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Estimating the Reliability Function of Parallel Stress-Strength Model for the Generalized Inverted Kumaraswamy Distribution

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Abstract

This paper discusses reliability of the stress-strength model, the reliability functions R_1 and R_2 for two and three components which has two and three strengths independently are exposed to a common stress respectively using Generalized inverted Kumaraswamy distribution(GIKD)with unknown shape parameter and known shape and scale parameters, after that the parameters are estimate from a stress- strength model. Estimate the reliability R_1, R_2 by two methods (ranked set sampling and Bayes). A numerical simulation study a comparison between the two estimators by Mean square error is performed. It is found that best estimator between the two estimators is Bayes estimation method.

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1. Introduction

The stress-strength model is an importance subject in reliability literature. In the statistical approach to the stress-strength model, the considerations depend on the assumption that the component strengths are independently and identically distributed and are subjected to a common stress [1].

Let X be a random variable refer to strength which follow Generalized inverted Kumaraswamy distribution (GIKD).

Let y is random variable refer to stress which follow Generalized inverted Kumaraswamy distribution (GIKD).

The system reliability using when X and Y are independent and identical.

Let X be a strength random variable subjected to a common stress Y then the reliability of the system contain one component is $R = P(X > Y)$. The reliability of a system is the probability that it is operating under stated environmental conditions [2].

The stress- strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y, when the stress applied to it exceeds its strength the component fails instantly and the component will function satisfactorily till $X > Y$ therefore $R = P(Y < X)$ is a measure of a component reliability. The stress-strength model is used in many applications in physics, in almost all areas of knowledge, especially in engineering [3].

2. The Reliability Mathematical Formula

In this model strength-stress random variables of components having strength (X_i; $i = 1,2,3$) and exposed to stress (Y), where X_i and Y are independently identically distributed Generalized inverted Kumaraswamy distribution (GIKD) random variables with common known shape and scale parameters α, γ respectively and unknown shape parameters β_i; $i = 1,2,3$ and b, such that b is shape parameter with stress random variables Y.

The system reliability $R1$ of two parallel components having strength variables X_i; $i = 1,2$ are subjected to a common random stress Y, is the probability of maximum two strengths under one stress, then the system reliability R_1 is given by:[2].

$$R_1 = p[y < \max\{x_1, x_2\}]$$

where X_1, X_2, Y are mutually independently identically distributed random variables.

The system reliability, R_1, can be written as:

$$R_1 = 1 - p[y > \max\{x_1, x_2\}]$$
$$= 1 - \int_0^\infty \int_0^y \int_0^y f(x_1, x_2, y)\, dx_1 dx_2 dy$$
$$= 1 - \int_0^\infty \int_0^y f(x_1) dx_1 \int_0^y f(x_2) dx_2\; f(y) dy$$
$$R_1 = 1 - \int_0^\infty Fx_1(y) Fx_2(y) f(y) dy \qquad (1)$$

Following the same paradigm, we have proposed a formula to find the reliability R_2 in case three parallel components having three independent strengths $X_1, X_2\ X_3$ and exposed to common stress Y and which is the probability of the maximum of the three strengths under one stress, then the system reliability R_2 is given by:

$$R_2 = p[y < \max\{x_1, x_2, x_3\}]$$
$$= 1 - p[y > \max\{x_1, x_2, x_3\}]$$
$$= 1 - \int_0^\infty \int_0^y \int_0^y \int_0^y f(x_1, x_2, x_3, y)\, dx_1\, dx_2\, dx_3\, dy$$
$$= 1 - \int_0^\infty \int_0^y f(x_1) dx_1 \int_0^y f(x_2) dx_2 \int_0^y f(x_3) dx_3\ f(y) dy$$
$$R_2 = 1 - \int_0^\infty Fx_1(y)\, Fx_2(y)\, Fx_3(y)\, f(y) dy \qquad (2)$$

3. Generalized Inverted Kumaraswamy Distribution (GIKD)

Kumaraswamy [5] obtained a distribution, which is derived from beta distribution after fixing some parameters in beta distribution. The distribution is appropriate to natural phenomena whose outcomes are bounded from both sides, such as the individual's heights, test scores, temperatures and hydrological daily data of rain fall.

The probability density function and distribution function of the Kumaraswamy distribution are given, respectively, by:

$$f(x;\ \alpha, \beta) = \alpha\beta\, x^{\alpha-1}[1 - x^\alpha)]^{\beta-1}, \alpha, \beta > 0, x \in (0,1)$$
$$F(x;\ \alpha, \beta) = 1 - [1 - x^\alpha)]^\beta$$

The inverted distribution have many applications in different fields such as; biological sciences, life testing problems, engineering sciences, environmental studies and econometrics. In the last two decades, the researchers proposed many inverted distributions due to its great applications; studied and [6] investigated the inverted Kumaraswamy distribution.

Abd Al-Fattah et al. [6] derived the inverted Kumaraswamy distribution from Kumaraswamy (Kum) distribution using the transformation $t = X^{-1} - 1$. When $X \sim$ Kum(α,β) where α and β are shape parameters, then probability density function and distribution function of the inverted Kumaraswamy distribution are given, respectively, by:

$$f(t, \alpha, \beta) = \alpha\beta(1 + t)^{-(\alpha+1)}[1 - (1 + t)^{-\alpha}]^{\beta-1}, \quad x > 0$$
$$F(t, \alpha, \beta) = [1 - (1 + t)^{-\alpha}]^\beta$$

where $\alpha > 0$ and $\beta > 0$ are two shape parameters.

