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Approximate Solution of Multi-Term Fractional Order Delay Differential Equations Using Homotopy Perturbation Method

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Abstract

In this paper the approximate solution of the non-linear equations of multi-term fractional order delay differential equations by using the homotopy perturbation method is considered. The fractional order derivative is communicated in the Caputo sense. In this methodology, the solutions are found in the form of a convergent power series with easily computed components. Finally, some examples are given to illustrate the obtained results, and then a comparison between the exact and the approximate results were given and they are presented in order to show the reliability and the accuracy of the proposed method.

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1. Introduction

The non-linear delay differential equation of multi-term fractional order derivative is given as

$$^{C}D_t^{\alpha} y(t) = f\left(t, y(t), {}^{C}D_t^{\beta} y\left(\Psi(t)\right)\right)$$

$$y^{(i)}(0) = y_0^i, i = 0,1,2, \dots, n-1$$

where $^{C}D_t^{\alpha}$ and $^{C}D_t^{\beta}$ are the Caputo fractional order derivatives $n-1 < \alpha \leq n, n \in \mathbb{N}, 0 < \beta < \alpha,$ $\Psi(t)$ is a continuous delay function, f is a nonlinear operator, and y_0^i are constants. Fractional order delay differential equations (DDEs) may be considered as a hybrids equation which encompass a combination between ordinary differential equations of fractional orders with time delay. Fractional order derivatives are used probably to model problems with memory effects that are considered non-local in nature in opposite to ordinary derivatives while time delays are used on modeling problems with history of an early state. Therefore, so many real-life problems may be modeled more accurate utilizing fractional derivatives and time delays [1-4]. An increasing interest in the theory of fractional order DDEs are payed in these days which is due to the fact that system dynamics may be expressed accurately using such type of equations in science and engineering. A lot of scientists' attention has been captured in modeling time delays in real world systems that may be recognized and implemented in everywhere [5]. The homotopy perturbation method (HPM) is a semi-analytical technique for solving a complicated linear and nonlinear differential equations, and that method it was discovered by J. He in 1999 which is, actually, a conjugation of the conventional perturbation method and homotopy in topology. In HPM, the solution is given by the summation of an infinite series, which usually converges rapidly to the exact solution [1]. The HPM is utilized to nonlinear oscillators [12], singular nonlinear differential equations [10], Volterra's integro-differential equation [11], nonlinear wave equations [9], branching of delay- differential equations [16], initial value problems [14], boundary value problems [13], and nonlinear coupled equations [10-15]. Moreover, in most cases the HPM is lead to every quickly convergence of the solution series [3]. Finally, we will be employing the HPM on the above equation to find the approximate solution then compared with the exact solution.

2. Preliminaries:

In this section, we mention the following basic definitions and some properties of fractional order utilized in the present paper.

Definition (1), [7]: The left Riemann-Liouville fractional integral operator of order $\alpha > 0$ are defined respectively by

$$^{R}I_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds$$

where Γ is the classical gamma function and $^{R}I_t^{0} y(t) = y(t)$.

Definition (2), [8]: The left Riemann-Liouville fractional derivative operator of order $\alpha > 0$. is defined as:

$$^{R}D_t^{\alpha} y(t) = \frac{1}{\Gamma(n-1)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} y(s) ds$$

where n is an integer and $n-1 < \alpha \leq n, n \in \mathbb{N}$.

