Al-Nahrain Journal of Science
ANJS, Vol.25 (1), March, 2022, pp. 35-40

Mixed Homotopy Integral Transform Method for Solving Non-Linear Integro-Differential Equation

Mohanad F. Kadhem* and Ali H. Alfayadh

Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University, Baghdad-Iraq

Article's Information

Received:
08.10.2021
Accepted:
09.02.2022
Published:
28.03.2022

Keywords:
Sawi transform
Integro-differential equations
Homotopy perturbation method
Integral transform
Linear differential equations

Abstract

In this paper, we have applied Sawi transform with homotopy perturbation method to obtain analytic approximation for non-linear integro-differential Equations. The proposed technique is compared with homotopy perturbation method and Abood transform homotopy perturbation method. The results show that Sawi transform homotopy perturbation is an efficient approach to solve non-linear integro-differential equations.

DOI: 10.22401/ANJS.25.1.06

*Corresponding author: moha_fat_1992@yahoo.com

1. Introduction

In recent years the principles of integro-differential equation have It inspired a tremendous amount of research work. Many methods can be used to solve integro-differential equations such as, Wavelet-Galerkin [1], Lagrange interpolation [2], compact finite difference [3], rationalize approximate solution [8], variation [7], conjugate gradient [6], quadrature difference [9], collocation [10], Homotopy perturbation [11] and Euler-Chebyshev method [12]. In the last several years Wazwaz [13] found that a brand-new combined style of the Laplace transform method with the Adomian decomposition method is being developed for analytical treatment of the nonlinear integro-differential equations. The combined system is capable of treating both first and second kind equations. Aruchunan and Sulaiman proposed a Numerical solution of first kind linear Fredholm integro-differential equation using conjugate gradient method [14]. In 2019 Maitama, S.; Zhao found a new integral transform and used to solve differential equation [15]. Singh, Kumar and Sushila proposed a hybrid variety of the Sumudu transform method with the homotopy perturbation method [16] to unravel nonlinear equations. The proposed scheme finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. Asem and Zarita used the homotopy perturbation with Sumudu transform method to seek out the solutions of nonlinear differential equations in an exceedingly series form. They discussed that this scheme avoids the round-off errors and finds the answer with none restrictive assumptions or discretization. After comparing figures between approximate, homotopy perturbation Sumudu transform, homotopy perturbation Laplace transform and exact solutions,they found that the solutions they need obtained are of high accuracy. Aggarwal et al. [17] applied Mahgoub transform for solving linear Volterra integral equations of first kind. Sawi transform is a new integral transform was introduced by Mohand and Abdelrahim to facilitate the process of solving differential equations, integral equations and integro-differential equations in the time domain [18]. In [19] they used Sawi transformation and decomposition Method to solving Volterra integral equation. In [20] they used Sawi Transform of Bessel's functions with application for evaluating definite integrals. In 2021, Higazy used Sawi transformation to solve system of ordinary differential equations [21]. Let us consider the general form of integro-differential equation of the following type:

$$y^{(n)} = f(x) + \int_a^b k(x.t)\, F\big(y(t)\big)dt \qquad (1)$$

$$y(0) = \alpha, 0 \le x \le 1 \qquad (2)$$

where the function $f(x)$ and the kernals $k(x.t)$ are given real-valued functions and the function $F(y(t))$ be non-linear function of t.

The advantage of the Sawi transform Homotopy Perturbation Method (SHPM) is its capability of combining the two powerful methods for obtaining exact solutions for linear and nonlinear integro-differential equations.

