Al-Nahrain Journal of Science
ANJS, Vol.25 (1), March, 2022, pp. 30-34

Estimate the Survival Function of the Power Lomax (POLO) Distribution by Using the Simulated Annealing Algorithm

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Article's Information

Received:
18.11.2021
Accepted:
11.01.2022
Published:
28.03.2022

Keywords:
The power Lomax distribution
POLO
Maximum likelihood method
The moment method
Simulated annealing algorithm
IMSE

Abstract

In this paper the survival function of the Power Lomax distribution is estimated by two methods of estimation, which are the maximum likelihood method and the moment method. The obtained estimators contain non-linear equations that cannot be solved by ordinary mathematical methods and do not represent the estimations clearly, so a simulated annealing algorithm was used to solve this problem, then simulation was used to compare the estimation methods based on the statistical comparison criterion mean squares of integral error (IMSE) and to get the best estimator for survival function. The results show the preference the maximum likelihood method than the moment method.

DOI: 10.22401/ANJS.25.1.05
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1. Introduction

In 2016, El-Houssainy et al. [1] introduced the Power Lomax (POLO) distribution, which is one of the extensions of the Lomax distribution. The structure and characteristics of the distribution were derived and studied, such as the probability density function, cumulative distribution function, survival function, hazard function, moments, arranged statistics, mode, flatness and entropy function. The parameters were estimated by the method of Maximum Likelihood Method (MLE) and applied real data for bladder cancer patients. In 2017 Moniem [2] studied the iterative relationships between the single moment and its product from the arranged statistics of the POLO distribution, and some results of the averages and variances of the arranged statistics of the POLO distribution were calculated and tabulated. Almetwally and Almongy in 2021 [3] estimated the stress function of the POLO distribution in several ways and compared them according to the (MSE) standard. The methods are the method of MLE, method (MPS) and method of Weighted Least Squares (WLS) and Baysian Estimator method using (LINEX) loss function, and through simulation and according to the comparison standard MSE, the results showed the preference of MPS method than the rest of the classical methods and it is a good alternative to MLE method in many cases, but the (bayes) estimator is the best among all the methods used in this paper. The analysis and study of the survival function one of the important branches of statistics, which has begun to increase in the past three decades through research work and publication in scientific journals, books and conferences, and by using various advanced statistical methods theoretically and practically through its estimation and obtaining the best efficient estimator.

The survival function is defined as the function that gives the probability of a person's survival from birth or the beginning of treatment until the occurrence of the "death or recovery" event [4]. Through this research, we will estimate the survival function of the POLO distribution, which is one of the extensions of the Lomax distribution. Through previous studies [1] it was shown that it's preference than the rest of the extensions of the Lomax distribution we will use estimation methods (Maximum Likelihood and Moments), but we have a problem of non-linear equations in the mentioned estimators that cannot be solved by ordinary mathematical methods, so we use simulated annealing algorithm to solve this problem.

2. Power Lomax Distribution

El-Houssainy et al., in 2016 introduced the Power Lomax (POLO) distribution. The POLO distribution is proposed by taking into account the power transformation [1]:

$$X = T^{1/\lambda}$$

where the random variable T follows Lomax distribution with parameters (θ,β), The distribution of X is known as the Power Lomax distribution, which is denoted symbolically by $X \sim$ POLO(θ,λ,β) to indicate that the random variable X has the power Lomax distribution with parameters θ, λ and β.

The PDF of the POLO distribution is defined by [1]:

$$f(x) = \theta\lambda\beta^{\theta}x^{\lambda-1}(\beta + x^{\lambda})^{-\theta-1} \quad (1)$$

where $x > 0, \theta, \beta, \lambda > 0$.

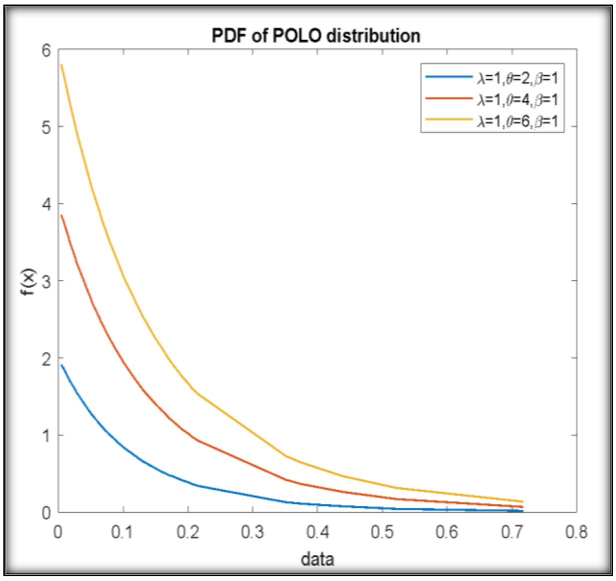

Figure 1. PDF of POLO distribution.