Iqbal [7] introduced an extension of the inverted Kumaraswamy distribution called the generalized inverted Kumaraswamy distribution (GIKD) by using transformation $t = x^\gamma$ with has probability density function and distribution function are given, respectively, by:

Let X be a random variable refer to strength then if $X \sim GIK(\alpha, \beta, \gamma)$, then the pdf of GIKD is [6], [7]:

$$f(x;\ \alpha, \beta, \gamma) = \alpha\beta\gamma\, x^{\gamma-1}\,(1 + x^\gamma)^{-(1+\alpha)}\,[1 - (1 + x^\gamma)^{-\alpha}]^{\beta-1} \qquad (3)$$

where $\alpha, \beta, \gamma > 0, x > 0, \alpha, \beta$ are shape parameters.

The CDF of GIKD is:

$$F(x;\ \alpha, \beta, \gamma) = [1 - (1 + x^\gamma)^{-\alpha}]^\beta \qquad (4)$$

Let y is random variable refer to stress then $Y \sim GIK(\alpha, b, \gamma)$. Also the pdf of GIKD is:

$$f(y;\ \alpha, b, \gamma) = \alpha b\gamma\, y^{\gamma-1}\,(1 + y^\gamma)^{-(1+\alpha)}\,[1 - (1 + y^\gamma)^{-\alpha}]^{b-1} \qquad (5)$$

such that $y > 0, \alpha, b, \gamma > 0$ and the cdf of GIKD is:

$$F(y;\ \alpha, b, \gamma) = [(1 - (1 + y^\gamma)^{-\alpha}]^b$$

3.1 Two Strength-one stress component reliability

Let X_i be the strength $\sim GIK(\alpha, \beta_i, \gamma)$, i = 1, 2 and Y is stress $\sim GIK(\alpha, b, \gamma)$

$$F(x_i;\ \alpha, \beta_i, \gamma) = [(1 - (1 + {x_i}^\gamma)^{-\alpha}]^{\beta_i},\ i = 1,2 \qquad (6)$$

Substitute (5) and (6) in (1)

$$R_1 = 1 - \int_0^\infty Fx_1(y) Fx_2(y) f(y) dy$$
$$= 1 - \int_0^\infty [(1 - (1 + y^\gamma)^{-\alpha}]^{\beta_1}\,[(1 - (1 + y^\gamma)^{-\alpha}]^{\beta_2}\ \alpha b\gamma\, y^{\gamma-1}\,(1 + y^\gamma)^{-(1+\alpha)}\,[1 - (1 + y^\gamma)^{-\alpha}]^{b-1}\, dy$$
$$R_1 = 1 - b\int_0^\infty [1 - (1 + y^\gamma)^{-\alpha}]^{\beta_1+\beta_2+b-1}\,\alpha(1 + y^\gamma)^{-(1+\alpha)}\ \gamma y^{\gamma-1}\, dy$$

By using:

$$\int_0^\infty [f(y)]^n\ \frac{d[f(y)]}{dy}\ dy = \frac{[f(y)]^{n+1}}{n+1}\Big|_0^\infty$$

where $\frac{d[f(y)]}{dy}$ is the derivative of $f(y)$.

In the same way $\alpha(1 + y^\gamma)^{-(1+\alpha)}\ \gamma y^{\gamma-1}$ is the derivative of $1 - (1 + y^\gamma)^{-\alpha}$

$$R_1 = 1 - \left\{\frac{b[(1-(1+y^\gamma)^{-\alpha}]^{\beta_1+\beta_2+b}}{\beta_1+\beta_2+b}\right\}\Big|_0^\infty$$
$$R_1 = 1 - \frac{b}{\beta_1+\beta_2+b} \qquad (7)$$

3.2 Three strength-one stress component reliability

If x_i is strength $\sim GIK(\alpha, \beta_i, \gamma)$, $i = 1,2,3$ and Y is stress $\sim GIK(\alpha, b, \gamma)$

$$F(x_i;\ \alpha, \beta, \gamma) = [(1 - (1 + {x_i}^\gamma)^{-\alpha}]^{\beta_i}, i = 1,2,3 \qquad (8)$$

Substitute (5) and (8) in (2)

$$R_2 = 1 - \int_0^\infty Fx_1(y) Fx_2(y)\, Fx_3(y) f(y) dy$$
$$= 1 - \int_0^\infty [(1 - (1 + y^\gamma)^{-\alpha}]^{\beta_1}\,[(1 - (1 + y^\gamma)^{-\alpha}]^{\beta_2}\,[(1 - (1 + y^\gamma)^{-\alpha}]^{\beta_3}\alpha b\gamma y^{\gamma-1}(1 + y^\gamma)^{-(1+\alpha)}\,[1 - (1 + y^\gamma)^{-\alpha}]^{b-1} dy$$
$$= 1 - b\int_0^\infty [(1 - (1 + y^\gamma)^{-\alpha}]^{\beta_1+\beta_2+\beta_3+b-1}\alpha(1 + y^\gamma)^{-(1+\alpha)}\,\gamma y^{\gamma-1}\, dy$$

By using $\int_0^\infty [f(y)]^n\ \frac{d[f(y)]}{dy}\ dy = \frac{[f(y)]^{n+1}}{n+1}\Big|_0^\infty$, where $\frac{d[f(y)]}{dy}$ is the derivative of $f(y)$.