Definition (3), [6]: For n to be the smallest integer that exceeds α, the Caputo fractional order derivative of order $\alpha > 0$, is defined as:

$$^{C}D_t^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds \\ \qquad\qquad , \text{for } n-1 < \alpha \leq n, n \in \mathbb{N} \\ \dfrac{d^n}{dt^n} y(t), \text{for } n = \alpha \end{cases}$$

Some properties concerning to fractional integrations and derivatives order for $\alpha > 0, n-1 < \alpha \leq n, n \in \mathbb{N}$, are listed beneath:

1. $^{C}D_t^{\alpha}\, {}^{R}I_t^{\alpha} y(t) = y(t)$.
2. $^{R}I_t^{\alpha}\, {}^{C}D_t^{\alpha} y(t) = y(t) - \sum_{p=0}^{n-1} y^{(p)}(0^+) \frac{t^p}{p!}$, $t > 0$.
3. $^{R}I_t^{\alpha} t^{v} = \frac{\Gamma(v+1)}{\Gamma(v+\alpha+1)} t^{v+\alpha}, v > -1, t > 0$.
4. $^{C}D_t^{\alpha} \sum_{i=0}^{k} c_i y(t) = \sum_{i=0}^{k} c_i\, {}^{C}D_t^{\alpha} y(t)$, c_i is a constant.
5. If $y^{(p)}(0) = 0$, $p = 0,1,\ldots,n-1$, $\alpha + \beta \leq n$, $0 < \beta < \alpha$, $n \in \mathbb{N}$, then:
$$^{C}D_t^{\alpha}\, {}^{R}I_t^{\beta} y(t) = {}^{R}I_t^{\beta}\, {}^{C}D_t^{\alpha} y(t)$$
$$= {}^{C}D_t^{\alpha-\beta} y(t) = {}^{R}I_t^{\beta-\alpha} y(t).$$

3. Homotopy Perturbation Method [2]:

The fundamental idea of the HPM, we deal with the following non-linear equation:

$$\mathcal{A}(y) = g(t), t \in \Omega \qquad \ldots(1)$$

where $\mathcal{A}$ is any operator, g is a known function of t. The operator $\mathcal{A}$ can generally speaking be divided into two parts $\mathcal{L}$ and $\mathcal{N}$ where $\mathcal{L}$ is a linear operator, and $\mathcal{N}$ is a non-linear operator. Therefore equation (1) can be rewritten as follows:

$$\mathcal{L}(y) + \mathcal{N}(y) = g(t)$$

According to [2], we can construct a homotopy $u(t,p): \Omega \times [0,1] \longrightarrow \mathbb{R}$ which satisfies the homotopy equation:

$$\mathcal{H}(u,p) = (1-p)\big[\mathcal{L}(u) - \mathcal{L}(y_0) + p[\mathcal{A}(u) - g(t)]\big] = 0$$

or

$$\mathcal{H}(u,p) = \mathcal{L}\big(u(t)\big) - \mathcal{L}\big(y_0(t)\big) + p\mathcal{L}\big(y_0(t)\big) + p\big[\mathcal{N}\big(u(t)\big) - g(t)\big] = 0 \qquad \ldots(2)$$

where $p \in [0,1]$, is an embedding parameter and y_0 is an initial approximation of the solution of equation (1). Obviously, from equation (2) we have:

$$\mathcal{H}(u,0) = \mathcal{L}(u) - \mathcal{L}(y_0) = 0$$
$$\mathcal{H}(u,1) = \mathcal{A}(u) - g(t) = 0$$

The variation practicability of p from 0 to 1 is just that of $u(t,p)$ from $y_0(t)$ to $y(t)$. In topology, this is called deformation, and $\mathcal{L}(u) - \mathcal{L}(y_0), \mathcal{A}(u) - g(t)$ are called homotopic. Suppose that the solution of equation (1) can be written as a power series in p as follows:

$$u(t,p) = \sum_{i=0}^{\infty} p^i u_i(t) \qquad \ldots(3)$$

where $u_0, u_1, \ldots$ are the unknown functions that must be determined. By setting $p = 1$ in equation (3) one can obtain:

$$y(t) = \lim_{p\to 1} u(t,p) = \sum_{i=0}^{\infty} u_i(t) \qquad \ldots(4)$$

Remark:

The infinite series given by equation (4) is convergent for most cases, however, the convergence rate depends on the non-linear operator $\mathcal{A}$. The following opinions are suggested by He J. to ensure that the convergence of the infinite series given by equation (4):

1. The second derivative of $\mathcal{N}(u)$ with respect to u must be small.
2. The norm of $\mathcal{L}^{-1}\frac{\partial \mathcal{N}}{\partial u}$ must be smaller than one.