2. Preliminaries

2.1 Sawi transform:

For a given function $f(t)$, $t \geq 0$ then Sawi transform is defined as [18]:

$$S[f(t)] = \frac{1}{\mu^2}\int_0^\infty f(t)e^{-\frac{1}{\mu}t}dt$$
$$= T(\mu), k_1 \leq \mu \leq k_2, k_1, k_2 > 0 \quad (3)$$

2.2 Sawi Transform of some function [18]:

For any function $f(t)$, we assume that the integral eq. (1) exist. The sufficient conditions for the existence of Sawi transform are that $S[(t)]$ for $t \geq 0$ be piecewise continuous and of exponential order, otherwise Sawi transform may or may not exist.

$$S[1] = \frac{1}{\mu} S[t]$$
$$= 1\, S[t^n]$$
$$= \mu^{n-1} n!\, S[e^{at}]$$
$$= \frac{1}{\mu(1-a\mu)}$$
$$S[\sin at] = \frac{a}{1+a^2\mu^2} S[\cos at]$$
$$= \frac{1}{v(1+a^2\mu^2)}$$

Theorem 1. If Sawi transform of $f(t)$ is given by $S\{f(t)\} = T(\mu)$, then:

i. $S[f'(t)] = \frac{T(\mu)}{\mu} - \frac{f(0)}{\mu^2}$.

ii. $S[f''(t)] = \frac{1}{\mu^2}T(\mu) - \frac{1}{\mu^2}f'(0) - \frac{1}{\mu^3}f(0)$.

iii. $S[f^{(n)}(t)] = \frac{T(\mu)}{\mu^n} - \sum_{k=0}^{n-1}\frac{f^k(0)}{\mu^{n-k+1}}$.

Proof.

i. Since:

$$S[f'(t)] = \frac{1}{\mu^2}\int_0^\infty f'(t)e^{-\frac{t}{\mu}}dt$$

Integrating by parts, we get:

$$S[f'(t)] = \frac{T(\mu)}{\mu} - \frac{f(0)}{\mu^2}$$

ii. Let $M(t) = f'(t)$. Then:

$$S[M'(t)] = \frac{T(\mu)}{\mu} - \frac{f(0)}{\mu^2}$$

We find that by using i,

$$S[f''(t)] = \frac{1}{\mu^2}T(\mu) - \frac{1}{\mu^2}f'(0) - \frac{1}{\mu^3}f(0)$$

iii. Its proved by Mathematical induction.

2.3 Inverse Sawi transformation:

Inverse Sawi transformation of $T(\mu)$, designated by $S^{-1}\{T(\mu)\}$, is another function $w(t)$ having the property that:

$$S[w(t)] = T(\mu)$$

i.e.

$$S^{-1}\{T(\mu)\} = w(t)$$

2.4 Sawi transform homotopy perturbation method:

In this section, we propose SHPM to the following integro-differential equation:

$$y^{(n)}(x) = f(x) + \lambda\int_a^b k(x.t)\, F\big(y(t)\big)dt \quad (4)$$

with initial conditions:

$$y(0) = z_0, y'(0) = z_1, \cdots, y^{(n-1)}(0) = z_{n-1} \quad (5)$$

for all $n = 0,1,...$

Take Sawi transform to both side of the eq. (4) and using eq. (5), we get:

$$\frac{1}{\mu^n}S\{y(x)\} = \frac{z_0}{\mu^{n+1}} + \frac{z_1}{\mu^n} + \cdots + \frac{z_{n-1}}{\mu^2} + S\{f(x)\} + \lambda S\left[\int_a^b k(x.t)\, F\big(y(t)\big)dt\right]$$

$$T(\mu) = \frac{1}{\mu}z_0 + z_1 + \cdots + \mu^{n-2}z_{n-1} + \mu^n S\{f(x)\} + \mu^n\lambda S\left[\int_a^b k(x.t)\, F\big(y(t)\big)dt\right] \quad (6)$$

where $T(\mu) = S\{y(x)\}$. Now apply Sawi inverse transform to th both sides of eq. (6), we get:

$$y(x) = z_0 + xz_1 + \cdots + \frac{x^{n-1}}{(n-1)!}z_{n-1} + S^{-1}\left\{\mu^n S\{f(x)\} + \mu^n\lambda S\left[\int_a^b k(x.t)\, F\big(y(t)\big)dt\right]\right\}$$