In the same way $\alpha(1 + y^\gamma)^{-(1+\alpha)}\ \gamma y^{\gamma-1}$ is derivative to:

$$1 - (1 + y^\gamma)^{-\alpha} = 1 - \left\{\frac{b[(1-(1+y^\gamma)^{-\alpha}]^{\beta_1+\beta_2+\beta_3+b}}{\beta_1+\beta_2+\beta_3+b}\right\}\Big|_0^\infty$$
$$R_2 = 1 - \frac{b}{\beta_1+\beta_2+\beta_3+b} \qquad (9)$$

4. Methods of Estimating the Reliability Function

In this sub section, we will estimate the unknown shape parameters $\beta, \beta_1, \beta_2, \beta_3, b$ of GIKD and estimate the system reliability functions R_1, R_2 have been estimated by two different methods of estimation; Ranked set sampling method and Bayes estimation method.

4.1 Ranked set sampling method (RSS)

McIntyre introduced the Ranked set sampling for improved the efficiency of the sample mean as an estimator of the population mean in situations when the characteristic of interest was difficult or expensive to measure [8] and [9].

Assume that we have random sample $X_1, X_2, \dots, X_n$ for GIKD. Let $X_{(1)}, X_{(2)} \dots, X_{(n)}$ be an order statistic, the pdf of $X_{(i)}$ is:

$$g_{(X_i)} = \frac{n!}{(i-1)!(n-i)!}\,[F(x_i)]^{i-1}\,[1-F(x_i)]^{n-i}\,f(x_i) \tag{10}$$

By substituting (3), (4) in (10)

$$g_{(X_i)} = \frac{n!}{(i-1)!(n-i)!}\left[[1-(1+x^\gamma)^{-\alpha}]^\beta\right]^{i-1}\left[1-[1-(1+x^\gamma)^{-\alpha}]^\beta\right]^{n-i}\alpha\beta\gamma\, x^{\gamma-1}(1+x^\gamma)^{-(1+\alpha)}\,[1-(1+x^\gamma)^{-\alpha}]^{\beta-1}$$

The likelihood function of $g_{(X_i)}$ is

$$L = \left[\frac{n!}{(i-1)!(n-i)!}\right]^n \alpha^n\beta^n\gamma^n \prod_{i=1}^n\left[[1-(1+x_i{}^\gamma)^{-\alpha}]^\beta\right]^{i-1}\prod_{i=1}^n\left[1-[1-(1+x_i{}^\gamma)^{-\alpha}]^\beta\right]^{n-i}\prod_{i=1}^n x_i{}^{\gamma-1}\prod_{i=1}^n(1+x_i{}^\gamma)^{-(1+\alpha)}\prod_{i=1}^n[1-(1+x_i{}^\gamma)^{-\alpha}]^{\beta-1} \tag{11}$$

The logarithm to relation (11) is:

$$Lnl = n\ln\left[\frac{n!}{(i-1)!(n-i)!}\right] + n\ln\alpha + n\ln\beta + n\ln\gamma + \sum_{i=1}^n\beta(i-1)\ln[1-(1+x_i{}^\gamma)^{-\alpha}] + \sum_{i=1}^n(n-i)\ln\left[1-[1-(1+x_i{}^\gamma)^{-\alpha}]^\beta\right] + (\gamma-1)\sum_{i=1}^n \ln x_i - (1+\alpha)\sum_{i=1}^n\ln(1+x_i{}^\gamma) + (\beta-1)\sum_{i=1}^n\ln[1-(1+x_i{}^\gamma)^{-\alpha}] \tag{12}$$

We want in this paper to estimate β, so taking partial derivatives to equation (12) with respect to β and equal to zero, such that β is unknown shape parameter, to get after that the reliability, we get:

$$\frac{dLnl}{d\beta} = \frac{n}{\beta} + \sum_{i=1}^n(i-1)\ln[(1-(1+x_i{}^\gamma)^{-\alpha}] - \sum_{i=1}^n(n-i)\frac{[(1-(1+x_i{}^\gamma)^{-\alpha}]^\beta \ln[(1-(1+x_i{}^\gamma)^{-\alpha}]}{1-[(1-(1+x_i{}^\gamma)^{-\alpha}]^\beta} + \sum_{i=1}^n\ln[1-(1+x_i{}^\gamma)^{-\alpha}] = 0$$

$$\hat{\beta}_{RSS} = \frac{-n}{\sum_{i=1}^n(i-1)\ln[(1-(1+x_i{}^\gamma)^{-\alpha}] - \sum_{i=1}^n(n-i)\frac{[(1-(1+x_i{}^\gamma)^{-\alpha}]^{\beta_o}\ln[(1-(1+x_i{}^\gamma)^{-\alpha}]}{1-[(1-(1+x_i{}^\gamma)^{-\alpha}]^{\beta_o}} + \sum_{i=1}^n\ln[1-(1+x_i{}^\gamma)^{-\alpha}]}$$

where β_o is initial value.

