Al-Nahrain Journal of Science
ANJS, Vol.23 (2), June, 2020, pp. 52 – 59

Backstepping Method for Stabilizing System of 2×2 Riccati Matrix Differential Equations

Akram H. Shather

Department of Communication Engineering, Technical College of Engineering, Sulaimani Polytechnic University, Sulaimani, Kurdistan Region, Iraq

Articles Information

Received:
10.03.2020
Accepted:
10.05.2020
Published:
04.06.2020

Keywords:
Backstepping method
Riccati differential equations
Lyapunov functions (Lf)
Asymptotically stable

Abstract

In this research paper, the backstepping method (BSM) will be proposed for stabilizing and solving system of 2×2 Riccati matrix ordinary differential equations. Such equations have many difficulties in the studying their solutions and stability. The basic idea behind of this approach is to use the BSM as a transformation method for transforming the original system into an equivalent one which is stabilizable and solvable based on constructing the Lyapunov function.

DOI: 10.22401/ANJS.23.2.08

* Corresponding author: dr_akram75@yahoo.com

1. Introduction

The BSM can be described as a particular approach to stabilize dynamic systems and non-linear control problems. So, it is a very useful approach for linearization feedback for non-linear systems with uncertainties. The procedure to do this is particularly driven by cases in which plant non-linearity and control input. So, it requires to recompense and make up for the impact of the non-linearity are in various equations.

The ability of BSM to transact with those systems whose feedback are neither linearizable nor even completely controllable. Also, it has the ability to transact with not only control synthesis defies, but also much limit classes of systems such as unmeasured values or states, unknown parameters or arguments, zero dynamics systems, and nuisance stochastic system.

The BS designs for boundary control in PDE's is given by Krstic et al. in 2008, [1]. Dynamic adaptive and non-linear BS resolves are explained in details by Krstic et al. in 1995, [2]. They characterized suitable techniques for output feedbacks and full state. They mentioned specific parameters for adaptation, tuning functions and design models. In addition, Similarly, Sepulchre et al. in 1997 [3], made the same extensions to forward, passivity and cascaded design models.

Fossen and Berge in 1997, introduced and studied the concept of the vectorial BS for the first time, where they described the structural properties of non-linear , multi-input and output systems [4]. This pliable them to spread design and analysis of non-linear system by the vectorial BSM. The stochastic systems presented by Krstic and Deng in 1998 [5], and specifically concentrated on the stability and the regulation for them. During 1999 till 2001, Loria et al. Making use of the BS designs, [6] and Fossen et al. in [7], gave two methods for "integral action" of non-linear systems. The development of the BS approach initially was for PDEs. The way of a continuum BS is developed for stabilizing parabolic linear PDEs was first presented and explained by Smyshlyaev and Krstic in 2004, [8]. In 2007 Vazquez and Krstic [9] designed BS for linearized Navier-Stokes equations. In 2008, Krstic made use of an infinite-dimensional BS transformation, to connect with Lf [10]. The results include in finite dimensional systems which consist of ODE plant state and delay state. The technique to design a least-squares estimator that uses unfiltered regress was presented and introduced by Krstic in 2009, [11].

Krstic considered the problem of dynamic adaptive non-linear control and introduced the 1st least-squares-based adaptive non-linear control design in addition. This comes from a Lf. In 2010, Krstic presented a procedure for recompensing input delay of arbitrary length in non-linear control systems [12]. This procedure came from the infinite dimensionality of the actuator dynamics and non-linear character of the plant which results in a non-linear feedback operator. This method is a non-linear version of the Smith predictor and its various predictor based modifications for linear plants. In addition, Krstic in 2010 developed a time varying Lyapunov functional equation for the feedback closed-loop system and constructed the exponential stability [13]. The state selection for a transport PDE which has propagation with a non-constant speed is the challenge in this case.

