On Commutativity of Rings with (σ, τ)-Biderivations

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Abstract

Let R be a prime ring with characteristic different from 2, $\mathcal{J}$ be a nonzero ideal of R. in this paper, for α,β,σ,τ as automorphisms of R, we present some results concerning the relationship between the commutativity of a ring and the existence of specific types of a (σ,τ)-Biderivation, we prove: (1) Suppose F:R×R⟶R is a nonzero(σ,τ)-Biderivation then R is a commutative ring if F satisfies one of the following conditions:

(i) $F(\mathcal{J}, \mathcal{J}) \subset C_{\alpha,\beta}$(ii) $[\mathrm{Im}F , \mathcal{J}]_{\alpha,\beta} = 0$ (iii) F(xω, y) = F(ωx, y)for all x, y, ω ∈ $\mathcal{J}$.

(2) Suppose F_1: R⟶R is a nonzero(σ, τ)-derivation and F_2:R×R⟶R is a (α, β)-Biderivation with $\mathrm{Im}F_2$=R, If $F_1F_2(\mathcal{J}, \mathcal{J})=0$ then $F_2=0$.

Keywords :Prime rings, Automorphisms, (σ, τ)-Biderivation.

1. Introduction

Throughout this paper *R* will be represent an associative ring with center *Z(R)*, and α,β,σ,τ are automorphisms of *R*. Recall that a ring is called prime if for any a,b ∈*R*, *aRb={0}* implies that either *a=0 or b=0*. The *(σ,τ)*-center of *R* denoted by $C_{\sigma,\tau}$ and defined by $C_{\sigma,\tau}=\{c\in R:\ c\sigma(r)=\tau(r)c,\ \text{for all } r\in R\}$. As usual [*x, y*] is denoted the commutator *xy – yx*, and we make use of the commutator identities [*xy, z*] = [*x, z*]*y* + *x*[*y, z*], [*x, yz*]= [*x, y*]*z* + *y*[*x, z*], *x,y,z*∈*R*. The symbol $[x, y]_{\alpha,\beta}$ stands for $x\alpha(y) - \beta(y)x$, also we will make extensive use of the following identities:

- $[xy, z]_{\alpha,\beta} = x[y, z]_{\alpha,\beta} + [x, \beta(z)]\, y$
$= x[y, \alpha(z)] + [x, z]_{\alpha,\beta}\, y$
- $[x, yz]_{\alpha,\beta} = \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$

A biadditive mapping *F: R*×*R*⟶*R* called a (*σ,τ*)-Biderivation if it satisfies the following:

i. $F(xy, z)=F(x, z)\sigma(y)+ \tau(x)F(y, z)$
ii. $F(x, yz)=F(x, y)\sigma(z)+ \tau(y)F(x, z)$

It is clear that the concept of a (*σ,τ*)-Biderivation includes the concept of Biderivation [9]. By $\mathbb{Q}_r$ we will denote the Martindale ring of quotient of *R*. It is known that this ring introduced by Martindale in [10], can be characterized by the following four properties.

(i) $R \subseteq \mathbb{Q}_r$.
(ii) for every $q \in \mathbb{Q}_r$ there exist a nonzero ideal $\mathcal{J}$ of R such that such that $q\mathcal{J} \subseteq R$.
(iii) if$q \in \mathbb{Q}_r$ and $\mathcal{J}$ is a nonzero ideal $\mathcal{J}$ of *R* such that $q\mathcal{J} = 0$, then *q=0*.
(iv) if$\mathcal{J}$ is an ideal of *R* and $h: \mathcal{J} \longrightarrow R$ is a right *R*-module map, then there exist $q \in \mathbb{Q}_r$ such that *h(u) =qu* for all $u \in \mathcal{J}$.