To find $\hat{R}_{1\,RSS}$ and $\hat{R}_{2\,RSS}$, we must estimate b, β_1, β_2 and β_3 since existing in relations. In the same way in case $\hat{\beta}_{RSS}$, we will find $\hat{b}_{RSS}$.

Let Y stress random sample has $GIK(\alpha, b, \gamma)$ distribution with sample size m where b is unknown parameter, the RSS estimator for ($\hat{b}_{RSS}$), can be derived as:

$$\hat{b}_{RSS} = \frac{-m}{\sum_{j=1}^m(j-1)\ln[(1-(1+y_j{}^\gamma)^{-\alpha}] - \sum_{j=1}^m(m-j)\frac{[(1-(1+y_j{}^\gamma)^{-\alpha}]^{b_o}\ln[(1-(1+y_j{}^\gamma)^{-\alpha}]}{1-[(1-(1+y_j)^{-\alpha}]^{b_o}} + \sum_{j=1}^m\ln[1-(1+y_j{}^\gamma)^{-\alpha}]} \tag{13}$$

where b_o is initial value.

Let $X_{1_{i_1}}; i_1 = 1, 2, \dots, n_1$, $X_{2_{i_2}}; i_2 = 1, 2, \dots, n_2$ and $X_{3_{i_3}}; i_3 = 1, 2, \dots, n_3$ are strengths random sample which has $GIK(\alpha, \beta_1, \gamma)$, $GIK(\alpha, \beta_2, \gamma)$, $GIK(\alpha, \beta_3, \gamma)$ distributions, with sample size n_1, n_2 and n_3, respectively, where β_1, β_2 and β_3 are unknown parameters and by using the same manner,the RSS estimators to the unknown parameters β_1, β_2 and β_3 are:

$$\hat{\beta}_{k_{RSS}} = \frac{-n_k}{\sum_{i_k=1}^{n_k}(i_k-1)\ln[(1-(1+x_{k_{i_k}}{}^\gamma)^{-\alpha}] - \sum_{i_k=1}^{n_k}(n_k-i_k)\frac{[(1-(1+x_{k_{i_k}}{}^\gamma)^{-\alpha}]^{\beta_{k_o}}\ln[(1-(1+x_{k_{i_k}}{}^\gamma)^{-\alpha}]}{1-[(1-(1+x_{k_{i_k}}{}^\gamma)^{-\alpha}]^{\beta_{k_o}}} + \sum_{i_k=1}^{n_k}\ln[1-(1+x_{k_{i_k}}{}^\gamma)^{-\alpha}]} \tag{14}$$

where $k = 1,2,3$.

By substituting (13) and (14) into relations (7) and (9), respectively, we get the reliability estimators $\hat{R}_{1\,RSS}$ and $\hat{R}_{2\,RSS}$

$$\hat{R}_{1\,RSS} = \frac{\hat{b}_{RSS}}{\hat{\beta}_{1\,RSS}+\hat{\beta}_{2\,RSS}+\hat{b}_{RSS}} \tag{15}$$

$$\hat{R}_{2\,RSS} = \frac{\hat{b}_{RSS}}{\hat{\beta}_{1\,RSS}+\hat{\beta}_{2\,RSS}+\hat{\beta}_{3\,RSS}+\hat{b}_{RSS}} \tag{16}$$

4.2 Bayes estimation methods

We have to find the Bayes estimator for β using non informative prior distribution $f(\beta)$ based on modified extension of Jeffery Prior and square error loss function [10] and [11].

To get the prior distribution to the parameter β using the modified extension of Jeffery, prior can be found by:

$$f(\beta) \propto \left(-nE\left[\frac{d^2\ln f}{d\beta^2}\right]\right)^c \tag{17}$$

To find $\left(-nE\left[\frac{d^2\ln f}{d\beta^2}\right]\right)$ take natural logarithm to pdf in relation (3)

$$Lnf = \ln\alpha + \ln\beta + \ln\gamma + (\gamma-1)\ln x - (1+\alpha)\ln(1+x^\gamma) + (\beta-1)\ln[1-(1+x^\gamma)^{-\alpha}]$$

Taking partial derivative with respect to β

$$\frac{d\ln f}{d\beta} = \frac{1}{\beta} + \ln[1-(1+x^\gamma)^{-\alpha}]$$

Again taking second partial derivative with respect to β, we get:

$$\frac{d^2 lnf}{d\beta^2} = \frac{-1}{\beta^2}$$
$$E\left(\frac{d^2 lnf}{d\beta^2}\right) = E\left(\frac{-1}{\beta^2}\right)$$
$$= \frac{-1}{\beta^2} - nE\left(\frac{d^2 lnf}{d\beta^2}\right)$$
$$= \frac{n}{\beta^2} \tag{18}$$

Substitute (18) in (17), we get $f(\beta) \propto \left(\frac{n}{\beta^2}\right)^c$

$$f(\beta) = kn^c \beta^{-2c} \tag{19}$$

which is the prior distribution.