In 2013, the non-linear dynamic systems with time delays was given by Bekiaris-Liberis and Krstic, the delay is considered as a non-linear function of the state at a previous time and depends on the delay parameter itself [14]. Also, Krstic and Bekiaris-Liberis in 2013 [14], revised many token methods but in general results on non-linear control in the infinite-dimensional precision. First, they presented certain designs for non-linear ODEs with constant time-varying or state-dependent input delays that appeared in considerable applications of networks control. Second, they gave a design for non-linear ODEs with a wave (string) PDE at its input, which is motivated by the drilling dynamics in petroleum engineering. Third, they gave design for systems of two coupled non-linear first-order hyperbolic PDEs, which are motivated by slugging flow dynamics in petroleum production in off-shore facilities. The main objective of this research article is to apply the BSM for stabilizating system and solving such as certain type of Riccati matrix DEs which are considered here to be 2×2 system of ODEs.

2.BSM for Dynamical Systems

There are specific theories and methods that are using to ensure the stability of the Non-linear systems without regarding the inner dynamics of the system. In 1990, Petar V. Kokotovic and others developed the BSM as a technique for designing and stabilizing controls for a special class of non-linear dynamical systems. These systems are driven from subsystems that radiate out from an irreducible subsystem that may be stabilized using some other methods. Due to this recursive structure, the designer can first start the design process at the known-stable system and "back out" new controllers that gradually stabilize each outer subsystem. The process stops when the final external control is obtained. Hence, this process is known as BS. BSM is a particular approach to stabilize dynamical systems and is particularly successful in the area of non-linear control problems while the idea of integrator BSM seems to be appeared simultaneously and often implicitly in [3] and [4]. Studying the stabilizability through an integrator BS introduced by Kokotovic and Sussmann in 1989, [5]. Integrator BS approach appeared as a recursive design technique used by Saberi et al. (1990), etc. [6]
For illustration purpose, consider the order dynamical system:

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) + u_1 \\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n) + u_2 \\ \dot{x}_3 &= f_3(x_1, x_2, \ldots, x_n) + u_3 \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) + u_n \end{aligned}\right\} \quad (1)$$

where $x(t) \in R^n$ is the state vector of the system, f_i, i = 1, 2, …, n; are either linear or non-linear functions and i u_i, = 1, 2, …, n; are the controller input.

The BS design of system (1) is a recursive method, which guarantees the global stable performance of the system. By using the BS design at the ith step the ith order subsystem may be stabilized with respect to the Lf V_i, by designing certain V_i, in addition to the design of the virtual control x_i and a control input function u_i. The analysis of this method may be summarized by follows:

Step 1: We consider the stability of the first equation in (1)

$$\dot{x}_1 = f_1(x_1, x_2, ..., x_n) + u_1 \qquad (2)$$

where x_2 is regarded as a virtual controller, then define $z_1 = x_1$ and derive the dynamics of the new coordinates transformation as:

$$\dot{z}_1 = \dot{x}_1 = f_1(z_1, x_2, ..., x_n) + u_1 \qquad (3)$$

for the design of $\alpha_1(z_1)$ to stabilizes system (2).
Now, construct the first Lf in quadratic form as follows:

$$V_1(z_1) = z_1^T p_1 z_1 \qquad (4)$$

Then the derivative of V_1 with respect to the time is given by:

$$\dot{V}_1 = -z_1^T Q_1 z_1 \qquad (5)$$

where Q_1 is a positive definite matrix. Then $\dot{V}_1$ is a negative definite function in R^n. Thus by Lyapunov stability theory, system (2) is asymptotically stable. Clear that the virtual control $x_2 = \alpha_1(z_1)$ and the state feedback input u1 makes the system (1) asymptotically stable. The function $\alpha_1(z_1)$ should be estimated while z_2 is considered as controller.

Step 2: Define the error z_2 between x2 and $\alpha_1(z_1)$ to be defined as:

$$z_2 = x_2 - \alpha_1(z_1) \qquad (6)$$

Consider the z_1, z_2-subsystem given by:

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, x_2, ..., x_n) + u_1 \\ \dot{z}_2 &= f_2(z_1, z_2, x_3, ..., x_n) - \dot{\alpha}_1(z_1) + u_2 \end{aligned} \qquad (7)$$

where x_3 is a virtual controller of subsystem (7), and assume that it is equal to $\alpha_2(z_1, z_2)$ and it makes the subsystem (7) asymptotically stable. Consider the Lf defined by:

$$V_2(z_1, z_2) = V_1(z_1) + z_2^T p_2 z_2 \qquad (8)$$

The derivative of V_2 is:

$$\dot{V}_2 = -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 < 0 \qquad (9)$$

where Q_1, Q_2 are positive definite matrices. Then $\dot{V}_2$ is a negative definite function on R^n. Thus by Lyapunov theory, the subsystem (7) is asymptotically stable.