Remarks:

1- The center of $\mathbb{Q}_r$, which denote by *C*, is called the extended centroid of *R*.
2- *C* is a field and $Z \subseteq C$.
3- The sub ring of $\mathbb{Q}_r$ generated by *R* and *C* called the central closure of *R* and denoted by R_c.
4- The sub ring $\mathbb{Q}_s$ of $\mathbb{Q}_r$ where:
$\mathbb{Q}_s=\{q\in \mathbb{Q}_r: \mathcal{J}q\subseteq R$ for some nonzero ideal $\mathcal{J}$ of $R\}$ is called the symmetric Martindale ring of quotient.
5- If $q_1Rq_2 = 0$ with $q_1,q_2\in \mathbb{Q}_r$ implies that $q_1=0$ or $q_2=0$.

The study of the commutativity of prime rings with derivation initiated by E. C. Posner [3]. Over the last three decades, a lot of work has been done on this subject. Many authors have investigated the properties of prime or semiprime rings with a (*σ,τ*)-derivation.

Our objective in the present paper is to generalize some results in [2], [7] and [8], further we introduce other results, for instance: Ashraf and Rehman proved in [5] that, if d_1 and $d_2$are two (*σ,τ*)-derivations of *R* such that $d_1\sigma= \sigma d_1$, $d_2\sigma = \sigma d_2$, $d_1\tau = \tau d_1$, $d_2\tau = \tau d_2$ and $d_1d_2(R)=0$, then d_1=0 or $d_2=0$. Here we prove, if *U* is a nonzero ideal of *R*, F_1 is a (*σ,τ*)-derivation and F_2 is a (*σ,τ*)-Biderivation with

$ImF_2=R$. If $F_1F_2(U, U)=0$ then either $F_1=0$ or $F_2=0$.

2. Preliminaries

In this section we recall some basic definition gather together a few results of general interest that will be needed.

Definition: [6]

Let R be ring. An automorphismσ of R is said to be X-inner if, there exists an invertible element $a \in \mathbb{Q}_s$ such that $\sigma(r)=ara^{-1}$ for all $r\in R$**.**

Lemma 2.1: [6]

Let M be any set. Suppose that $H, G{:}M \longrightarrow \mathbb{Q}_r$ satisfy $H(s)\ x\ G(t)= G(s)\ x\ H(t)$, for all $s,t \in M$ and all x in some nonzero $\mathcal{I}$ ideal of R. if $H\neq\{0\}$, then there exists $\lambda \in$C such that $G(s)= \lambda\ H(t)$.

Lemma 2.2: [1]

Let R be ring. Suppose σ is an automorphism of R. if there exist nonzero elements $a_1,\ a_2,\ a_3,a_4\in \mathbb{Q}_r$ such that $a_1r\ a_2=a_3\ \sigma(r)a_4$ for all $r \in R$, thenσ is X-inner.

Lemma 2.3: [4]

Let R be a semiprime ring, and let $\mathcal{I}$ be a right ideal of R, then $Z(\mathcal{I})\subset Z(R)$.

Lemma 2.4: [4]

Let R be semiprime ring, $\mathcal{I}$ a right ideal of R. If the ideal $\mathcal{I}$ is a commutative, then $\mathcal{I} \subset Z(R)$.In addition if R is a prime ring then R must be commutative.

For prove of our results in this study, we need to introduce some preliminary lemmas.

Lemma 2.5 :

Let R be ring and S be a subring of R. if $F{:}\ S\times S\longrightarrow R$ is a $(\sigma,\ \tau)$-Biderivation, then for any $x,y,z,u,v\in S$ we have:

$F(x\ ,\ y)\ \sigma(z)[\sigma(u),\ \sigma(v)] = [\tau(x),\ \tau(y)]\tau(z)\ F(u\ ,v)$

Proof:

We compute $F(xu,\ yv)$ in two different ways. Since F is a $(\sigma,\ \tau)$-Biderivation in the first argument, then we have:

$F(xu, yv) = F(x, yv)\ \sigma(u) + \tau(x)F(u, yv)$

Using the fact that F is a $(\sigma,\ \tau)$-Biderivation in the second argument, it follows:

$$F(xu, yv) = F(x, y)\sigma(v)\ \sigma(u) + \tau(y)F(x,v)\ \sigma(u) + \tau(x)F(u, y)\ \sigma(v) + \tau(x)\ \tau(y)F(u, y)$$

On the other hand, we have:

$$F(xu, yv) = F(xu, y)\ \sigma(v) + \tau(y)F(xu, v) = F(x, y)\sigma(u)\ \sigma(v) + \tau(x)F(u, y)\ \sigma(v) + \tau(y)D(x, v)\ \sigma(u) + \tau(y)\ \tau(x)F(u, y)$$

Comparing the relations so obtained for $F(xu, yv)$, we get:

$F(x\ ,y)\ [\sigma(u),\ \sigma(v)] = [\tau(x),\ \tau(y)]\ F(u\ ,v)$, for all $x,y,u,v\in S$.

Putting zu for u, and using the identity $[s\omega,\ t] =[s,\ t]\,\omega + s[\,\omega,t]$, we obtain the assertion of the lemma. ■

O. Glbasi and N. Aydin Showed in [8 lemma 2] that: Let R be prime ring, $\mathcal{I}$ be a nonzero ideal of R and D is a (σ,τ)-derivation of R. f D is trivial on $\mathcal{I}$ then D itself is trivial. n the next lemma we extend this result to $(\sigma,\ \tau)$-Biderivation.

Lemma 2.6:

Let R be prime ring, $\mathcal{I}$ be a nonzero right ideal of R. Suppose that $F{:}\ R\times R\longrightarrow R$ is a $(\sigma,\ \tau)$-Biderivation. If $F(\mathcal{I},\mathcal{I}) =0$ then $F =0$.

Proof:

For any $u,v\in\ \mathcal{I},\ r \in R$, we have:

$0= F(ur, v)=F(u, v)\ \sigma(r) + \tau(u)\ F(r, v)$

That is

$\tau(u)\ F(r, v) =0$, for all$u,v\in \mathcal{I},\ r \in R$.

Putting $us,\ s \in R$ instead of u in above relation gives:

$\tau(u)\ \tau(s)\ F(r, v)=0$, for all$u,v\in\ \mathcal{I},\ r,s \in R$.

Hence

$u\ R\tau^{-1}(F(r, v) =0$, for all$u,v\in \mathcal{I},\ r \in R$.

Using the primeness of R, since $\mathcal{I}$ is a nonzero left ideal of R, we conclude that:

$F(r, v) =0$, for all$v \in \mathcal{I},\ r \in R$.

Replacing v by $vt,\ t \in R$, we get:

$\tau(v)\ F(r, t)=0$, for all $v \in\ \mathcal{I},\ r,t \in R$.

This means that:

$\mathcal{I}\tau^{-1}(F(r, t)) =0$, for all$r,t\in R$.

Since R is a prime ring and $\mathcal{I}$ is a nonzero right ideal of R, it follows that $F =0$. ■

3. (σ, τ)-Biderivation and commutativity of prime rings

Theorem3.1:

Let R be a non-commutative prime ring and $F{:}R\times R\longrightarrow R$ be a nonzero (σ,τ)-Biderivation, then there exists an invertible element $b\in\mathbb{Q}_s$ such that $\sigma^{-1}(F(x,y)) = b\,[x,y]$.

Proof:

According to lemma (2.6), the mapping F satisfies that:

$F(x\ ,y)\ \sigma(z)[\sigma(u),\ \sigma(v)] = [\tau(x),\ \tau(y)]\tau(z)\ F(u\ ,v)$, for all $x,y,z,u,v\in R$.

