma^n(r)) \delta_n(\sigma^n(a,y,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,b,c))$$

Since R' is prime, we obtain:

$$\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c)) = 0 \quad (4)$$

Replacing c+z for c in equation (4), we get:

$$\delta_n(\sigma^n(a,b,c+z))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c+z)) = 0$$

Hence:

$$\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c)) + \delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,z)) + \delta_n(\sigma^n(a,b,z))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c)) + \delta_n(\sigma^n(a,b,z))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,z)) = 0$$

We can use equation (4), we get:

$$\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,z)) + \delta_n(\sigma^n(a,b,z))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,c)) = 0$$

Therefore, we get:

$$0 = \delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,z)) \phi_n(\sigma^n(r))\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r)) B_n(\tau^n(x,y,z)) = -\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,z)) \phi_n(\sigma^n(r))\delta_n(\sigma^n(a,b,z))\phi_n(\sigma^n(r)) B_n(\tau^n(x,y,c))$$

Since R' is prime, we obtain:

$$\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,z)) = 0 \quad \blacksquare$$

Now, we will prove the principal theorem of this section which is an extension of result in [8].

Theorem 3.4. Let $F = (f_i)_{i\in\mathbb{N}}$ be a GJT (σ,τ)-HH of a ring R into prime ring R' and $\theta = (\phi_i)_{i\in\mathbb{N}}$ be a related JT (σ,τ)-HH, then F is GT(σ,τ)-HH or triple (σ,τ)-HAH.

Proof. Since F is a GJT (σ,τ)-HH. Then by Theorem 3.3, we have:

$$\delta_n(\sigma^n(a,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(x,y,z)) = 0$$

Since R' is prime, therefore either $\delta_n(\sigma^n(a,b,c)) = 0$ or $B_n(\tau^n(x,y,z)) = 0$, for each $n \in \mathbb{N}$ and for all $a,b,c,x,y,z \in R$.
If $B_n(\tau^n(x,y,z)) = 0$, then by Lemma 2.2, we obtain F is triple (σ,τ)- higher anti-homomorphism.
But if $\delta_n(\sigma^n(a,b,c)) = 0$, then by Remark 2.10, we obtain F is GT (σ,τ)-HH. ■

Theorem 3.5. Let $F = (f_i)_{i\in\mathbb{N}}$ be a GJT (σ,τ)-HH of R into a 2-torsion-free prime ring R' and $\theta = (\phi_i)_{i\in\mathbb{N}}$ be a related JT (σ,τ)-HH. Then F G(σ,τ)-HH or G(σ,τ)-HAH.

Proof. Since $F = (f_i)_{i\in\mathbb{N}}$ is a GJT (σ,τ)-HH of R into R' there exist $\theta = (\phi_i)_{i\in\mathbb{N}}$ be a related JT (σ,τ)-HH. Then by Lemma 2.4, we have ϕ is a Triple (σ,τ) -HH or Triple(σ,τ)-HAH. Therefore we obtain two cases

Case I: If $\theta = \phi$, where Φ is triple (σ,τ)-HH. Then for all $a,b,c \in R$, we have:

$$\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))$$

Since:

$$A_n(a,b,c) = \phi_n(abc) - \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))$$

this means $A_n(a,b,c) = 0$. By Lemma 3.1, we get:

$$\delta_n(\sigma^n(a\,b,c))\phi_n(\sigma^n(r))B_n(\tau^n(a\,b,c)) = 0$$

Since Ŕ is prime ring, then either $\delta_n(\sigma^n(a\,b,c)) = 0$ or $B_n(\tau^n(a\,b,c)) = 0$.
If $B_n(\tau^n(a\,b,c)) = 0$, then we get ϕ is triple (σ,τ)-HAH, and this will be a contradiction with assumption. Therefore $\delta_n(\sigma^n(a\,b,c)) = 0$, for all $a,b,c \in R$, that is:

$$f_n(abc) = \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))$$

We proceed by induction on $n \in \mathbb{N}$. f n = 1, let $W = f_1(abxab)$. Since ϕ JT (σ,τ)-HH. Then by Lemma 1.3, we have ϕ is a (σ,τ)-HH. Hence:

$$\begin{aligned} W &= f_1(abxab) \\ &= f_1(\sigma(ab))\phi_1(\sigma(x))\phi_1(\tau(ab)) \\ &= f_1(\sigma(ab))\phi_1(\sigma(x))\phi_1(\tau(a))\phi_1(\tau(b)) \qquad (5) \end{aligned}$$