Take homotopy perturbation method [22] to the above equation, we get:

$$y_0(x) = z_0 + xz_1 + \cdots + \frac{x^{n-1}}{(n-1)!}z_{n-1} + S^{-1}\{\mu^n S\{f(x)\}\} \quad (7)$$

$$\sum_{n=0}^\infty p^n\, y_n(x) = S^{-1}\left\{\mu^n\lambda S\left[\int_a^b k(x.t)\, F\big(y(t)\big)dt\right]\right\}p \quad (8)$$

From eq. (7), we get $y_0(x)$, and from finding the coefficient of p in eq. (8), we get $y_1.y_2.\cdots$ and then the series solution is:

$$y = \lim_{p\to 1}\left[\sum_{n=0}^\infty p^n y_n\right]$$

which is the solution of eq. (4).

3. Examples and Discussions

Example 1, [22]. Consider the linear integro-differential equation of second kind:

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 xty(t)dt, y(0) = 0 \quad (9)$$

Take Sawi transform on both side of eq. (9), we get:

$$Sy'(x) = S\left[1 - \frac{1}{3}x\right] + S\left[\int_0^1 xty(t)dt\right]$$
$$\frac{T(\mu)}{\mu} - \frac{y(0)}{\mu^2} = \frac{1}{\mu} - \frac{1}{3} + S\left[\int_0^1 xty(t)dt\right]$$
$$T(\mu) = 1 - \frac{1}{3}\mu + S\mu\left[\int_0^1 xty(t)dt\right]$$

Now, take the inverse of Sawi transform on both side of the above equation, we get:

$$y(x) = x - \frac{x^2}{6} + S^{-1}\left[S\mu\left[\int_0^1 xty(t)dt\right]\right] \quad (10)$$

Take the homotopy perturbation method to eq. (10) on both sides we get,

$$\sum_{n=0}^\infty p^n\, y_n(x) = pS^{-1}\left\{\mu S\left[x\int_0^1 \sum_{n=0}^\infty p^n H_n(x)\, dx\right]\right\} \quad (11)$$

Equating the identical powers of p in eq.(11), we get:

$$y_0(x) = x - \frac{x^2}{6}$$
$$y_1(x) = S^{-1}\left[S\mu\left[\int_0^1 xty_0(t)dt\right]\right]$$
$$y_1(x) = \frac{29}{100}\, S^{-1}(\mu)$$

$y_1(x) = \frac{29}{200}x^2$

Similarly, we find:

$y_2(x) = \frac{29}{1600}x^2$

$y_3(x) = \frac{29}{12800}x^2$

$y_4(x) = \frac{29}{51200}x^2$

$y_5(x) = \frac{29}{204800}x^2$

$y(x) = \sum_{n=0}^{\infty} y_n(x)$

$= x - \frac{x^2}{6} + \frac{29}{200}x^2 + \frac{29}{1600}x^2 + \frac{29}{12800}x^2 + \frac{29}{51200}x^2 + \frac{29}{204800}x^2 + \cdots$

$y(x) = x - \frac{97}{204800}x^2$

Table 1. Comparison between SHPM, exact and ATHPM solutions [22].

x	Exact	SHPM	ATHPM [22]	|Exact – SHPM|
0	0	0.0000000	0.0000000	0.0000000
0.1	0.1	0.0999943	0.1000000	5.6803385×10^{-6}
0.2	0.2	0.1999773	0.2000000	2.2721354×10^{-5}
0.3	0.3	0.2999489	0.2999999	5.1123047×10^{-5}
0.4	0.4	0.3999091	0.3999998	9.0885417×10^{-5}
0.5	0.5	0.4998580	0.4999998	1.4200846×10^{-4}
0.6	0.6	0.5997955	0.5999997	2.0449219×10^{-4}
0.7	0.7	0.6997217	0.6999995	2.7833659×10^{-4}
0.8	0.8	0.7996365	0.7999994	3.6354167×10^{-4}
0.9	0.9	0.8995399	0.8999992	4.6010742×10^{-4}
1	1	0.9994320	0.9999990	5.6803385×10^{-4}