That is

$\sigma^{-1}(F(x,y))\ z[u,\ v]=\theta([x,y])\theta(z)\sigma^{-1}(F(u\ ,v))$ for all $x,y,z,u,v\in R$, where $\theta=\sigma^{-1}\circ\tau$ is an automorphism of R.

Since R is a non-commutative ring and $F\neq 0$, we can find $a_1=\sigma^{-1}(F(x,\ y))\neq 0$, $a_2=[u,\ v]\neq 0$, $a_3=[\theta(x),\theta(y)]\neq 0$ and $a_4=\sigma^{-1}(F(u,v))\neq 0$, then we have:
$a_1 z\ a_2=a_3\ \theta(z)\ a_4$, for all $z\in R$.

Hence by lemma (2.2) we conclude that θ is X-inner, that is $\theta(s) = asa^{-1}$ for some $a\in\mathbb{Q}_s$. Therefore:

$\sigma^{-1}(F(x,y))z[u,v]=a\quad[x,y]\ za^{-1}\sigma^{-1}(F(u,v))$, for all $x,y,z,u,v\in R$.

Left multiplication by a^{-1} leads to:
$a^{-1}\sigma^{-1}(F(x,y))z[u,v]=[x,y]\ za^{-1}\sigma^{-1}(F(u,v))$, for all $x,y,z,u,v\in R$.

Let $M=R\times R$, note that maps $H,G{:}\ M\longrightarrow\mathbb{Q}_r$ defined by $H(x,y)=[x,y]$,
$G(x,y)=a^{-1}\sigma^{-1}(F(x,y))$ satisfy all the requirements of lemma(2.1).So there exist $\lambda\in C$ such that:

$G(x,y)=\lambda\ H(x,y)$.

That is:

$a^{-1}\sigma^{-1}(F(x,y)) = \lambda[x,y]$, for all $x,y\in R$.

Equivalently
$\sigma^{-1}(F(x,y))=b[x,y]$, for all $x,y\in R$, and $b=\lambda a$.

Note that $b\neq 0$ for $F\neq 0$, whence b is invertible.

Theorem 3.2:

Let R be a prime ring, $\mathcal{I}$ be a nonzero ideal of R. suppose that $F{:}\ R\times R\longrightarrow R$ is anonzero(σ,τ)-Biderivation such that $F(\mathcal{I},\mathcal{I})\subset C_{\alpha,\beta}$. Then R is a commutative ring.

Proof:

According to hypothesis, for any $u,v,\omega\in\mathcal{I}$ we have:

$[F(u\omega,v),r]_{\alpha,\beta}=0$, for all $r\in R$. .... (1)

Equivalently

$F(u,\ v)[\sigma(\omega),\ \alpha(r)]+[F(u,\ v),\ r]_{\alpha,\beta}\sigma(\omega)+\tau(u)[F(\omega,\ v),\ r]_{\alpha,\beta}+[\tau(u),\ \beta(r)]\ F(\omega,v)=0$

According to (1) the above relation reduces to:
$F(u,v)\ [\sigma(\omega),\ \alpha(r)]+[\tau(u),\ \beta(r)]\ F(\omega,v)=0$, for all $u,v,\omega\in\mathcal{I}$, $r\in R$.

Taking $\theta(\omega)$ instead of r in the above relation where $\theta=\alpha^{-1}\sigma$, we get:

$[\tau(u),\ \beta\theta(\omega)]F(\omega,v)=0$, for all $u,v,\omega\in\mathcal{I}$. .... (2)

Putting uz instead of u in (2) and using (2), we arrive at:

$[\tau(u),\ \beta\theta(\omega)]\tau(z)\ F(\omega,v)=0$, for $u,v,\omega,z\in\mathcal{I}$.

Using the primeness of R and the fact that $\tau(\mathcal{I})\neq\{0\}$ is an ideal of R, we conclude:

$[\tau(u),\ \beta\theta(\omega)]=0$, for all $u\in\mathcal{I}$ **or** $F(\omega,v)=0$.