The likelihood function to pdf in relation (3) is:

$$L(x_i, \alpha, \beta, \gamma) = \alpha^n \beta^n \gamma^n \prod_{i=1}^n x_i^{\gamma-1} \prod_{i=1}^n (1 + x_i^{\gamma})^{-(1+\alpha)} \prod_{i=1}^n [1 - (1 + x_i^{\gamma})^{-\alpha}]^{\beta-1} \tag{20}$$

let $L(x_i, \alpha, \beta, \gamma) = L$, then the posterior distribution is:

$$f(\beta/x_1, x_2, \ldots, x_n) = \frac{L\, f(\beta)}{\int L\, f(\beta) d\beta} \tag{21}$$

Substitute (19) and (20) in (21), we get:

$$f(\beta/x_1, x_2, \ldots, x_n) = \frac{\alpha^n \beta^n \gamma^n \prod_{i=1}^n x_i^{\gamma-1} \prod_{i=1}^n (1+x_i^{\gamma})^{-(1+\alpha)} \prod_{i=1}^n [1-(1+x_i^{\gamma})^{-\alpha}]^{\beta-1}\, kn^c \beta^{-2c}}{\int \alpha^n \beta^n \gamma^n \prod_{i=1}^n x_i^{\gamma-1} \prod_{i=1}^n (1+x_i^{\gamma})^{-(1+\alpha)} \prod_{i=1}^n [1-(1+x_i^{\gamma})^{-\alpha}]^{\beta-1}\, kn^c \beta^{-2c}\, d\beta}$$

$$f(\beta/x_1, x_2, \ldots, x_n) = \frac{\beta^{n-2c} \prod_{i=1}^n [1-(1+x^{\gamma})^{-\alpha}]^{\beta-1}}{\int \beta^{n-2c} \prod_{i=1}^n [1-(1+x^{\gamma})^{-\alpha}]^{\beta-1} d\beta} \tag{22}$$

To simplify (22), we take $\prod_{i=1}^n [1 - (1 + x_i{}^{\gamma})^{-\alpha}]^{\beta-1}$

$$\prod_{i=1}^n [1 - (1 + x_i{}^{\gamma})^{-\alpha}]^{\beta-1} = e^{\sum_{i=1}^n ln[1-(1+x_i{}^{\gamma})^{-\alpha}]^{\beta-1}} = e^{\beta \sum_{i=1}^n ln[1-(1+x_i{}^{\gamma})^{-\alpha}]}\ e^{-\sum_{i=1}^n ln[1-(1+x_i{}^{\gamma})^{-\alpha}]}$$

$$\prod_{i=1}^n [1 - (1 + x_i{}^{\gamma})^{-\alpha}]^{\beta-1} = e^{-\sum_{i=1}^n \beta ln[1-(1+x_i{}^{\gamma})^{-\alpha}]^{-1}}\ e^{-\sum_{i=1}^n 1 ln[1-(1+x_i{}^{\gamma})^{-\alpha}]} \tag{23}$$

by using the rule $a^x = e^{lna^x} = e^{xlna} = e^{-xlna^{-1}}$. Substitute (23) in (22), we get:

$$f(\beta/x_1, x_2, \ldots, x_n) = \frac{\beta^{n-2c}\, e^{-\sum_{i=1}^n \beta ln[1-(1+x^{\gamma})^{-\alpha}]^{-1}}\, e^{-\sum_{i=1}^n 1 ln[1-(1+x^{\gamma})^{-\alpha}]}}{\int \beta^{n-2c}\, e^{-\sum_{i=1}^n \beta ln[1-(1+x^{\gamma})^{-\alpha}]^{-1}}\, e^{-\sum_{i=1}^n 1 ln[1-(1+x^{\gamma})^{-\alpha}]}\, d\beta}$$

Since the integral to β and similar the $\frac{e^{-\sum_{i=1}^n 1 ln[1-(1+x^{\gamma})^{-\alpha}]}}{e^{-\sum_{i=1}^n 1 ln[1-(1+x^{\gamma})^{-\alpha}]}}$, so we get:

$$f(\beta/x_1, x_2, \ldots, x_n) = \frac{\beta^{n-2c}\, e^{-\sum_{i=1}^n \beta ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}}}{\int \beta^{n-2c}\, e^{-\sum_{i=1}^n \beta ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}} d\beta} = \frac{\beta^{n-2c}\, e^{-\sum_{i=1}^n \beta ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}}}{\frac{\Gamma(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+1}}} \tag{24}$$

We can find $\hat{\beta}_{bays}$ by using the loss function. The square loss function which is defined as bellow:

$$L(\hat{\beta}, \beta) = (\hat{\beta} - \beta)^2$$

then the risk function $R(\hat{\beta}, \beta)$ appears as:

$$R(\hat{\beta}, \beta) = E[\mathrm{L}(\hat{\beta}, \beta)]$$
$$= E[(\hat{\beta} - \beta)^2]$$
$$= \int_0^{\infty} (\hat{\beta} - \beta)^2 f(\beta/x_1, x_2, \ldots, x_n) \mathrm{d}\beta \tag{25}$$