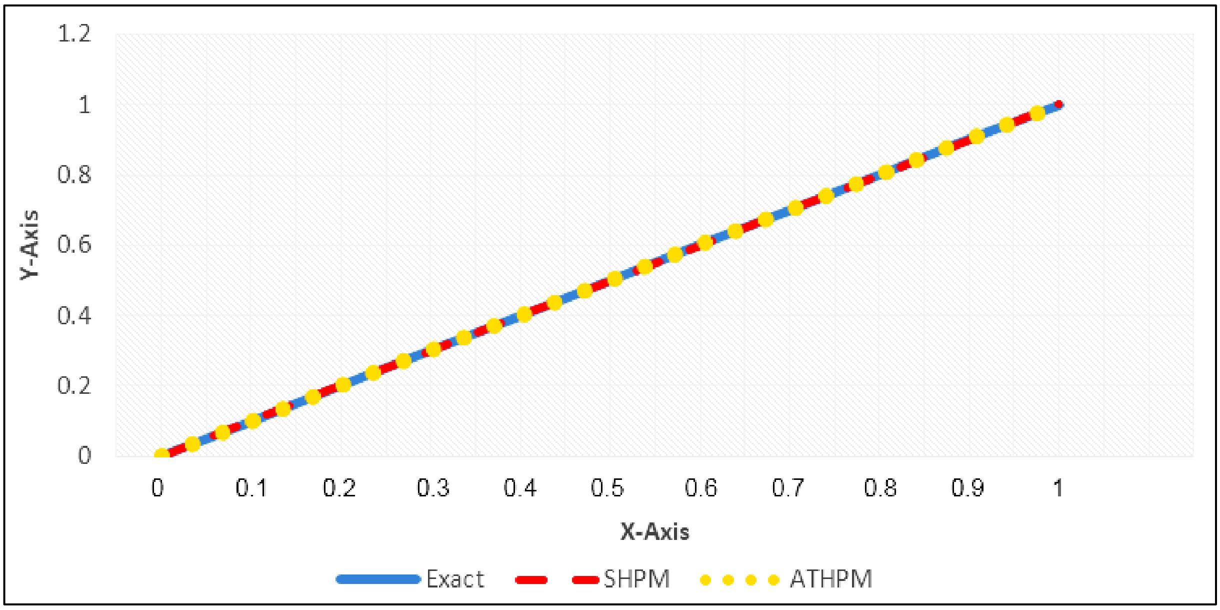

Figure 1. Comparison between SHPM, Exact and ATHPM solutions [22].

From Figure 1 and Table 1, we see that the new method is very accurate by comparing the obtained results with the exact and ATHPM [22] and it's made the process easily.

Example 2, [22]. Consider the nonlinear integro-differential equation:

$$y'(x) = 1 - \frac{1}{3}x^3 + \int_0^1 x^3 y^2(t)dt, y(0) = 0 \quad (12)$$

Take Sawi transform on both side of eq. (12), we get:

$$Sy'(x) = S\left[1 - \frac{1}{3}x^3\right] + S\left[\int_0^1 x^3 y^2(t)dt\right]$$

$$\frac{T(\mu)}{\mu} - \frac{y(0)}{\mu^2} = \frac{1}{\mu} - \frac{3!}{3}\mu^2 + S\left[\int_0^1 x^3 y^2(t)dt\right]$$

$$T(\mu) = 1 - 2!\,\mu^3 + S\mu\left[\int_0^1 x^3 y^2(t)dt\right]$$

Now, take the inverse of Sawi transform on both sides of the above equations, we get:

$$y(x) = x - \frac{x^4}{12} + S^{-1}\left[S\mu\left[\int_0^1 x^3 y^2(t)dt\right]\right] \quad (13)$$

Take homotopy perturbation method to eq. (13) on both sides, we get:

$\sum_{n=0}^{\infty} p^n y_n(x) =$
$pS^{-1}\left\{\mu S\left[x^3 \int_0^1 \sum_{n=0}^{\infty} p^{n-1} H_{n-1}(x)\, dt\right]\right\}$ (14)

where $H_n(t)$ are the Adomain polynomial for the nonlinear term $y^2(t)$.