Consequently, since $\tau(\mathcal{I})$ is a nonzero ideal implies that for any $\omega\in I$ we have:

$\omega\in Z(R)$ **or** $F(\omega,v)=0$

If $F(\mathcal{I},\mathcal{I})=0$ then $F=0$ by lemma (2.6). So according to the hypothesis it must be $F(\mathcal{I},\mathcal{I})\neq 0$.

A consideration of Brauer's trick leads to $\mathcal{I}\subset Z(R)$,hence R is commutative by lemma (2.5).

Theorem 3.3:

Let R be a prime ring, $\mathcal{I}$ be a right ideal of R. Suppose $F{:}\mathcal{I}\times\mathcal{I}\longrightarrow R$ is a nonzero (σ,τ)-Biderivation such that $ImF\subset Z(R)$, then R is a commutative ring.

Proof:

Since $ImF\subset Z(R)$, and F is a nonzero, there exists nonzero elements $u,v\in\mathcal{I}$ such that
$F(u,v)\in Z(R)$. This means:

$[F(u, v), r] = 0$, for any $r \in R$. .... (1)

Replacing u by un in (1) and using (1), we arrive at:
$F(u, v)\ [\sigma(n), r] + [\tau(u), r]\ F(n, v)=0$, for all $u,v,n\in \mathcal{J}$, $r \in R$.

Taking $\sigma(n)=F(z, \omega)$, $z, \omega \in \mathcal{J}$ implies that:
$[\tau(u), r]\ F(\sigma^{-1}F(z, \omega), v)=0$, for all $u,v,z, \omega \in \mathcal{J}$, $r \in R$. .... (2)

Putting sr for r in (2) and using (2) leads to:
$[\tau(u), s]\ r\ F(\sigma^{-1}F(z, \omega), v)=0$, for all $u,v,z, \omega \in \mathcal{J}$, $r,s\in R$.

That is
$[\tau(u), s]\ R\ F(\sigma^{-1}F(z, \omega), v)=0$, for all $u,v,z, \omega \in \mathcal{J}$, $s \in R$.

But R is a prime ring and F is nontrivial, so we have $[\tau(u), s]=0$, for all $u \in \mathcal{J}$, $s \in R$.

Therefore R is a commutative ring by lemma (2.5).

Theorem 3.4:

Let R be a prime ring, $\mathcal{J}$ a nonzero ideal of R. Suppose $F: R\times R\longrightarrow R$ is a nonzero (σ,τ)-Biderivation. If there exists an element $\omega \in \mathcal{J}$ satisfying $[F(u, v), \omega]_{\sigma,\tau} = 0$, for all $u,v\in \mathcal{J}$ then $\omega \in Z(\mathcal{J})$.

Proof:

If R is commutative, then there is nothing to prove, so we can suppose R is non-commutative.

Let ω be an element of $\mathcal{J}$ with:
$[F(u, v), \omega]_{\sigma,\tau} = 0$, for all $u,v \in \mathcal{J}$.

That is
$F(u, v)\sigma(\omega)- \tau(\omega)F(u, v)=0$, for all $u,v \in \mathcal{J}$. .... (1)

Putting uz instead of u leads to:
$F(u, v)\sigma(z)\sigma(\omega)+\tau(u)F(z, v)\ \sigma(\omega)- \tau(\omega)F(u,v)\ \sigma(z) - \tau(\omega)\ \tau(u)F(z, v)=0$, for all $u,v,z \in \mathcal{J}$.

In view of (1) the above relation reduces to:
$F(u, v)[\ \sigma(z), \sigma(\omega)]-[\tau(\omega), \tau(u)]F(z, v)=0$, for all $u,v,z\in \mathcal{J}$. .... (2)

Replacing u by ωu in (2) and using (2) implies that:
$D(\omega, y)\sigma(u)[\sigma(z),\sigma(\omega)]=0$, for all $u,v,z\in \mathcal{J}$.