On the other hand,

$$\begin{aligned} W &= f_1(a(bxa)b) \\ &= f_1(\sigma(a))\phi_1(\sigma(bxa))\phi_1(\tau(b)) \\ &= f_1(\sigma(a))\phi_1(\sigma(bx))\phi_1(\tau(a))\phi_1(\tau(b)) \\ &= f_1(\sigma(a))\phi_1(\sigma(b))\phi_1(\sigma(x))\phi_1(\tau(a))\phi_1(\tau(b)) \qquad (6) \end{aligned}$$

Comparing (5) and (6), we get the result.

We can assume that f_m is true for all $a,b,c,r \in R, n \in \mathbb{N}$ and $m < n$. Let:

$$W = f_n(abxab)$$
$$= \sum_{i=1}^{n} f_i\left(\sigma^i(ab)\right)\phi_i\left(\sigma^i\tau^{n-i}(x)\right)\phi_i\left(\tau^i(ab)\right)$$
$$= \sum_{i=1}^{n} f_i\left(\sigma^i(ab)\right)\phi_i\left(\sigma^i\tau^{n-i}(x)\right) \sum_{j=1}^{i}\phi_j\left(\sigma^j(a)\right).\phi_j\left(\tau^j(b)\right)$$
$$= f_n(\sigma^n(ab))\phi_n(\sigma^n(x))\sum_{j=1}^{i}\phi_j\left(\sigma^j(a)\right). + \phi_j\left(\tau^j(b)\right)\sum_{i=1}^{n-1} f_i\left(\sigma^i(ab)\right) \phi_i\left(\sigma^i\tau^{n-i}(x)\right)\sum_{j=1}^{i}\phi_j\left(\sigma^j(a)\right)\phi_j\left(\tau^j(b)\right) \quad (7)$$

On the other hand:

$$W = f_n(a(bxa)b)$$
$$= \sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(bxa)\right)\phi_i\left(\tau^i(b)\right)$$
$$= \sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\left(\sum_{j=1}^{i}\phi_j\left(\sigma^j(b)\right) \phi_j\left(\sigma^j\tau^{i-j}(xa)\right)\right)\phi_i\left(\tau^i(b)\right)$$
$$= \sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right) \phi_i\left(\sigma^j\tau^{i-j}(xa)\right)\phi_i\left(\tau^i(b)\right)$$
$$= \sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right) \left(\sum_{j=1}^{i}\phi_j\left(\sigma^j(x)\right)\phi_j\left(\sigma^j\tau^{i-j}(a)\right)\right)\phi_i\left(\tau^i(b)\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{i} f_j\left(\sigma^j(a)\right)\phi_i\left(\sigma^i(b)\right)\phi_i\left(\sigma^i\tau^{n-i}(x)\right) \left(\sum_{j=1}^{i}\phi_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j\tau^{i-j}(b)\right)\right)$$
$$= \sum_{j=1}^{n} f_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j(b)\right)\phi_j\left(\sigma^j\tau^{n-j}(x)\right) \left(\sum_{j=1}^{i}\phi_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j\tau^{i-j}(b)\right)\right) + \sum_{i=1}^{n-1}\sum_{j=1}^{i} f_j\left(\sigma^j(a)\right)\phi_i\left(\sigma^i(b)\right)\phi_i\left(\sigma^i\tau^{n-i}(x)\right) \left(\sum_{j=1}^{i}\phi_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j\tau^{i-j}(b)\right)\right) \quad (8)$$

By comparing (7) and (8), we get:

$$0 = \sum_{i=1}^{n-1}\left(f_i\left(\sigma^i(ab)\right) - \sum_{j=1}^{i} f_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j(b)\right)\right) \phi_j\left(\sigma^j\tau^{n-j}(x)\right)\left(\sum_{j=1}^{i}\phi_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j\tau^{i-j}(b)\right)\right) + \left(f_n\left(\sigma^n(ab)\right) - \sum_{j=1}^{n} f_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j(b)\right)\right) \phi_j\left(\sigma^j\tau^{n-j}(x)\right)\left(\sum_{j=1}^{i}\phi_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j\tau^{i-j}(b)\right)\right)$$