Equating the identical powers of p in eq. (11), we get:

$y_0(x) = x - \frac{x^4}{12}$

$y_1(x) = S^{-1}\left[S\mu\left[\int_0^1 x^3 t y_0{}^2(t) dt\right]\right]$

$y_1(x) = \frac{397}{1287} S^{-1}(3!\,\mu^3)$

$y_1(x) = \frac{397}{5183} x^4$

Similarly, we find:

$y_2(x) = \frac{6749}{1119744} x^4$

$y(x) = \sum_{n=0}^{\infty} y_n(x)$

$= x - \frac{x^4}{12} + \frac{397}{5183} x^4 + \frac{6749}{1119744} x^4 + \cdots$

$y(x) = x - \frac{4117661}{5803633152} x^4$

Table 2. Comparison between SHPM, Exact and HPM solutions.

x	Exact	SHPM	ATHPM [22]	|Exact − SHPM|
0	0	0.0000000	0.0000000	0.0000000
0.1	0.1	0.0999999	0.0999996	7.0949712×10^{-8}
0.2	0.2	0.1999988	0.1999939	1.1351954×10^{-6}
0.3	0.3	0.2999941	0.2999691	5.7469267×10^{-6}
0.4	0.4	0.3999815	0.3999024	1.8163126×10^{-5}
0.5	0.5	0.4999547	0.4997617	4.4343570×10^{-5}
0.6	0.6	0.5999061	0.5995059	9.1950827×10^{-5}
0.7	0.7	0.6998261	0.6990846	1.7035026×10^{-4}
0.8	0.8	0.7997033	0.7984384	2.9061002×10^{-4}
0.9	0.9	0.8995248	0.8974987	4.6550106×10^{-4}
1	1	0.9992757	0.9961876	7.0949712×10^{-4}