The Cumulative Distribution Function (CDF) of POLO distribution is given by [1]:

$$F(x) = 1 - \beta^{\theta}(x^{\lambda} + \lambda)^{-\theta} \quad (2)$$

where $x > 0, \theta, \beta, \lambda > 0$.

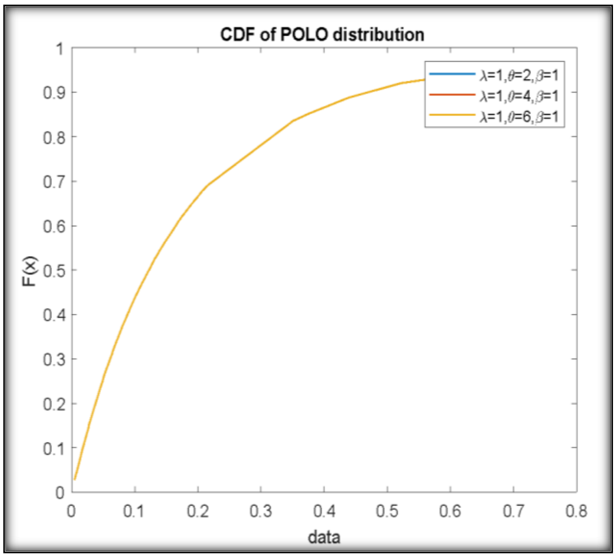

Figure 2. CDF of POLO distribution.

The survival function $S(t)$ of POLO distribution is given by [1]:

$$S(x) = \beta^{\theta}(x^{\lambda} + \lambda)^{-\theta} \quad (3)$$

where $x > 0, \theta, \beta, \lambda > 0$.

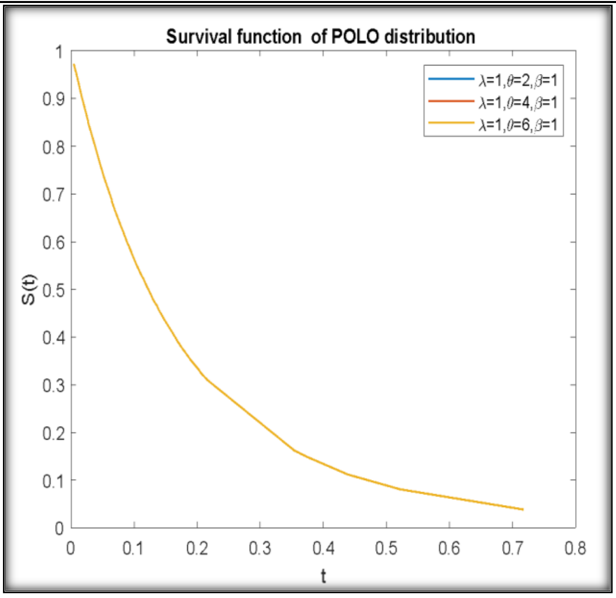

Figure 3. Survival function of POLO distribution.

3. Estimation Methods

3.1 Maximum likelihood method:

The Maximum Likelihood Method is regarded as one of the most important estimation methods because it has some excellent properties such as sufficiency and consistency in some cases, and it is more accurate than other estimation methods, particularly in large sample sizes [5].

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a POLO distribution with a density function (eq. (1)). We maximize the probability density function (eq. (1)) for the POLO distribution as follows:

$$L(x;\theta,\lambda,\beta) = \prod_{i=1}^{n}\theta\lambda\beta^{\theta}x_i^{\lambda-1}(\beta + x_i^{\lambda})^{-\theta-1} \quad (4)$$

For the purpose of estimating the maximum possibility function, we take the logarithm of both sides of the equation and partially derive with respect to the parameters θ, λ, β and to get:

$$\frac{\partial}{\partial\theta}L(\psi) = \frac{n}{\theta} + n\ln\beta - \sum_{i=1}^{n} ln(\beta + x_i^{\lambda}) \quad (5)$$

$L(\theta,\lambda,\beta) = L(\psi)$

Equalizing the derivative to zero, we get:

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n}(\hat{\beta} + x_i^{\hat{\lambda}}) - n ln\hat{\beta}} \quad (6)$$