That is
$\sigma^{-1}(F(\omega, v))\mathcal{J}\ [z, \omega]=0$, for all $u,v,z\in \mathcal{J}$.

Since $\mathcal{J}$ an ideal of R, we conclude that:
$\sigma^{-1}(F(\omega, v))\ \mathcal{J}\ R\ [z, \omega]=0$, for all $u,v,z\in \mathcal{J}$.

Using the primeness of R, either
$\sigma^{-1}(F(\omega,v))\ \mathcal{J} = 0$ or $[z,\omega]=0$, for all $u,v,z\in \mathcal{J}$.

If $[z,\omega]=0$, for all $z \in \mathcal{J}$ then as a direct conclusion we have $\omega \in Z(\mathcal{J})$.

On the other hand

If $\sigma^{-1}(F(\omega,v))\,\mathcal{J}=0$, since $\mathcal{J}$ is a nonzero ideal of R, again the primeness of R leads to:

$\sigma^{-1}(F(\omega, v))=0$, for all $v \in \mathcal{J}$.

By theorem (3.1) there exists an invertible element $b\in \mathbb{Q}_s$ such that:

$\sigma^{-1}(F(\omega, v)) = b\ [\omega, v]$, for all $v \in \mathcal{J}$.

Consequently we get $[\omega, v]=0$ for all $v \in \mathcal{J}$, and hence $\omega \in Z(\mathcal{J})$. ■

Theorem 3.5:

Let R be a prime ring, $\mathcal{J}$ a nonzero ideal of R. Suppose $F_1: R\longrightarrow R$ is a (σ,τ)-derivation and $F_2: R\times R\longrightarrow R$ is a (α, β)-Biderivation such that $ImF_2=R$. If $F_1F_2(\mathcal{J},\mathcal{J})=0$, then $F_1=0$ or $F_2=0$.

Proof:

For any $u,v, \omega \in \mathcal{J}$ we have:

$0=F_1F_2(u\omega,v)$
$=F_1\mathbf{(}F_2(u, v)\alpha(\omega) + \beta(u)F_2(\omega, v)\mathbf{)}$
$=F_1F_2(u, v)\sigma\alpha(\omega) + \tau F_2(u, v)F_1\alpha(\omega)$
$+F_1\beta(u)\sigma F_2(\omega, v) + \tau\beta(u)F_1F_2(\omega, v)$

According to our hypothesis, the above relation reduces to:

$\tau F_2(u, v)F_1\alpha(\omega)+F_1\beta(u)\sigma F_2(\omega, v)=0$, for all $u,v, \omega \in \mathcal{J}$. .... (1)

Replacing u by ru, $r \in R$ in (1), we get:
$\tau F_2(r,v)\ \tau\alpha(u)F_1\alpha(\omega)+ \tau\beta(r)\ \ \tau F_2(u,v)F_1\alpha(\omega) +F_1\beta(r)\ \sigma\beta(u)\sigma F_2(\omega,v) + \tau\beta(r)F_1\beta(u)\ \sigma F_2(\omega, v) =0$, for all $u,v, \omega \in \mathcal{J}$ and $r \in R$.

In view of (1) the above relation becomes:
$\tau F_2(r, v)\ \tau\alpha(u)F_1\alpha(\omega)+F_1\beta(r)\sigma\beta(u)\sigma F_2(\omega, v) =0$, for all $u,v, \omega \in \mathcal{J}$, $r\in R$. .... (2)

Putting $r=\beta^{-1}F_2(z, v)$, $z \in \mathcal{J}$ in (2) leads to:
$\tau F_2(\beta^{-1}F_2(z, v), v)\ \tau\alpha(u)F_1\alpha(\omega)=0$, for all $u,v,z, \omega \in \mathcal{J}$.