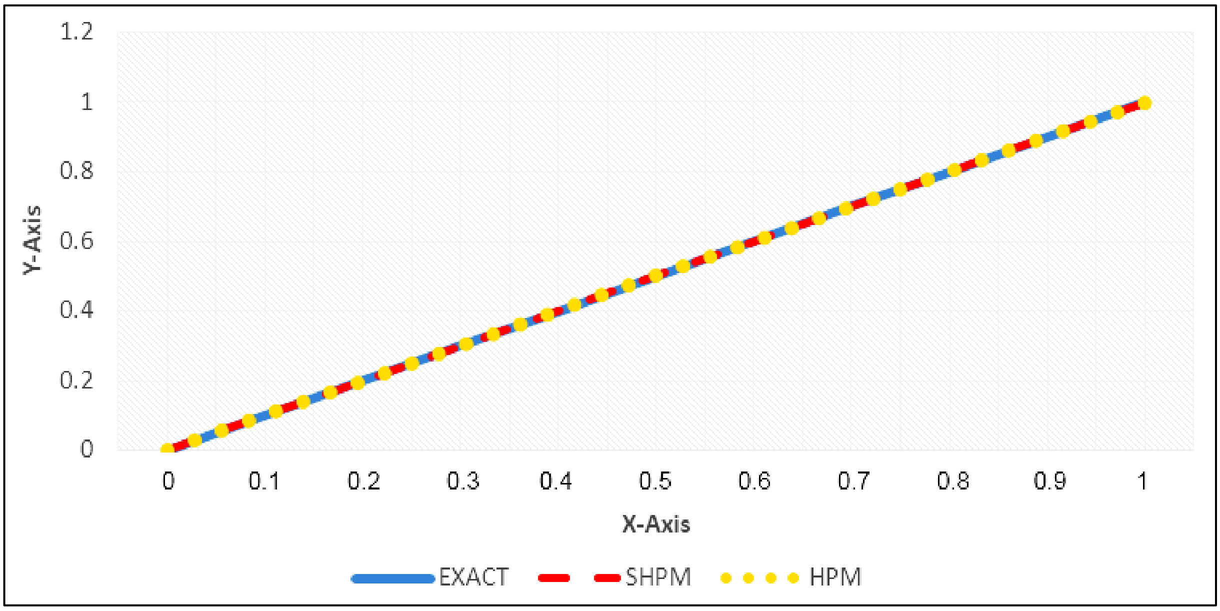

Figure 2. Comparison between SHPM, Exact and ATHPM solutions [22].

From Figure 2 and Table 2, we see that the new method is very accurate by comparing the obtained results with the exact and ATHPM [22] and it's made the process easily.

Example 3, [23]. Consider the following nonlinear integro-differential equation:

$y'(x) = -\frac{1}{2} + \int_0^x y'^2(t)dt, y(0) = 0$ (15)

Take Sawi transform on both side of eq. (15), we get:

$Sy'(x) = S\left[-\frac{1}{2}\right] + S\left[\int_0^x y'^2(t)dt\right]$

$\frac{T(\mu)}{\mu} - \frac{y(0)}{\mu^2} = -\frac{1}{2\mu} + S\left[\int_0^x y'^2(t)dt\right]$

$T(\mu) = -\frac{1}{2}\mu + S\mu\left[\int_0^x y'^2(t)dt\right]$

Now, take inverse of Sawi transform on both side of the above eq. we get

$$y(x) = -\frac{x}{2} + S^{-1}\left[S\mu\left[\int_0^x y'^2(t)dt\right]\right] \quad (16)$$

Take homotopy perturbation method to eq. (16) on both sides, we get:

$$\sum_{n=0}^{\infty} p^n y_n(x) = pS^{-1}\left\{\mu S\left[x^3 \int_0^1 \sum_{n=0}^{\infty} p^{n-1} H_{n-1}(x)\, dt\right]\right\} \quad (17)$$

where $H_n(t)$ are the adomain polynomial for the nonlinear term $y'^2(t)$.

Equating the identical powers of p in eq. (17), we get:

$$y_0(x) = -\frac{x}{2}$$

$$y_1(x) = \frac{1}{8}x^2$$

$$y_2(x) = \frac{-1}{24}x^3$$

$$y_3(x) = \frac{1}{64}x^4$$

$$y_4(x) = \frac{-7}{160}x^5$$

$$y_5(x) = \frac{1}{320}x^6$$

$$y_6(x) = \frac{-1}{896}x^7$$

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = -\frac{x}{2} + \frac{1}{8}x^2 - \frac{1}{24}x^3 + \frac{1}{64}x^4 - \frac{7}{160}x^5 + \frac{1}{320}x^6 - \frac{1}{896}x^7 + \cdots$$

Table 3. Comparison between SHPM, Exact and HPM solutions.

x	Exact	SHPM	HPM [23]	|SHPM − Exact|
0	0.0000000	0.0000000	0.0000000	0.0000000
0.1	−0.0487902	−0.0487902	−0.0487902	5.1615824×10^{-10}
0.2	−0.0953102	−0.0953101	−0.0953101	3.2185277×10^{-8}
0.3	−0.1397619	−0.1397616	−0.139761	3.5141534×10^{-7}
0.4	−0.1823216	−0.1823197	−0.1823197	1.8615559×10^{-6}
0.5	−0.2231436	−0.2231370	−0.2231370	6.5767978×10^{-6}
0.6	−0.2623643	−0.2623464	−0.2623464	1.7821610×10^{-5}
0.7	−0.3001046	−0.3000648	−0.3000648	3.9790627×10^{-5}
0.8	−0.3364722	−0.3363962	−0.3363962	7.6046145×10^{-5}
0.9	−0.3715636	−0.3714371	−0.3714371	1.2649605×10^{-4}
1	−0.4054651	−0.4052827	−0.4052827	1.8237001×10^{-4}