4. The Methodology:

Consider the non-linear multi-term fractional order DDE, in this section, we will propose the HPM to find the approximate solution.

$$^{C}D_t^{\alpha} y(t) = g(t) + f\big(y(t), {}^{C}D_t^{\beta} y(\Psi(t))\big),$$

$n-1<\alpha\le n, 0<\beta<\alpha, n\in\mathbb{N}, t\in[0,1]$ ...(5)

$y^{(i)}(0)=y_0^i, i=0,1,2,\dots,n-1$ ...(6)

Consider the following non-linear differential equation:

$\mathcal{A}(y)=g(t)\ t\in[0,1]$

where $\mathcal{A}$ is any differential operator, g is a known function of t

$\mathcal{A}(y)-g(t)=0$ ...(7)

and $\mathcal{A}(y)$ can be divided into two parts $\mathcal{L}(y)$ and $\mathcal{N}(y)$ such as equation (7) becomes:

$\mathcal{L}(y)+\mathcal{N}(y)-g(t)=0$

where

$\mathcal{L}(y)={}^{C}D_t^{\alpha}$

$\mathcal{N}(y)=f\left(y(t,p),{}^{C}D_t^{\beta}y(\Psi(t,p))\right)$

According to [13], we construct a homotopy $u(t,p):[0,1]\times[0,1]\longrightarrow\mathbb{R}$ which satisfies:

$$\mathcal{H}(u,p)=(1-p)[u(t,p)-y_0(t)]+p[\mathcal{A}(u)-g(t)]=0$$

$$\mathcal{H}(u,p)=(1-p)[u(t,p)-y_0(t)]+p[{}^{C}D_t^{\alpha}u(t,p)-f\left(u(t,p),{}^{C}D_t^{\beta}u(\Psi(t,p))\right)-g(t)] \quad ...(8)$$

where $p\in[0,1]$, and y_0 is the initial approximation for the solution of equation (5).

By using equation (8) it follows that:

$\mathcal{H}(u,0)=u(t,0)-y_0(t)=0$

$\mathcal{H}(u,1)={}^{C}D_t^{\alpha}u(t,1)-f\left(u(t,1),{}^{C}D_t^{\beta}u(\Psi(t,1))\right)-g(t)=0$

and the variation practicability of p from 0 to1 is just that of $u(t,p)$ from $y_0(t)$ to $y(t)$. In topology, this called deformation, and

${}_0^{C}D_t^{\alpha}(u)-{}^{C}D_t^{\alpha}(y_0)$,

${}^{C}D_t^{\alpha}(u)+f\left(u(t),{}^{C}D_t^{\beta}u(\Psi(t))\right)-g(t)$

are called homotopic.

Next, we assume that the solution of equation (8) can be expressed as

$u(t,p)=\sum_{i=0}^{\infty}p^iu_i(t,p)$ ...(9)

Therefore, the approximated solution of the equation (5) can be obtained as follows:

$y(t)=\lim_{p\to1}u(t,p)=\sum_{i=0}^{\infty}u_i(t)$ ...(10)

By substituting the approximated solution given by equation (9) into equation (8) one can get:

$$\sum_{i=0}^{\infty}p^iu_i(t)-y_0(t)+py_0(t)+p\left[\sum_{i=0}^{\infty}p^if\left(u_i(t),{}^{C}D_t^{\beta}u_i(\Psi(t))\right)-g(t)\right]=0.$$

Then by equating the terms with identical powers of p one can have:

$p^0: {}^{C}D_t^{\alpha}u_0(t)-{}^{C}D_t^{\alpha}y_0(t)=0$

$p^1: {}^{C}D_t^{\alpha}u_1(t)+{}^{C}D_t^{\alpha}y_0(t)+f\left(u_0(t),{}^{C}D_t^{\beta}u_0(\Psi(t))\right)-g(t)=0$