By the assumption of $m < n$, reduces the last equation to:

$$\left(f_n\left(\sigma^n(ab)\right) - \sum_{j=1}^{n} f_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j(b)\right)\right) \phi_j\left(\sigma^j\tau^{n-j}(x)\right)\left(\sum_{j=1}^{i}\phi_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j\tau^{i-j}(b)\right)\right) = 0$$

This implies that, for all $a,b,x \in R, n \in \mathbb{N}$.

$$\left(f_n\left(\sigma^n(ab)\right) - \sum_{j=1}^{i} f_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j(b)\right)\right)R' = 0$$

Since R' is prime, then we get:

$$f_n\left(\sigma^n(ab)\right) = \sum_{j=1}^{i} f_j\left(\sigma^j(a)\right)\phi_j\left(\sigma^j(b)\right)$$

this means F is G(σ,τ)-HH.

Case II: If Φ is triple (σ,τ)-HAH, hence:

$$\phi_n(abc) = \sum_{i=1}^{n}\phi_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right) \phi_i\left(\tau^i(a)\right)$$

By the same way of proof case I, we get:

$$\delta_n\left(\sigma^n(a\,b,c)\right) = 0 \text{ or } B_n\left(\tau^n(a\,b,c)\right) = 0$$

If $B_n\left(\tau^n(a\,b,c)\right) = 0$, then we get ϕ is triple (σ,τ)-HAH, and this will be a contradiction with assumption. Therefore $\delta_n\left(\sigma^n(a\,b,c)\right) = 0$, for all $a,b,c \in R$, that is:

$$f_n(abc) = \sum_{i=1}^{n} f_i\left(\sigma^i(a)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(c)\right)$$

Since:

$$B_n(a,b,c) = \phi_n(abc) - \sum_{i=1}^{n}\phi_i\left(\sigma^i(c)\right) \phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

this means $B_n(a,b,c) = 0$. By Lemma 2.1, we get:

$$\gamma_n\left(\sigma^n(a,b,c)\right)\phi_n\left(\sigma^n(r)\right)A_n\left(\tau^n(a\,b,c)\right) = 0$$

Since Ŕ is prime ring, then either $\gamma_n\left(\sigma^n(a,b,c)\right) = 0$ or $A_n\left(\tau^n(a\,b,c)\right) = 0$. If $A_n\left(\tau^n(a\,b,c)\right) = 0$, then we get ϕ is triple (σ,τ)-HH, and this will be a contradiction with assumption. Therefore $\gamma_n\left(\sigma^n(a\,b,c)\right) = 0$, for all $a,b,c \in R$, that is:

$$f_n(abc) = \sum_{i=1}^{n} f_i\left(\sigma^i(c)\right)\phi_i\left(\sigma^i\tau^{n-i}(b)\right)\phi_i\left(\tau^i(a)\right)$$

As in the proof case I, we get:

$$\left(f_n\left(\sigma^n(ab)\right) - \sum_{j=1}^{n} f_j\left(\sigma^j(b)\right)\phi_j\left(\sigma^j(a)\right)\right) \phi_j\left(\sigma^j\tau^{n-j}(x)\right)\left(\sum_{j=1}^{i}\phi_j\left(\sigma^j(b)\right)\phi_j\left(\sigma^j\tau^{i-j}(a)\right)\right) = 0$$

This implies that, for all $a,b,x \in R, n \in \mathbb{N}$.

$$\left(f_n\left(\sigma^n(ab)\right) - \sum_{j=1}^{i} f_j\left(\sigma^j(b)\right)\phi_j\left(\sigma^j(a)\right)\right)R' = 0$$

Since R' is prime, then we get:

$f_n\left(\sigma^n(ab)\right) = \sum_{j=1}^{i} f_j\left(\sigma^j(b)\right)\phi_j\left(\sigma^j(a)\right)$, this means F is G(σ,τ)-HAH. ■

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