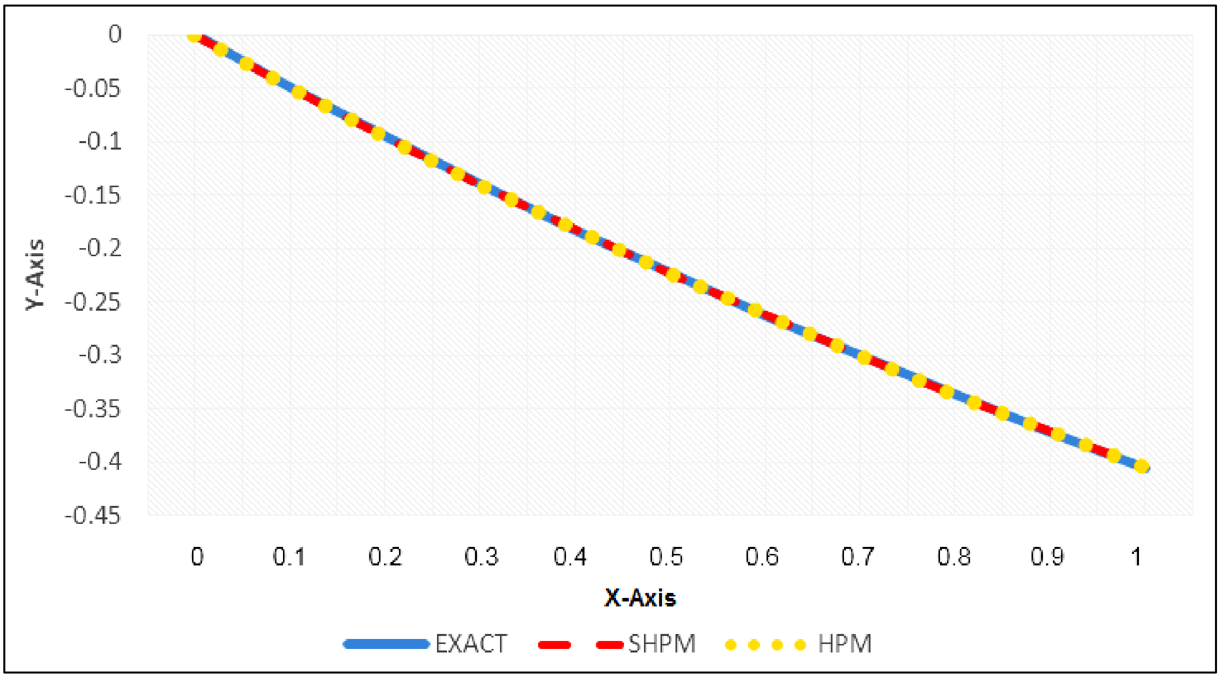

Figure 3. Comparison between SHPM, Exact and HPM solutions [23].

From Figure 3 and Table 3, we see that the new method is very accurate by comparing the obtained results with the exact and HPM [23] and its made the process easily.

4. Conclusion

The mixed of homotopy perturbation method with Sawi transform shows that it's a very effective method to solve non-linear integro-differential equations. We have compared the result obtained by Sawi homotopy Perturbation method (SHPM) with exact solution and ATHPM [22] and HPM [23]. We conclude that the new suggested method is an efficient to handle non-linear integro-differential equations.

Acknowledgments
We would like to thanks Al-Nahrain University for their support.

Conflicts of Interest
The authors declare that there is no conflict of interest.