$p^2: {}^{C}D_t^{\alpha}u_2(t)+f\left(u_1(t),{}^{C}D_t^{\beta}u_1(\Psi(t))\right)=0$

$p^j: {}^{C}D_t^{\alpha}u_j(t)+f\left(u_{j-1}(t),{}^{C}D_t^{\beta}u_{j-1}(\Psi(t))\right)=0, j=3,4,\dots$

Consequently, by applying the operators ${}^{R}I_t^{\alpha}$ to the above differential equations. Then we get $u_0,u_1,u_2,\dots$ and by using (10) we get the approximate solution.

5. Illustrative Examples:

Some illustrative examples are given in this section, in order to show the adequacy and effectiveness of the proposed method.

Example (1): Consider the linear multi-term fractional order DDE

$${}^{C}D_t^{1.5}y(t)+{}^{C}D_t^{0.5}y(\sin t)=g(t), 0\le t\le1$$

subject to the initial values $y(0)=0, y'(0)=0$ and $\alpha=1.5,\beta=0.5$ with the exact solution $y(t)=t^2$, where

$$g(t)=\frac{\Gamma(3)}{\Gamma(1.5)}t^{0.5}+\frac{\Gamma(3)}{\Gamma(2.5)}t^{1.5}-\frac{\Gamma(5)}{3\Gamma(4.5)}t^{3.5}+\frac{2\Gamma(7)}{45\Gamma(6.5)}t^{5.5}$$

First choose the initial approximation $y_0(t)$ to be:

$y_0(t)=0$

$\mathcal{L}(y)={}^{C}D_t^{1.5}y(t), \mathcal{N}(y)=0, p\in[0,1],$

$t\in[0,1]$

$$\mathcal{H}(u,p)=(1-p)[{}^{C}D_t^{1.5}\,u(t)-{}^{C}D_t^{1.5}\,y_0(t)] + p[{}^{C}D_t^{1.5}u(t)+{}^{C}D_t^{0.5}u(\sin t) - g(t)]=0$$

$$\sum_{i=0}^{\infty}p^{i\,C}D_t^{1.5}u_i(t)-{}^{C}D_t^{1.5}y_0(t)+ \quad p^{C}D_t^{1.5}y_0(t)+ p[\sum_{i=0}^{\infty}p^{i\,C}D_t^{0.5}u_i(sint)-g(t)]=0$$

$$p^0:\,{}^{C}D_t^{1.5}u_0(t)-{}^{C}D_t^{1.5}y_0(t)=0$$

$$p^1:\,{}^{C}D_t^{1.5}u_1(t)+{}^{C}D_t^{1.5}y_0(t) + {}^{C}D_t^{0.5}u_0\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}\right)-g(t) =0$$

$$p^2:\,{}^{C}D_t^{1.5}u_2(t)+{}^{C}D_t^{0.5}u_1\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}\right)=0$$

$$p^3:\,{}^{C}D_t^{1.5}u_3(t)+{}^{C}D_t^{0.5}u_2\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}\right)=0$$

$$\vdots$$

Consequently, by applying the operators ${}^{R}I_t^{1.5}$ to the above equations. For simplicity, let

$$u_0(t)=y_0(t)=0$$

$$u_1(t)=t^2+\frac{2}{\Gamma(4)}t^3-\frac{\Gamma(5)}{3\Gamma(6)}t^5+\frac{2\Gamma(7)}{45\Gamma(8)}t^7$$

$$u_2(t)= -\left[\frac{t^3}{315}(2t^4-21t^2+105)+\frac{2t^2}{\Gamma(4)}\left(\frac{13t^4-80t^4+240}{960}\right)-\frac{\Gamma(5)}{3\Gamma(6)}t^6\left(\frac{23t^4-75t^2+120}{720}\right)+\frac{2\Gamma(7)}{45\Gamma(8)}t^8\left(\frac{77t^4-168t^2+180}{1440}\right)\right]$$

$$\vdots$$

So

$$y(t)=u_0(t)+\sum_{i=1}^{\infty}u_i(t)$$

Table (1) presents a comparison between the approximate solution and the exact solution of example (1) utilized HPM up to three terms as well as the absolute error evaluated at certain discrete points, over the unit interval [0,1]. It is noticed that we are made an approximation to *sint* using Maclurian series expansion up to three terms.