Similarly, the virtual control $x_3 = \alpha_2(z_1, z_2)$ may be defined and the state feedback input u_2 make the subsystem (7) asymptotically stable.

Step n: So on, proceeding similarly as in steps 1 and 2, define the error variable zn as:

$$z_n = x_n - \alpha_{n-1}(z_1, z_2, \dots, z_n) \qquad (10)$$

consider $(z_1, z_2, z_3 ..., z_n)$ subsystem given by:

$$\left.\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2) + u_1 \\ \dot{z}_2 &= f_2(z_1, z_2, z_3) + u_2 \\ &\vdots \\ \dot{z}_n &= f_n(z_1, z_2, ..., z_n) + \alpha_{n-1}(z_1, z_2, ..., z_n) + u_n \end{aligned}\right\} \qquad (11)$$

and consider the Lf defined by:

$$V_n(z_1, z_2, z_3 ..., z_n) = V_{n-1}(z_1, z_2, ..., z_{n-1}) + z_n^T p_n z_n \qquad (12)$$

Therefore, derivative of V_n is:

$$\dot{V}_n = -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 - \ldots - z_n^T Q_n z_n < 0 \qquad (13)$$

where $Q_1, Q_2, Q_3, \ldots Q_n$ are positive definite matrices. Then $\dot{V}_n$ is a negative definite function on R^n. Thus by Lyapunov stability theory the subsystem (11) is stable. The virtual control $x_n = \alpha_{n-1}(z_1, z_2, \ldots, z_{n-1})$ and the state feedback input u_n makes the subsystem (11) asymptotically stable. Thus by Lyapunov stability theory the system (1) is globally asymptotically stable for all initial conditions $x_i(0) \in R^n$.

3. Application of the Method for 2×2 Riccati Matrix Differential Equations

In this section, the BSM which is proposed above in section (2) will be used to stabilize the following Riccati $n \times n$ system:

$$\dot{X}(t) + X(t)A + A^T X(t) - X(t)BX(t) + C(t) = 0$$

For simplicity and illustration, we will take n = 2, and apply the BSM to derive the related transformed system which is asymptotically stable.

Now, introduce control functions as auxiliary variables to stabilize this system, i.e., for 2×2 system; consider the Riccati system:

$$\dot{X}(t) + X(t)A + A^T X(t) - X(t)BX(t) + C(t) - u = 0$$

or equivalently:

$$\begin{pmatrix} \dot{x}_{11} & \dot{x}_{12} \\ \dot{x}_{21} & \dot{x}_{22} \end{pmatrix} + \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^T \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

$$+\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} - \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which implies:

$$\begin{pmatrix} \dot{x}_{11} & \dot{x}_{12} \\ \dot{x}_{21} & \dot{x}_{22} \end{pmatrix} + \begin{pmatrix} a_{11}x_{11} + a_{21}x_{12} & a_{12}x_{11} + a_{22}x_{12} \\ a_{11}x_{21} + a_{21}x_{22} & a_{12}x_{21} + a_{22}x_{22} \end{pmatrix} + \begin{pmatrix} a_{11}x_{11} + a_{21}x_{21} & a_{11}x_{12} + a_{21}x_{22} \\ a_{12}x_{11} + a_{22}x_{21} & a_{12}x_{12} + a_{22}x_{22} \end{pmatrix} -$$

$$\begin{pmatrix} x_{11}(b_{11}x_{11} + b_{12}x_{21}) + x_{12}(b_{21}x_{11} + b_{22}x_{21}) & x_{11}(b_{11}x_{12} + b_{12}x_{12}) + x_{12}(b_{21}x_{12} + b_{22}x_{22}) \\ x_{21}(b_{11}x_{11} + b_{12}x_{21}) + x_{22}(b_{21}x_{11} + b_{22}x_{22}) & x_{21}(b_{11}x_{12} + b_{12}x_{22}) + x_{21}(b_{11}x_{12} + b_{22}x_{22}) \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} - \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore, carrying out matrix operations, the following related system of ODE's with control functions is obtained:

$$\dot{x}_{11} + a_{11}x_{11} + a_{21}x_{21} + a_{11}x_{11} + a_{21}x_{12} - x_{11}(b_{11}x_{11} + b_{12}x_{21}) - x_{12}(b_{21}x_{11} + b_{22}x_{21}) + c_{11} - u_1 = 0$$
$$\dot{x}_{12} + a_{11}x_{12} + a_{21}x_{22} + a_{12}x_{11} + a_{22}x_{12} - x_{11}(b_{11}x_{12} + b_{12}x_{22}) - x_{12}(b_{21}x_{12} + b_{21}x_{22}) + c_{12} - u_2 = 0$$
$$\dot{x}_{21} + a_{12}x_{11} + a_{22}x_{21} + a_{11}x_{21} + a_{21}x_{22} - x_{21}(b_{11}x_{11} + b_{12}x_{21}) - x_{22}(b_{21}x_{11} + b_{22}x_{21}) + c_{21} - u_3 = 0$$
$$\dot{x}_{22} + a_{12}x_{12} + a_{22}x_{22} + a_{12}x_{21} + a_{22}x_{22} - x_{21}(b_{11}x_{12} + b_{12}x_{22}) - x_{22}(b_{21}x_{12} + b_{22}x_{22}) + c_{22} - u_4 = 0$$

and carrying out some simplification, the last system will be reduced to:

$$\left.\begin{aligned}
\dot{x}_{11} &= -2a_{11}x_{11} + b_{11}x^2_{11} - a_{21}x_{21} - a_{21}x_{12} + b_{12}x_{11}x_{21} + b_{21}x_{11}x_{12} + b_{22}x_{12}x_{21} - c_{11} + u_1 \\
\dot{x}_{12} &= -(a_{11} + a_{22})x_{12} - a_{12}x_{22} - a_{12}x_{11} + b_{11}x_{12}x_{11} + b_{12}x_{11}x_{22} + b_{22}x_{12}x_{22} - c_{12} + u_2 \\
\dot{x}_{21} &= -a_{12}x_{11} - (a_{11} + a_{22})x_{21} - a_{21}x_{22} + b_{11}x_{11}x_{21} + b_{12}x^2_{21} + b_{21}x_{11}x_{22} + b_{22}x_{21}x_{22} - c_{21} + u_3 \\
\dot{x}_{22} &= -a_{12}x_{12} - 2a_{22}x_{22} - a_{12}x_{21} + b_{11}x_{12}x_{21} + b_{12}x_{21}x_{22} + b_{21}x_{12}x_{22} + b_{22}x^2_{22} - c_{22} + u_4
\end{aligned}\right\} \qquad (14)$$

where x_{11}, x_{12}, x_{21} and x_{22} are system states and u_1, u_2, u_3 and u_4 are the control inputs. The objective is to design a state feedback control to stabilize the original Riccati system. The analysis of this criterion may be carried out as in the following steps:

Step 1: In this step, consider the stability of the first equation of the system (14), by defining $z_{11} = x_{11}$ and derive the dynamics of the new coordinates as:

$$\dot{z}_{11} = \dot{x}_{11}$$

$$= -2a_{11}z_{11} + b_{11}z^2{}_{11} - a_{21}x_{21} - a_{21}x_{12} + b_{12}z_{11}x_{21} + b_{21}z_{11}x_{12} + b_{22}x_{12}x_{21} - c_{11} + u_1 \quad (15)$$

and suppose that, the first Lf as follows:

$$V_1 = \frac{1}{2}z^2{}_{11} \quad (16)$$

The time derivative of V_1 becomes:

$$\dot{V}_1 = z_{11}\dot{z}_{11}$$

$$= z_{11}\left[-2a_{11}z_{11} + b_{11}z^2{}_{11} - a_{21}x_{21} - a_{21}x_{12} + b_{12}z_{11}x_{21} + b_{21}z_{11}x_{12} + b_{22}x_{12}x_{21} - c_{11} + u_1\right] \quad (17)$$