By substitute equation (24) in (25), we get:

$$R(\hat{\beta}, \beta) = \int_0^{\infty} (\hat{\beta}^2 - 2\hat{\beta}\beta + \beta^2) \frac{\beta^{n-2c}\, e^{-\sum_{i=1}^n \beta ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}}}{\frac{\Gamma(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+1}}} \mathrm{d}\beta$$

$$= \int_0^{\infty} \left(\hat{\beta}^2 \frac{\beta^{n-2c}\, e^{-\sum_{i=1}^n \beta ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}}}{\frac{\Gamma(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+1}}} - 2\hat{\beta} \frac{\beta^{n-2c+1}\, e^{-\sum_{i=1}^n \beta ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}}}{\frac{\Gamma(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+1}}} + \frac{\beta^{n-2c+2}\, e^{-\sum_{i=1}^n \beta ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}}}{\frac{\Gamma(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+1}}} \right) d\beta$$

$$R(\hat{\beta}, \beta) = \hat{\beta}^2 \frac{\frac{\Gamma(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+1}}}{\frac{\Gamma(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+1}}} - 2\hat{\beta} \frac{\frac{\Gamma(n-2c+2)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+2}}}{\frac{\Gamma(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+1}}} + \frac{\frac{\Gamma(n-2c+3)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+3}}}{\frac{\Gamma(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^{n-2c+1}}}$$

$$\mathrm{R}(\hat{\beta}, \beta) = \hat{\beta}^2 - 2\hat{\beta} \frac{(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)} + \frac{(n-2c+2)(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)^2}$$

Taking partial derivatives with respect to $\hat{\beta}$ and equating the results to the zero:

$$\frac{dR}{d\hat{\beta}} = 2\hat{\beta} - 2\frac{(n-2c+1)}{\left(\sum_{i=1}^n ln[1-(1+x_i^{\gamma})^{-\alpha}]^{-1}\right)} = 0$$

Then the Bayes estimator is as bellow:

$$\hat{\beta}_{bays} = \frac{n-2c+1}{\sum_{i=1}^{n} ln[1-(1+x_i{}^{\gamma})^{-\alpha}]^{-1}}$$

In the same way, let Y stress random sample has $GIK(\alpha, b, \gamma)$ distribution with sample size m where b is unknown parameter, the Bays estimator for $(\hat{b}_{bays})$, can be derived as:

$$\hat{b}_{bays} = \frac{m-2c+1}{\sum_{j=1}^{m} ln[1-(1+y_j{}^{\gamma})^{-\alpha}]^{-1}} \quad (26)$$

Let $X_{1_{i_1}}; i_1 = 1,2,\dots,n_1$, $X_{2_{i_2}}; i_2 = 1,2,\dots,n_2$ and $X_{3_{i_3}}; i_3 = 1,2,\dots,n_3$ are strengths random sample has $GIK(\alpha, \beta_1, \gamma)$, $GIK(\alpha, \beta_2, \gamma)$, $GIK(\alpha, \beta_3, \gamma)$ distributions, with sample size n_1, n_2 and n_3, respectively, where β_1, β_2 and β_3 are unknown parameters. By using the same manner, the Bayes estimators to the unknown parameters β_1, β_2 and β_3 are:

$$\hat{\beta}_{k\ bays} = \frac{n_k-2c+1}{\sum_{i_k=1}^{n_k} ln[1-(1+x_{k i_k}{}^{\gamma})^{-\alpha}]^{-1}} \quad (27)$$

such that $k = 1,2,3$, and c is a constant.

By substitute the equations (26) and (27) in relations (7) and (9), respectively, we get the reliability estimators $\hat{R}_{1\ bays}$ and $\hat{R}_{2\ bays}$

$$\hat{R}_{1\ bays} = \frac{\hat{b}_{bays}}{\hat{\beta}_{1\ bays}+\hat{\beta}_{2\ bays}+\hat{b}_{bays}} \quad (28)$$

$$\hat{R}_{2\ bays} = \frac{\hat{b}_{bays}}{\hat{\beta}_{1\ bays}+\hat{\beta}_{2\ bays}+\hat{\beta}_{3\ bays}+\hat{b}_{bays}} \quad (29)$$

5. Simulation

A simulation study is a technique used to estimate the shape parameter of GIK, then utilizing shape parameter to estimate the reliability function through to two strength-one stress component and three strength-one stress component and comparing the results by utilizing mean squares error technique.

Generating Random Variables: Assume that U be a random variable with the Uniform distribution in (0,1),the samples random data are generated to follow Generalized Inverted Kumaraswamy distribution using the cdf of Generalized Inverted Kumaraswamy distribution and find the inverse of the distribution function as follow $U = F(X) \rightarrow X = F^{-1}(U)$ by using equation (4), which is:

$$F(x;\ \alpha, \beta, \gamma) = [1-(1+x^{\gamma})^{-\alpha}]^{\beta}$$

$$(F)^{\frac{1}{\beta}} = 1-(1+x^{\gamma})^{-\alpha}$$

$$(1+x^{\gamma})^{-\alpha} = 1-(F)^{\frac{1}{\beta}}$$

$$1+x^{\gamma} = \left[1-(F)^{\frac{1}{\beta}}\right]^{\frac{-1}{\alpha}}$$

$$x^{\gamma} = \left[1-(F)^{\frac{1}{\beta}}\right]^{\frac{-1}{\alpha}} - 1$$

$$x = \left[\left[1-\left(F(x;\ \alpha,\beta,\gamma)\right)^{\frac{1}{\beta}}\right]^{\frac{-1}{\alpha}} - 1\right]^{\frac{1}{\gamma}}$$

by letting $U = F(x;\ \alpha, \beta, \gamma)$, where U is a Uniform continuous random variable defined on the interval (0,1)

$$x = \left[\left[1-(U)^{\frac{1}{\beta}}\right]^{\frac{-1}{\alpha}} - 1\right]^{\frac{1}{\gamma}} \quad (30)$$

We can use the generate data in technique (30) to get the random samples of the generalized inverted Kumaraswamy.