Table 1. Exact and approximate results using HPM of example (1).

t	Approximate solution	Exact solution	Absolute error
0	0	0	0
0.1	0.0100	0.01	1.382×10^{-7}
0.2	0.0400	0.04	3.505×10^{-6}
0.3	0.0900	0.09	2.002×10^{-5}
0.4	0.1600	0.16	5.989×10^{-5}
0.5	0.2501	0.25	1.24×10^{-4}
0.6	0.3602	0.36	2.167×10^{-4}
0.7	0.4904	0.49	4.197×10^{-4}
0.8	0.6410	0.64	1.077×10^{-3}
0.9	0.8131	0.81	3.146×10^{-3}
1	1.0087	1	8.735×10^{-3}

Example (2): Consider the linear multi-term fractional order DDE ${}^{C}D_t^{1.4}y(t)+{}^{C}D_t^{0.3}y\left(\frac{t}{2}\right)+y(t)=g(t)$ subject to the initial values $y(0)=0, y'(0)=0, 0\le t\le 1$, and $\alpha=1.4, \beta=0.3$ with the exact solution $y(t)=t^2+t^5$ where $g(t)=t^2+t^5+\frac{\Gamma(3)}{\Gamma(1.6)}t^{0.6}+\frac{\Gamma(6)}{\Gamma(4.6)}t^{3.6}+\frac{\Gamma(3)}{4\Gamma(2.7)}t^{1.7}+\frac{\Gamma(6)}{32\Gamma(5.7)}t^{4.7}$

First choose the initial approximation $y_0(t)$ to be:

$$y_0(t)=0$$

$$\mathcal{L}(y)={}^{C}D_t^{1.4}y(t), \mathcal{N}(y)=0, p\in[0,1]$$

$$\mathcal{H}(u,p)=(1-p)[{}^{C}D_t^{1.4}\,u(t)-{}^{C}D_t^{1.4}\,y_0(t)] + p\left[{}^{C}D_t^{1.4}u(t)+{}^{C}D_t^{0.3}u\left(\frac{t}{2}\right)+u(t) - g(t)\right]=0$$

$$\sum_{i=0}^{\infty}p^{i\,C}D_t^{1.4}u_i(t)-{}^{C}D_t^{1.4}y_0(t)+p^{C}D_t^{1.4}y_0(t)+ p\left[\sum_{i=0}^{\infty}p^{i\,C}D_t^{0.3}u_i\left(\frac{t}{2}\right)+\sum_{i=0}^{\infty}p^i\,u(t)-g(t)\right]=0$$

$$p^0:\,{}^{C}D_t^{1.4}u_0(t)-{}^{C}D_t^{1.4}y_0(t)=0$$

$$p^1:\,{}^{C}D_t^{1.4}u_1(t)+{}^{C}D_t^{1.4}y_0(t) + {}^{C}D_t^{0.3}u_0\left(\frac{t}{2}\right)+u_0(t)-g(t)=0$$

$$p^2:\,{}^{C}D_t^{1.4}u_2(t)+{}^{C}D_t^{0.3}u_1\left(\frac{t}{2}\right)+u_1(t)=0$$

$$p^3:\,{}^{C}D_t^{1.4}u_3(t)+{}^{C}D_t^{0.3}u_2\left(\frac{t}{2}\right)+u_2(t)=0$$

$$\vdots$$

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Consequently, by applying the operators ${}^{R}I_t^{1.4}$ to the above equations. For simplicity, let