Assume that the controller $x_{12} = \alpha_1(z_{11})$. If:

$$u_1 = -b_{11}z^2{}_{11} + a_{21}x_{21} + a_{21}x_{12} - b_{12}z_{11}x_{21} - b_{21}z_{11}x_{12} - b_{22}x_{12}x_{21} + c_{11} \quad (18)$$

and letting $\alpha_1(z_{11}) = 0,$ then equation (18) will take the form:

$$\dot{V}_1 = -2a_{11}z^2{}_{11} \quad (19)$$

Hence, the zero solution is asymptotically stable.

Step 2: Define the variable $z_{12} = x_{12} - \alpha_1(z_{11}),$ so we can write the second equation of system (14) as:

$$\dot{z}_{12} = -(a_{11} + a_{22})z_{12} - a_{12}x_{22} - a_{12}x_{11} + b_{11}z_{12}x_{11} + b_{12}x_{11}x_{22} + b_{21}z^2{}_{12} + b_{22}z_{12}x_{22} - c_{12} + u_2$$

and a second Lf may be chosen to by:

$$V_2 = V_1 + \frac{1}{2}z_{12}^2$$

Therefore, the time derivative of which becomes:

$$\dot{V}_2 = \dot{V}_1 + z_{12}\dot{z}_{12}$$

$$= -2a_{11}\dot{z}_{11} + z_{12}^2[-(a_{11} + a_{22})z_{12} - a_{12}x_{22} - a_{12}x_{11} + b_{11}z_{12}x_{11} + b_{12}x_{11}x_{22} + b_{21}z_{12}^2 + b_{22}z_{12}x_{22} - c_{12} + u_2]$$

Now, if:

$$u_2 = a_{12}x_{22} + a_{12}x_{11} - b_{11}z_{12}x_{11} - b_{12}x_{11}x_{22} - b_{21}z^2{}_{12} - b_{22}z_{12}x_{22} + c_{12} \quad (20)$$

and $\alpha_2(z_{11}, z_{12}) = 0,$ then:

$$\dot{V}_2 = -2a_{11}z_{11}^2 - (a_{11} + a_{22})z_{12}^2$$

Step 3: Define a new variable $z_{21} = x_{21} - \alpha_2(z_{11}z_{12}),$ so we can rewrite the third equation of system (14) as:

$$\dot{z}_{21} = -a_{12}x_{11} - (a_{11} + a_{22})z_{21} - a_{21}x_{22} + b_{11}x_{11}z_{21} + b_{12}z_{21}^2 + b_{21}x_{11}x_{22} + b_{22}z_{21}x_{22} - c_{21} + u_3$$

and choosing the Lf to be define as:

$$V_3 = V_2 + \frac{1}{2}z_{21}^2$$

Then the derivative of V_3 is given by:

$$\dot{V}_3 = \dot{V}_2 + z_{21}\dot{z}_{21}$$

$$=-2a_{11}z_{11}^2-(a_{11}+a_{22})z_{12}^2+z_{21}[-a_{12}x_{11}-(a_{11}+a_{22})z_{21}-a_{21}x_{22}+b_{11}x_{11}z_{21}+ b_{12}z_{21}^2+b_{21}x_{11}x_{22}+b_{22}z_{21}x_{22}-c_{21}+u_3]$$

Also, if:

$$u_3=a_{12}x_{11}+a_{21}x_{22}-b_{11}x_{11}z_{21}-b_{12}z_{21}^2-b_{21}x_{11}x_{22}-b_{22}z_{21}x_{22}+c_{21} \quad (21)$$

and letting $\alpha_3(z_{11},z_{12},z_{13})=0,$ then:

$$\dot{V}_3=-2a_{11}z_{11}^2-(a_{11}+a_{22})z_{12}^2-(a_{11}+a_{22})z_{21}^2$$

Step 4: Define the variable $z_{22}=x_{22}-\alpha_3(z_{11},z_{22},z_{13})$ and hence we can rewrite the last equation of system (15) as:

$$\dot{z}_{22}=-a_{12}x_{12}-2a_{22}z_{22}-a_{12}x_{21}+b_{11}x_{12}x_{21}+b_{12}x_{21}z_{22}+b_{21}x_{12}z_{22}+b_{22}z_{22}^2-c_{22}+u_4$$

choose the fourth Lf:

$$V_4=V_3+\frac{1}{2}z^2{}_{22}$$

and hence the time derivative of V_4 is given by:

$$\dot{V}_4=\dot{V}_3+z_{22}\dot{z}_{22}$$

$$=-2a_{11}z_{11}^2-(a_{11}+a_{22})z_{12}^2-(a_{11}+a_{22})z_{21}^2+z_{22}(-a_{12}x_{12}-2a_{22}z_{22}-a_{12}x_{21}+ b_{11}x_{12}x_{21}+b_{12}x_{21}z_{22}+b_{21}x_{12}z_{22}+b_{22}z_{22}^2-c_{22}+u_4)$$

If:

$$u_4=a_{12}x_{12}+a_{12}x_{21}-b_{11}x_{12}x_{21}-b_{12}x_{21}-b_{21}x_{12}z_{22}-b_{22}z_{22}^2+c_{22} \quad (22)$$

then:

$$\dot{V}_4=-2a_{11}z_{11}^2-(a_{11}+a_{22})z_{12}^2-(a_{11}+a_{22})z_{21}^2-2a_{22}z_{22}^2$$

Finally, substitute u_1, u_2, u_3 and u_4 given by equations (18), (20), (21) and (22) back in system (14), where: $z_{11}=x_{11},\ z_{12}=x_{12}-\alpha_1(z_{11}),\ z_{21}=x_{21}-\alpha_2(z_{11}z_{12}),\ z_{22}=x_{22}-\alpha_3(z_{11},z_{22},z_{13})$ and so we get the feedback control functions:

$$\left.\begin{aligned}
u_1&=-b_{11}x_{11}^2+a_{21}x_{21}+a_{21}x_{12}-b_{12}x_{11}x_{21}-b_{21}x_{11}x_{12}-b_{22}x_{12}x_{21}+c_{11}\\
u_2&=a_{12}x_{22}+a_{12}x_{11}-b_{11}x_{12}x_{11}-b_{12}x_{11}x_{22}-b_{21}x_{12}^2-b_{22}x_{12}x_{22}+c_{12}\\
u_3&=a_{12}x_{11}+a_{21}x_{22}-b_{11}x_{11}x_{21}-b_{12}x_{21}^2-b_{21}x_{11}x_{22}-b_{22}x_{21}x_{22}+c_{21}\\
u_4&=a_{12}x_{12}+a_{12}x_{21}-b_{11}x_{12}x_{21}-b_{12}x_{21}-b_{21}x_{12}z_{22}-b_{22}x_{22}^2+c_{22}
\end{aligned}\right\} \quad (23)$$

Then we get the following transformed system of ODE's related or equivalent to (14):

$$\left.\begin{aligned}
\dot{x}_{11}&=-2a_{11}x_{11}\\
\dot{x}_{12}&=-(a_{11}+a_{22})x_{12}\\
\dot{x}_{21}&=-(a_{11}+a_{22})x_{21}\\
\dot{x}_{22}&=-2a_{22}x_{22}
\end{aligned}\right\} \quad (24)$$

System (24) may be written in matrix form as:

$$\begin{pmatrix}\dot{x}_{11}\\ \dot{x}_{12}\\ \dot{x}_{21}\\ \dot{x}_{22}\end{pmatrix}=\begin{pmatrix}-2a_{11} & 0 & 0 & 0\\ 0 & -(a_{11}+a_{22}) & 0 & 0\\ 0 & 0 & -(a_{11}+a_{22}) & 0\\ 0 & 0 & 0 & -2a_{22}\end{pmatrix}\begin{pmatrix}x_{11}\\ x_{12}\\ x_{21}\\ x_{22}\end{pmatrix}$$

or:

$$\dot{X}=AX$$

The matrix A is a diagonal matrix and hence have the following eigenvalues:

$$\lambda_1=-2a_{11},\ \lambda_2=-(a_{11}+a_{22})=\lambda_3 \text{ and } \lambda_4=-2a_{22}$$