The simulation process has been designed in basic stages, which are important and necessary to find the estimation reliability of the shape parameter for the generalized Inverted Kumaraswamy distribution.

Step 1: Different sizes of the samples has been selected which are proportional with the effect of the sample size on the accuracy of the results obtained by using the two approaches of this paper, where the small sample size is choosing to be (15), medium sample size to be (30) and the large sample size to be (90).

The random samples $y_1, y_2, \dots, y_m, x_{11}, x_{12}, \dots, x_{1n1}, x_{21}, x_{22}, \dots, x_{2n2}, x_{31}, x_{32}, \dots, x_{3n3}$, of sizes ($m, n_1, n_2, n_3$), such that the sample sizes are (15,15,15,15), (30,30,30,30), (90,90,90,90),(15,30,30,90), (30,90,90,30) and (15,30,90,15).

Step 2: Real parameters values are selected for 6 experiments $(b, \beta_1, \beta_2, \beta_3, \alpha, \gamma)$ in the following table:

Experiment	***b***	***β₁***	***β₂***	***β₃***	***α***	***γ***
1	2	1	3	1.5	1	1
2	2	1	3	1.5	2	1
3	2	1	3	1.5	1	2
4	3	2.5	3.1	2.8	1	1
5	3	2.5	3.1	2.8	2	1
6	3	2.5	3.1	2.8	1	2

Step 3: We make estimate of shape parameters $b, \beta_1, \beta_2, \beta_3$ of GIKD by (ranked set sampling and Bayes) methods as in (13), (14), (26), (27), respectively.

Step 4: We get estimation of Reliability as in: (15), (16), (28) and (29).

Step 5: We Calculate mean by: Mean=$\frac{\sum_{i=1}^{L} \hat{R}_i}{L}$

Step 6: Make comparison between the two different methods of estimation by using Mean Square Error:

$$\text{MSE}(\hat{R}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{R}_i - R)^2$$

where L represents number of replications for any experiment. The number of iterations is selected to be L = 500, where $\hat{R}$ is the estimation of the R.

Simulation Results: Using the proposed estimation methods, the results presented in table (1) are obtained:

Table 1. The ($\hat{R}_1$, $\hat{R}_2$ and MSE) values for experiment (1), such that R_1 – REAL = 0.666667, R_2 – REAL = 0.733333

Sample size	(15,15,15,15)	(30,30,30,30)	(90,90,90,90)	(15,30,30,90)	(30,90,90,30)	(15,30,90,15)
R̂1-RSS	0.682477	0.662789	0.664249	0.460147	0.373669	0.361179
MSE	0.215099	0.005798	0.001192	0.061152	0.090361	0.110197
R̂1BAYS	0.668592	0.67122	0.668742	0.65865	0.662659	0.650191
MSE	0.004392	0.002439	0.000734	0.00425	0.00204	0.004687
R̂2-RSS	0.672071	0.748176	0.747907	0.492403	0.611991	0.622924
MSE	2.788304	0.004322	0.000952	0.07783	0.022828	0.549213
R̂2BAYS	0.751187	0.75265	0.751876	0.740252	0.748101	0.739332
MSE	0.003223	0.001969	0.000795	0.002915	0.001703	0.003379

Table 2. The ($\hat{R}_1$, $\hat{R}_2$ and MSE) values for experiment (2), such that R_1 – REAL = 0.666667, R_2 – REAL = 0.733333

Sample size	(15,15,15,15)	(30,30,30,30)	(90,90,90,90)	(15,30,30,90)	(30,90,90,30)	(15,30,90,15)
R̂1-RSS	0.69341	0.666679	0.668798	0.469369	0.379752	0.354138
MSE	0.955491	0.006958	0.001391	0.055282	0.086978	0.112599
R̂1BAYS	0.668336	0.665359	0.665222	0.654052	0.657681	0.657483
MSE	0.005085	0.002443	0.000894	0.003849	0.002277	0.003641
R̂2-RSS	0.678088	0.74925	0.751515	0.502276	0.618043	0.621868
MSE	4.289437	0.005443	0.001225	0.070349	0.01994	0.34652
R̂2BAYS	0.749374	0.749026	0.748979	0.736536	0.743813	0.746596
MSE	0.00364	0.001857	0.000817	0.002498	0.001678	0.002859

Table 3. The ($\hat{R}_1$, $\hat{R}_2$ and MSE) values for experiment (3), such that R_1 – REAL = 0.666667, R_2 – REAL = 0.733333