$$u_0(t) = y_0(t) = 0$$

$$u_1(t) = t^2 + t^5 + \frac{\Gamma(3)}{4\Gamma(4.1)}t^{3.1} + \frac{\Gamma(6)}{32\Gamma(7.1)}t^{6.1} + \frac{\Gamma(3)}{\Gamma(4.4)}t^{3.4} + \frac{\Gamma(6)}{\Gamma(7.4)}t^{6.4}$$

$$u_2(t) = -0.00955t^{4.5} - 0.00431t^{6.1} - 0.02335t^{4.8} - 0.07339t^{3.1} - 0.07785t^{6.4} - 0.00026t^{7.5} - 0.00455t^{7.8} - 0.19731t^{3.4} - \Gamma\left(\frac{2}{5}\right)0.00004t^{7.5} - \Gamma\left(\frac{1}{10}\right)7.59896\times 10^{-7}t^{7.2} - \Gamma\left(\frac{1}{10}\right)0.00018t^{4.2} - \Gamma\left(\frac{2}{5}\right)0.00163t^{4.5}$$

$$\vdots$$

So

$$y(t) = u_0(t) + \sum_{i=1}^{\infty} u_i(t)$$

Table (2) presents a comparison between the approximate solution and the exact solution of example (2) utilized HPM up to three terms as well as the absolute error evaluated at certain discrete points, over the unit interval [0,1].

Table 2. Exact and approximate results using HPM of example (2)

t	Approximate solution	Exact solution	Absolute error
0	0	0	0
0.1	0.0100	0.01	3.531×10^{-9}
0.2	0.0403	0.04	2.187×10^{-7}
0.3	0.0924	0.092	2.473×10^{-6}
0.4	0.1702	0.17	1.391×10^{-5}
0.5	0.2813	0.281	5.335×10^{-5}
0.6	0.4379	0.438	1.607×10^{-4}
0.7	0.6584	0.658	4.1×10^{-4}
0.8	0.9686	0.968	9.273×10^{-4}
0.9	1.4024	1.4	1.915×10^{-3}
1	2.0036	2	3.682×10^{-3}

Example (3): Consider the nonlinear multi-term fractional order DDE

$${}^{C}D_t^{1.9}y(t) + {}^{C}D_t^{0.1}y\left(\frac{t}{2}\right)e^{y(t)} = g(t)$$

Subject to the initial values $y(0) = 0\,, y'(0) = 0, 0 \le t \le 1$, and $\alpha = 1.9, \beta = 0.1$ with the exact solution $y(t) = t^2$, where:

$$g(t) = \frac{\Gamma(3)}{\Gamma(1.1)}t^{0.1} + \frac{\Gamma(3)}{4\Gamma(2.9)}t^{1.9} + \frac{\Gamma(5)}{4\Gamma(4.9)}t^{3.9} + \frac{\Gamma(7)}{16\Gamma(4.9)}t^{5.9}$$

First choose the initial approximation $y_0(t)$ to be:

$$y_0(t) = 0$$

$$\mathcal{L}(y) = {}^{C}D_t^{1.9}y(t), \mathcal{N}(y) = {}^{C}D_t^{0.1}y\left(\frac{t}{2}\right)e^{y(t)},$$

$$p \in [0,1], t \in [0,1]$$

$${}^{C}D_t^{1.9}u(t) - {}^{C}D_t^{1.9}y_0(t) + p\,{}^{C}D_t^{1.9}y_0(t) + p\left[{}^{C}D_t^{0.1}y\left(\frac{t}{2}\right)e^{y(t)} - g(t)\right] = 0$$

$$\sum_{i=0}^{\infty} p^i\,{}^{C}D_t^{1.9}u_i(t) - {}^{C}D_t^{1.9}y_0(t) + p\,{}^{C}D_t^{1.9}y_0(t) + p\left[{}^{C}D_t^{0.1}\left(\sum_{i=0}^{\infty} p^i u_i\left(\frac{t}{2}\right)\right)\left(\sum_{i=0}^{\infty} p^i e^{y_i(t)}\right) - g(t)\right] = 0$$