Therefore, if $a_{11}>0$ and $a_{22}>0,$ then $\lambda_1,\ \lambda_1,\ \lambda_1$ and λ_4 will be negative and then the system is asymptotically stable, i.e., using the feedback control functions (23) imply that system (14) is asymptotically stable and the solution is given by:

$$\begin{aligned}x_{11}(t)&=e^{-2a_{11}t}\\ x_{12}(t)&=e^{-(a_{11}+a_{22})t}\\ x_{21}(t)&=e^{-(a_{11}+a_{22})t}\\ x_{22}(t)&=e^{-2a_{22}t}\end{aligned} \qquad (24)$$

4. Illustration Examples

As an illustration for the above approach the following examples will be given:

Example (1):

Consider the Riccati matrix differential equation (14) with $A=\begin{bmatrix}2 & 1\\ 0 & 4\end{bmatrix}$; $B=\begin{bmatrix}2 & 4\\ 4 & 0\end{bmatrix}$; and $C=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$. Hence, using the above suggested approach, the solution that given by (24) will become:

$$x_{11}(t)=e^{-2a_{11}t}=e^{-4t};\ x_{12}(t)=e^{-(a_{11}+a_{22})t}=e^{-6t};$$
$$x_{21}(t)=e^{-(a_{11}+a_{22})t}=e^{-6t};\text{ and } x_{22}(t)=e^{-2a_{22}t}=e^{-8t}$$

which may be represented in figure (1), and it is very clearly that the Riccati matrix differential equation is stabilizable.

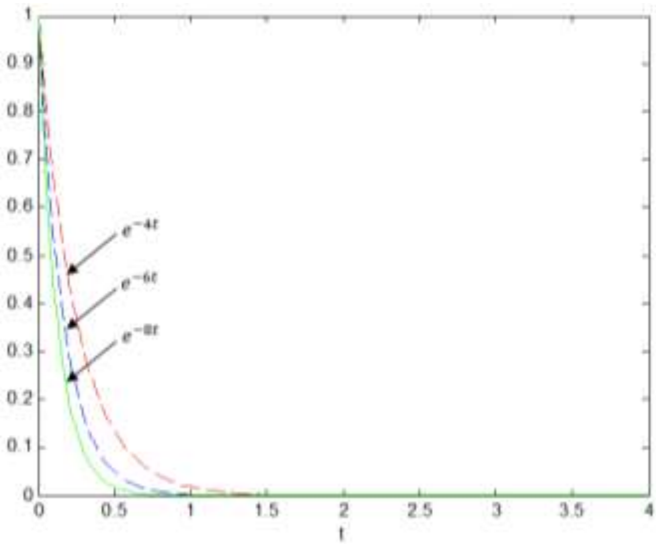

Figure 1. The asymptotic solutions of example (1).

Example (2):
Consider the Riccati matrix differential equation (14) with $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 14 \\ 14 & 2 \end{bmatrix}$; and $C = \begin{bmatrix} 10 & 2 \\ 2 & 40 \end{bmatrix}$.

Hence using the above suggested approach, the solution given by (24) will become:

$x_{11}(t) = e^{-2a_{11}t} = e^{-4t}$; $x_{12}(t) = e^{-(a_{11}+a_{22})t} = e^{-3t}$; $x_{21}(t) = e^{-(a_{11}+a_{22})t} = e^{-3t}$; and $x_{22}(t) = e^{-2a_{22}t} = e^{-2t}$

which may be represented in figure (2), and it is also stabilizable.

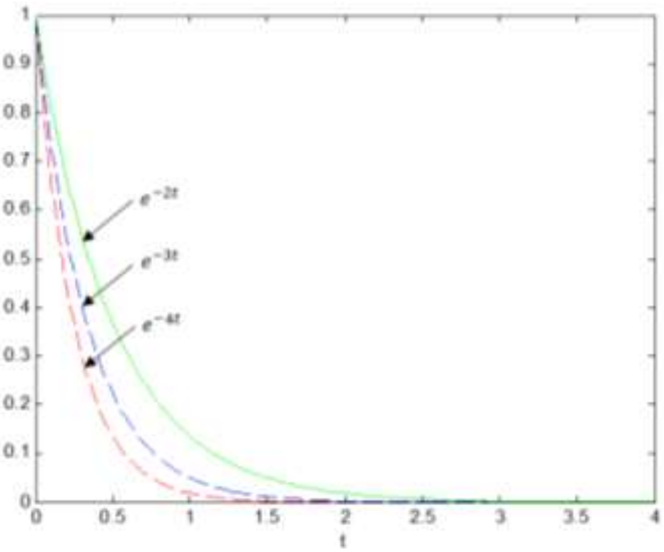

Figure 2. The asymptotic solutions of example (2).