References
[1] Avudainayagam A. and Vani C.; "Wavelet-Galerkin method for integro-differential equations", Applied Math. Comput., 32: 247-254, 2000.
[2] Rashed M. T.; "Lagrange interpolation to compute the numerical solutions differential and integrodifferential equations", Applied Math. Comput., 151: 869-878, 2003.
[3] Zhao and Corless R.M.; "Compact finite difference method for integro-differential equations", Applied Math. Comput., 177: 325-328, 2006.
[4] Maleknejad K.; Mirzaee F. and Abbasbandy S.; "Solving linear integro-differential equations system by using rationalized Haar functions method", Applied Math. Comput., 155: 317-328, 2004.
[5] Sweilam N. H.; "Fourth order integro-differential equations using variational iteration method", Comput. Math. Appli., 54: 1086-1091, 2007.
[6] Khosla P. K. and Rubin S. G.; "A conjugate gradient iterative method", Comput. Fluids, 9: 109-121, 1981.
[7] Saad Y. and Schultz M.H.; "A generalized minimal residual algorithm for solving nonsymmetric linear systems", SIAM J. Sci. Stat. Comput., 7: 856-869, 1986.
[8] Hosseini S. M. and Shahmorad S.; "Numerical piecewise approximate solution of Fredholm integro-differential equations by the Tau method", Applied Math. Model., 29, 2005.
[9] Fedotov A. I., "Quadrature-difference methods for solving linear and nonlinear singular integrodifferential equations", Nonlinear Anal., 71: 303-308, 2009.
[10] Aguilar M. and Brunner H.; "Collocation methods for second-order Volterra integro-differential equations", Applied Numer. Math., 4: 455-470, 1988.
[11] Yildirim A.; "Solution of BVPs for fourth-order integro-differential equations by using homotopy perturbation method", Comput. Math. Appli., 32: 711-1716, 2008.
[12] Van der Houwen P.J. and Sommeijer B.P.; "Euler-Chebyshe methods for integro-differential equations", Applied Numer. Math., 24: 203-218, 1997.
[13] Wazwaz A.; "The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations", Applied Mathematics and Computation, 216 (2020): 1304-1309, 2009.
[14] Aruchunan E. and Sulaiman J.; "Numerical solution of first kind linear Fredholm Integrodifferential equation using conjugate gradient method", Proceedings of the Curtin Sarawak 1st International Symposium on Geology, Sept., 2009: 11-13, 2009.
[15] Maitama S. and Zhao W.; "New Integral Transform: Shehu Transform a Generalization of Sumudu and Laplace Transform for Solving Differential Equations", Int. J. Anal. Appl., 17 (2): 167-190, 2019.
[16] Singh J.; Kumar D. and Sushila, "Homotopy Perturbation Sumudu Transform Method for Nonlinear Equations", Adv. Theor. Appl. Mech., 4 (4): 165-175, 2011.
[17] Aggarwal S.; Sharma N. and Chauhan R.; "Application of Mahgoub transform for solving linear Volterra integral equations of first kind", Global Journal of Engineering Science and Researches, 5(9): 154-161, 2018.
[18] Mohand M. and Abdelrahim M.; "The New Integral Transform "Sawi Transform", Advances in Theoretical and Applied Mathematics, 14 (1): 81-87, 2019.
[19] Higazy M.; Sudhanshu A. and Taher A. N.; "Sawi Decomposition Method for Volterra Integral Equation with Application", Journal of Mathematics, Hindawi, 18, December, 2020.
[19] Sudhanshu A.; Swarg D. S. and Aakansha V.; "Sawi Transform of Bessel’s Functions with Application for Evaluating Definite Integrals", International Journal of Latest Technology in Engineering, Management & Applied Science (IJLTEMAS), IX (VII), July, 2020.
[20] Higazy M. and Aggarwal S.; "Sawi transformation for system of ordinary differential equations with application", Ain Shams Engineering Journal, 2 April, 2021.
[21] Chandrali B. and Jayaprakasha P. C.; "Numerical solution for Fredhlom integro-differential Equation by Aboodh transform homotopy perturbation method and variational iteration method", Proteus Journals, 11 (11), 2020.
[22] Ghasemi M.; Tavassoli K. M. and Babolian E.; "Application of He’s homotopy perturbation method to nonlinear integro-differential equations", Applied Mathematics and Computation 188: 538-548, 2007.