$$p^0: {}^{C}D_t^{1.9}u_0(t) - {}^{C}D_t^{1.9}y_0(t) = 0$$

$$p^1: {}^{C}D_t^{1.9}u_1(t) + {}^{C}D_t^{1.9}y_0(t) + {}^{C}D_t^{0.1}u_0\left(\frac{t}{2}\right)e^{u_0(t)} - g(t) = 0$$

$$p^2: {}^{C}D_t^{1.9}u_2(t) + {}^{C}D_t^{0.1}u_0\left(\frac{t}{2}\right)e^{u_0(t)} + {}^{C}D_t^{0.1}u_1\left(\frac{t}{2}\right)e^{u_0(t)} = 0$$

$$p^3: {}^{C}D_t^{1.9}u_3(t) + {}^{C}D_t^{0.1}u_0\left(\frac{t}{2}\right)e^{u_3(t)} + {}^{C}D_t^{0.1}u_1\left(\frac{t}{2}\right)e^{u_1(t)} + {}^{C}D_t^{0.1}u_2\left(\frac{t}{2}\right)e^{u_0(t)} = 0$$

$$\vdots$$

Consequently, by applying the operators ${}^{R}I_t^{1.9}$ to the above equations. For simplicity, let

$$u_0(t) = y_0(t) = 0$$

$$u_1(t) = t^2 + \frac{\Gamma(3)}{4\Gamma(4.8)}t^{3.8} + \frac{\Gamma(5)}{4\Gamma(6.8)}t^{5.8} + \frac{\Gamma(7)}{16\Gamma(8.8)}t^{7.8}$$

$$u_2(t) = -0.02803t^{3.8} - \Gamma\left(\frac{4}{5}\right)0.000005t^{7.6} - \Gamma\left(\frac{4}{5}\right)1.21497\times 10^{-7}t^{9.6} - \Gamma\left(\frac{4}{5}\right)0.00008t^{5.6}$$

$$\vdots$$

So

$$y(t) = u_0(t) + \sum_{i=1}^{\infty} u_i(t)$$

Table (3) presents a comparison between the approximate solution and the exact solution of example (3) utilized HPM up to two terms as well as the absolute error evaluated at certain discrete points, over the unit interval [0,1]. It is noticed that we are made an approximation to exponential using Maclurian series expansion up to three terms.

Table 3. Exact and approximate results using HPM of example (3)

t	Approximate solution	Exact solution	Absolute error
0	0	0	0
0.1	0.0100	0.01	1.891×10^{-8}
0.2	0.0400	0.04	1.06×10^{-6}
0.3	0.0900	0.09	1.122×10^{-5}
0.4	0.1600	0.16	6.016×10^{-5}
0.5	0.2502	0.25	2.223×10^{-4}
0.6	0.3606	0.36	6.5×10^{-4}
0.7	0.4916	0.49	1.618×10^{-3}
0.8	0.6435	0.64	3.58×10^{-3}
0.9	0.8172	0.81	7.248×10^{-3}
1	1.0136	1	0.014

Example (4): Consider the nonlinear multi-term fractional order DDEs

$$ {}^{C}D_t^{3.25}y(t) + {}^{C}D_t^{0.25}y^2\left(\frac{t}{2}\right) = g(t) $$

subject to the initial values $y(0)=0, y'(0)=0, y''(0)=0, y'''(0)=0, 0\le t\le 1$ and $\alpha=3.25, \beta=0.25$ with the exact solution $y(t)=t^4$, where:

$$ g(t) = \frac{\Gamma(5)}{\Gamma(1.75)}t^{0.75} + \frac{315}{2\Gamma(8.75)}t^{7.75} $$