Sample size	(15,15,15,15)	(30,30,30,30)	(90,90,90,90)	(15,30,30,90)	(30,90,90,30)	(15,30,90,15)
R̂1-RSS	0.677165	0.665116	0.669564	0.462873	0.382886	0.348717
MSE	0.208522	0.006668	0.00112	0.059008	0.084184	0.11974
R̂1BAYS	0.66922	0.669425	0.664688	0.657258	0.657042	0.656722
MSE	0.00442	0.002539	0.000731	0.003915	0.001963	0.004021
R̂2-RSS	0.772751	0.749515	0.752857	0.49588	0.622336	0.579007
MSE	0.254879	0.004658	0.001094	0.074584	0.01744	0.166795
R̂2BAYS	0.752265	0.752219	0.748124	0.73912	0.74346	0.744718
MSE	0.00305	0.001965	0.000683	0.00259	0.001445	0.003154

Table 4. The ($\hat{R}_1$, $\hat{R}_2$ and MSE) values for experiment (4), such that R_1 – REAL = 0.651163, R_2 – REAL = 0.736842

Sample size	(15,15,15,15)	(30,30,30,30)	(90,90,90,90)	(15,30,30,90)	(30,90,90,30)	(15,30,90,15)
R̂1-RSS	0.664226	0.666889	0.666586	0.474359	0.388022	0.377091
MSE	0.003605	0.001308	0.000523	0.034378	0.07008	0.078056
R̂1BAYS	0.669317	0.666299	0.667077	0.660967	0.660283	0.649599
MSE	0.005441	0.002512	0.000969	0.004262	0.001812	0.004275
R̂2-RSS	0.747617	0.749744	0.749766	0.510649	0.619613	0.613732
MSE	0.002452	0.000885	0.000358	0.05423	0.014887	0.019238
R̂2BAYS	0.752593	0.750752	0.750419	0.742441	0.746379	0.739827
MSE	0.003532	0.001612	0.000665	0.002786	0.001436	0.003184

Table 5. The ($\hat{R}_1$, $\hat{R}_2$ and MSE) values for experiment (5), such that R_1 – REAL = 0.651163, R_2 – REAL = 0.736842

Sample size	(15,15,15,15)	(30,30,30,30)	(90,90,90,90)	(15,30,30,90)	(30,90,90,30)	(15,30,90,15)
R̂1-RSS	0.665548	0.668007	0.666433	0.482746	0.385462	0.380036
MSE	0.00363	0.001387	0.000534	0.031316	0.071621	0.075891
R̂1BAYS	0.667748	0.665751	0.666357	0.652302	0.663003	0.650421
MSE	0.00514	0.002748	0.000981	0.004246	0.002113	0.0036
R̂2-RSS	0.749777	0.750851	0.749564	0.518847	0.61718	0.619999
MSE	0.002507	0.000876	0.000356	0.050392	0.015644	0.017028
R̂2BAYS	0.749839	0.749288	0.750107	0.735491	0.748952	0.738806
MSE	0.003311	0.00176	0.00066	0.002833	0.001566	0.002865

Table 6. The ($\hat{R}_1$, $\hat{R}_2$ and MSE) values for experiment (6), such that R_1 – REAL = 0.651163, R_2 – REAL = 0.736842

Sample size	(15,15,15,15)	(30,30,30,30)	(90,90,90,90)	(15,30,30,90)	(30,90,90,30)	(15,30,90,15)
R̂1-RSS	0.665742	0.664359	0.666531	0.47403	0.389742	0.377586
MSE	0.003469	0.001348	0.000571	0.03451	0.06929	0.077032
R̂1BAYS	0.670999	0.669295	0.666936	0.662551	0.657044	0.654584
MSE	0.005128	0.002657	0.001096	0.00423	0.001986	0.003456
R̂2-RSS	0.749104	0.747805	0.750195	0.510324	0.622943	0.61731
MSE	0.002348	0.000895	0.000386	0.054394	0.014069	0.01716
R̂2BAYS	0.753297	0.752612	0.749792	0.743591	0.743186	0.74289
MSE	0.003446	0.001784	0.000689	0.002879	0.001368	0.00262

6. Conclusions

The reliability, $R_1 = P[Y < \max\{X_1, X_2\}]$, of two parallel components having X_1, X_2 Generalized Inverted Kumaraswamy strengths and exposed to Y Generalized Inverted Kumaraswamy common stress with unknown shape parameter β and common known shape α and scale γ parameters is obtained. Then derived the reliability, $R_2 = P[Y < \max\{X_1, X_2, X_3\}]$, of three parallel components having X_1, X_2, X_3 Generalized Inverted Kumaraswamy strengths and exposed to Y Generalized Inverted Kumaraswamy common stress.

MSE criteria used to make a comparison between two different estimators for each reliability function, where Bayes give the best performance for R_1 and R_2 after that ranked set sampling method is second order in the best.

I noticed and studied all experiments (tables) in simulation. We found $\hat{R}_1$, $\hat{R}_2$ in all sample sizes A,B,C,D,E,F in experiments (Tables 1,2,3) that MSE is the least in Bayes method therefore Bayes method is first order in the best, again I found $\hat{R}_1$, $\hat{R}_2$ in sample sizes D,E,F in experiments 4, 5, 6 that MSE is the least in Bayes method therefore Bayes method is the best in the case while we found $\hat{R}_1$, $\hat{R}_2$ in sample sizes A, B, C in experiments 4, 5, 6 that MSE is the least in method ranked set sampling method therefore ranked set sampling method is the best in the case.

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