Equivalently

$F_2(\beta^{-1}F_2(z,v),v)\ \alpha(\mathcal{I})\tau^{-1}F_1\alpha(\mathcal{I})=\{0\}$, for all $v,z\in \mathcal{I}$.

Since $\alpha(\mathcal{I})$ is a nonzero ideal of R, using the primeness of R we get either $F_1\alpha(\mathcal{I})=\{0\}$ and consequently $F_1=0$ by [8, lemma 2].

Otherwise

$F_2(\beta^{-1}F_2(z, v), v)=0$, for all $v,z \in \mathcal{I}$. .... (3)

The substitution $F_2(z, v)\beta(u)$ for $F_2(z, v)$ in (3) and using (3), we arrive at:

$(F_2(z,v))^2=0$, for all $v,z \in \mathcal{I}$.

Again using the primeness of R leads to:

$F_2(z, v)=0$, for all $v,z \in \mathcal{I}$.

By application of lemma (2.6), we have $F_2=0$. ∎

Theorem 3.6:

Let R be a prime ring, $a \in R$. Suppose that $F:R\times R\longrightarrow R$ is a nonzero (σ,τ)-Biderivation satisfies that $[ImF, a]_{\alpha,\beta}=0$, then $a \in Z(R)$ or $F(\tau^{-1}\beta(a), t)=0$.

Proof:

Define $h: R\longrightarrow R$ by $h(x)=[x, a]_{\alpha,\beta}$ for all $x \in R$ then:

$h(xy)= h(x)\, y + x\, f_1(y) = f_2(x)y + x\, h(y)$, for all $x,y\in R$.

Where $f_1(x) = [x, \alpha(a)]$, $f_2(x)= [x, \beta(a)]$, for all $a \in R$, therefore we have:

$h(F(r, t)) = 0$, for all $r,t\in R$. .... (1)

Replacing r by rs in (1), we get:

$0= h(F(r, t)\, \sigma(s) + \tau(r)F(s, t))$

$= hF(r, t)\sigma(s) +F(r, t)\, f_1\sigma(s) + f_2\tau(r)\, F(s, t) + \tau(r)hF(s, t)$, for all $r,s,t\in R$

According to (1) the above relation reduces to:

$F(r, t)\, f_1\sigma(s)+f_2\tau(r)\, F(s, t)=0$, for all $r,s,t \in R$.

That is

$F(r, t)\ [\sigma(s), \alpha(a)] + [\tau(r), \beta(a)]F(s, t)=0$, for all $r,s,t\in R$.

The substitution $\tau^{-1}\beta(a)$ for r leads to:

$F(\tau^{-1}\beta(a), t)\ [\sigma(s), \alpha(a)]=0$, for $s,t\in R$. ...... (2)

Putting sc instead of s in (2) and using (2), we arrive at:

$F(\tau^{-1}\beta(a), t)\, \sigma(s)\ [\sigma(c), \alpha(a)] =0$, for all $c,s,t\in R$.

That is

$F(\tau^{-1}\beta(a), t)\ R[\sigma(c), \alpha(a)] =0$, for all $c,t\in R$.

Using the primeness of R we get the assertion of theorem.

Corollary 3.7:

Let R be a prime ring, $\mathcal{I}$ a nonzero ideal of R. Suppose that $F: R\times R\longrightarrow R$ is a nonzero (σ, τ)- Biderivation such that $[ImF , \mathcal{I}]_{\alpha,\beta} =0$, then R is commutative ring.

Proof:

Let $[ImF ,\mathcal{I}]_{\alpha,\beta} =0$, then for all $u\in \mathcal{I}$, $t \in R$, we have either $u \in Z(R)$ or $F(\tau^{-1}\beta(u), t)=0$.

If $F(\tau^{-1}\beta(u), t)=0$ for all $u \in \mathcal{I}$, $t \in R$, since $U=\tau^{-1}\beta(\mathcal{I})$ is a nonzero ideal then in particularly we have $F(\mathcal{I}, \mathcal{I})=0$, using lemma (2.6) it follows that $F=0$ which contradicts the hypothesis.