First choose the initial approximation y_0 to be:

$$ y_0(t) = 0 $$

$$ \mathcal{L}(y) = {}^{C}D_t^{3.25}y(t), \mathcal{N}(y) = {}^{C}D_t^{0.25}y^2\left(\frac{t}{2}\right), $$

$$ p\in[0,1], t\in[0,1] $$

$$ {}^{C}D_t^{3.25}u(t) - {}^{C}D_t^{3.25}y_0(t) + p\,{}^{C}D_t^{3.25}y_0(t) + p\left[{}^{C}D_t^{0.25}y^2\left(\frac{t}{2}\right) - g(t)\right] = 0 $$

$$ \sum_{i=0}^{\infty}p^i\,{}^{C}D_t^{3.25}u_i(t) - {}^{C}D_t^{3.25}y_0(t) + p\,{}^{C}D_t^{3.25}y_0(t) + p\left[{}^{C}D_t^{0.25}\left(\sum_{i=0}^{\infty}p^i u_i\left(\frac{t}{2}\right)\right)^2\right] = 0 $$

$$ p^0: {}^{C}D_t^{3.25}u_0(t) - {}^{C}D_t^{3.25}y_0(t) = 0 $$

$$ p^1: {}^{C}D_t^{3.25}u_1(t) + {}^{C}D_t^{3.25}y_0(t) + {}^{C}D_t^{0.25}u_0^2\left(\frac{t}{2}\right) - g(t) = 0 $$

$$ p^2: {}^{C}D_t^{3.25}u_2(t) + {}^{C}D_t^{0.25}u_0\left(\frac{t}{2}\right)u_1\left(\frac{t}{2}\right) + {}^{C}D_t^{0.25}u_1\left(\frac{t}{2}\right)u_0\left(\frac{t}{2}\right) = 0 $$

$$ p^3: {}^{C}D_t^{3.25}u_3(t) + {}^{C}D_t^{0.25}u_0\left(\frac{t}{2}\right)u_2\left(\frac{t}{2}\right) + {}^{C}D_t^{0.25}u_1^2\left(\frac{t}{2}\right) + {}^{C}D_t^{0.25}u_2\left(\frac{t}{2}\right)u_0\left(\frac{t}{2}\right) = 0 $$

$$ \vdots $$

Consequently, by applying the operators ${}^{R}I_t^{3.25}$ to the above equations. For simplicity, let

$$ u_0(t) = y_0(t) = 0 $$

$$ u_1(t) = t^4 + \frac{315}{2\Gamma(12)}t^{11} $$

$$ u_2(t) = 0 $$

$$ u_3(t) = -\left[\frac{1}{3717828145022238720000}t^{25} + \frac{1}{20329959260160}t^{18} + \frac{1}{253440}t^{11}\right] $$

$$ u_4(t) = 0 $$

$$ \vdots $$

So

$$ y(t) = u_0(t) + \sum_{i=1}^{\infty}u_i(t) $$

Table (4) presents a comparison between the approximate solution and the exact solution of example (4) utilized HPM up to four terms as well as the absolute error evaluated at certain discrete points, over the unit interval [0,1].

Table 4. Exact and approximate results using HAM of example (4)

t	Approximate solution	Exact solution	Absolute error
0	0	0	0
0.1	0.0001	0.0001	0
0.2	0.0016	0.0016	0
0.3	0.0081	0.0081	0
0.4	0.0256	0.0256	0
0.5	0.0625	0.0625	0
0.6	0.1296	0.1296	0
0.7	0.2401	0.2401	0
0.8	0.4096	0.4096	0
0.9	0.6560	0.6561	7.327×10^{-15}
1	0.99999	1	4.929×10^{-14}

Conclusions

The aim of this work is to show the HPM how to get the best results while used it to solve the multi-term fractional order DDEs. Thus, it has been shown that the HPM is suitable and reliable in treating linear as well as nonlinear equations, and we see that through the examples in this paper, when compered the numerical results obtained with the exact solution and then the differences are presented in tabulated. Obviously, the HPM is so powerful and efficient technique for achieving the approximation solutions of the suggested equation.

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