5. Conclusions

The results showed an improvement in the stability of the system after applying the BS control technique, and a reliable time performance is obtained. Noteworthy, that for systems with multiple inputs it is advisable to control them making use of this technique. Hence, the flexibility can be obtained in designing the control input law during the simulation. Recursively, the non-linearity which ensures asymptotic stability is ignored based on BS controller uses Lfs in each integrator level.

Acknowledgment
The author would to extend his thanks and appreciation to the Asst. Prof. Fadhel S. Fadhel and to the M.Sc. student Saja Faiq Nouman from the Department of Mathematics and Computer Applications, Faculty of Science, Al-Nahrain University, Baghdad-Iraq; for their efforts in reviewing and verifying the contents of this paper.

References

[1] Krstic, M.; Smyshlyaev, A.; "Boundary Control of PDEs: A Course on BS Designs", SIAM, USA, 2008.
[2] Krstic, M.; Kanellakopoulos, I.; Kokotovic, P. V.; "Non-linear and Adaptive Control Design", Wiley and Sons, Inc., New York, 1995.
[3] Sepulchre, R.; Jankovic, M.; Kokotovic, P.; "Constructive Non-linear Control", Springer-Verlag Berlin, 1997.
[4] Fossen, T. I.; Berge, S. P.; "Non-linear Vectorial BS Design for Global Exponential Tracking of Marine Vessels in the Presence of Actuator Dynamics", In: Proceedings of the IEEE Conference on Decision and Control (CDC'97), San Diego, CA., 4237–4242, 1997.
[5] Krstic, M.; Deng, H.; "Stabilization of Non-linear Uncertain Systems", Springer-Verlag. Berlin, 1998.
[6] Loria, A.; Fossen, T. I.; Teel, A.; "UGAS and ULES of Non Autonomous Systems: Applications to Integral Action Control of Ships and Manipulators", In: Proceedings of the 5th European Control Conference (ECC'99), Karlsruhe, Germany, 1999.
[7] Fossen, T. I.; Loria, A.; Teel, A.; "A Theorem for UGAS and ULES of (Passive) Nonautonomous Systems: Robust Control of Mechanical Systems and Ships", Int. J. Rob. Nonlin. Cont. 11, 95–108, 2001.
[8] Smyshlyaev, A.; Krstic, M.; "Closed form boundary state feedbacks for a class of 1D partial integro-differential equations", IEEE Trans. Auto. Cont. 49(12), pp. 2185–2202, 2004.
[9] Vazquez, R.; Krstic, M.; "A closed-form feedback controller for stabilization of the linearized 2D Navier-Stokes Poiseuille flow", IEEE Trans. Auto. Cont. 52, 2298–2312, 2007.
[10] Krstic, M.; "Lyapunov tools for predictor feedbacks for delay systems: Inverse optimality and robustness to delay mismatch", Automatic 44, 2930-2935, 2008.
[11] Krstic, M.; "On using least-squares updates without regress or filtering in identification and adaptive control of non-linear systems", Automatic, Vol. 45, pp. 731-735, 2009.
[12] Krstic, M.; "Input delay compensation for forward complete and feed forward non-linear systems", IEEE Trans. Auto. Cont. 55, 287-303, 2010.
[13] Krstic, M.; "Lyapunov stability of linear predictor feedback for time-varying input delay", IEEE Trans. Auto. Cont. 55, 554-559, 2010.
[14] Krstic, M.; Bekiaris-Liberis, N.; "Non-linear stabilization in infinite dimension", Ann. Rev. Cont. 37, 220-231, 2013.