Hence $\mathcal{I} \subset Z(R)$, which forces $\mathcal{I}$ to be commutative, consequently R is commutative by lemma (2.4).

Theorem 3.8:

Let R be a 2-torsion free prime ring, $\mathcal{I}$ a nonzero ideal of R. Suppose that $F: R\times R\longrightarrow R$ is a nonzero (σ, τ)-Biderivation such that $F(x\omega, y) = F(\omega x, y)$ for all $x,y, \omega \in \mathcal{I}$, then R is commutative ring.

Proof:

For any $u \in \mathcal{I}$ such that $F(u, y) =0$, for all $y \in \mathcal{I}$, like $u =[x, s]$ we have:

$F(\omega,y)\, \sigma(u) =F(\omega u,y) = F(u\omega,y) = \tau(u)F(\omega, y)$ for all $y,\omega \in \mathcal{I}$.

That is:

$[F(\omega, y), u]_{\sigma,\tau} =0$, for all $y,\omega \in \mathcal{I}$.

An application of theorem (2.6) implies that $u \in Z(\mathcal{I})$.

Hence the conclusion is: for any $u \in \mathcal{I}$ satisfy that $F(u, y) =0$, for all $y \in \mathcal{I}$, we get $u \in Z(\mathcal{I})$.

According to the above conclusion have:

$[x, s]\in Z(\mathcal{I})$, for all $x,s\in \mathcal{I}$ and consequently we have:

$[t, [x, s]] =0$, for all $x,s,t\in \mathcal{I}$.

The substitution *xs* for *s* in the above relation gives:

$[t, [x, xs]] = [t, x] [x, s] = 0$, for all$x,s,t \in \mathcal{J}$.

Putting *st* for s leads to:

$[t, x] s [x, t] = 0$, for all$x,s,t \in \mathcal{J}$.

Equivalently

$[t, x]\mathcal{J}[x, t] = 0$, for all$x,t \in \mathcal{J}$.

But $\mathcal{J}$an ideal of *R*, then:

$[t, x] R \mathcal{J}[x, t] = 0$, for all$x,t \in \mathcal{J}$.

The primness of *R* leads us to conclude that either $[t, x] = 0$ or $\mathcal{J} [x, t] = 0$ for all$x,t \in \mathcal{J}$.
If $\mathcal{J}[x, t]=0$ for all$x,t \in \mathcal{J}$, since $\mathcal{J}$ is a nonzero ideal of *R*, we have:

$[x, t] = 0$, for all $x, t \in \mathcal{J}$.

This means that $\mathcal{J}$ is commutative, and by application of lemma (2.3), we have $\mathcal{J} = Z(\mathcal{J}) \subset Z(R)$.
Finally using lemma (2.4), we conclude that *R* is commutative. ■

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الخلاصة

لتكن *R* حلقة أولية مميزها لايساوي $\mathcal{J},2$ $\neq\{0\}$ مثالي في *R*. في هذا البحث ولأجل $\alpha, \beta, \sigma, \tau$ تشاكلات تقابلية على *R*, قدمنا بعض النتائج المرتبطة بالعلاقة بين أبدالية الحلقة *R* ووجودية أنواع خاصة من ثنائيات المشتقات-(σ, τ).

برهنا إن الحلقة الأولية *R* تكون أبدالية إذا حققت ثنائية المشتقة-(σ, τ) غير الصفرية $F:R\times R\longrightarrow R$ أحد الشروط التالية:

(i) $F(\mathcal{J}, \mathcal{J}) \subset C_{\alpha,\beta}$
(ii) $[ImF, \mathcal{J}]_{\alpha,\beta} = 0$
(iii) $F(x\omega, y) = F(\omega x, y)$for all $x, y, \omega \in \mathcal{J}$.