Derive (eq. (5)) with respect to λ; to get:

$$\frac{\partial}{\partial\lambda}L(\psi) = \frac{n}{\lambda} + \sum_{i=1}^{n} ln\, x_i - (\theta+1)\sum_{i=1}^{n}\frac{x_i^{\lambda}\, ln\, x_i}{(\beta + x_i^{\lambda})} \quad (7)$$

Equalizing the derivative to zero, gives:

$$\hat{\lambda} = \frac{n}{(\hat{\theta}+1)\sum_{i=1}^{n}\frac{x_i^{\hat{\lambda}}\, ln x_i}{(\hat{\beta} + x_i^{\hat{\lambda}})} - \sum_{i=1}^{n} ln\, x_i} \quad (8)$$

Finally, derive (eq. (5)) with respect to β; to get:

$$\frac{\partial}{\partial\beta}L(\psi) = \frac{n\theta}{\beta} - (\theta+1)\sum_{i=1}^{n}\frac{1}{(\beta + x_i^{\lambda})} \quad (9)$$

Equalizing the derivative to zero, gives:

$$\hat{\beta} = \frac{n\hat{\theta}}{(\hat{\theta}+1)\sum_{i=1}^{n}\frac{1}{(\hat{\beta} + x_i^{\hat{\lambda}})}} \quad (10)$$

Since we cannot find the estimators for parameters (θ,λ,β) from eqs. (6), (8) and (10), because it is difficult to solve this nonlinear equations simultaneously, so we resort to using the simulated annealing algorithm.

3.2 The moment method:

The Moments estimation method is one of the widely used method for estimating parameters of statistical models. It is based on the principle of equating the population moment with the sample Moment, and then estimating the parameter [6].

Let $x_1, x_2, \dots, x_n$ be a random sample of size n from a POLO distribution with a density function given in (eq. (1)). We equate the population moment to the sample moment, as follows [1]:

$$\mu_1' = \frac{\theta\beta^{\frac{1}{\lambda}}\Gamma\left[\theta-\frac{1}{\lambda}\right]\Gamma\left[\frac{1}{\lambda}\right]}{\lambda\Gamma[1+\theta]} \quad (11)$$

$$m_1 = \frac{\sum_{i=1}^{n} x_i}{n} \quad (12)$$

$$\mu_1' = m_1$$

$$n\theta\beta^{\frac{1}{\lambda}}\Gamma\left[\theta-\frac{1}{\lambda}\right]\Gamma\left[\frac{1}{\lambda}\right] = \lambda\Gamma[1+\theta]\sum_{i=1}^{n} x_i \quad (13)$$

$$\mu_2' = \frac{\theta\beta^{2/\lambda}\Gamma\left[\theta-\frac{2}{\lambda}\right]\Gamma\left[\frac{2+\lambda}{\lambda}\right]}{\Gamma[1+\theta]} \quad (14)$$

$$m_2 = \frac{\sum_{i=1}^{n} x_i^2}{n} \quad (15)$$

$$\mu_2' = m_2$$

$$n\hat{\theta}\hat{\beta}^{2/\hat{\lambda}}\Gamma\left[\hat{\theta}-\frac{2}{\hat{\lambda}}\right]\Gamma\left[\frac{2+\hat{\lambda}}{\hat{\lambda}}\right] = \Gamma\left[1+\hat{\theta}\right]\sum_{i=1}^{n} x_i^2 \quad (16)$$

$$\mu_3' = \frac{\theta\beta^{3/\lambda}\Gamma\left[\theta-\frac{3}{\lambda}\right]\Gamma\left[\frac{3+\lambda}{\lambda}\right]}{\Gamma[1+\theta]} \quad (17)$$

$$m_3 = \frac{\sum_{i=1}^{n} x_i^3}{n} \quad (18)$$

$$\mu_3' = m_3$$

$$n\hat{\theta}\hat{\beta}^{3/\hat{\lambda}}\Gamma\left[\hat{\theta}-\frac{3}{\hat{\lambda}}\right]\Gamma\left[\frac{3+\hat{\lambda}}{\hat{\lambda}}\right] = \Gamma\left[1+\hat{\theta}\right]\sum_{i=1}^{n} x_i^3 \quad (19)$$

Equations (13), (16) and (19) are nonlinear, so we resort to using the simulated annealing algorithm to get parameters estimators.

4. Simulated Annealing

The Simulated Annealing algorithm is considered as the oldest among the metaheuristic algorithms, and it is one of the first algorithms to have explicit strategies for escaping from the local minimum through random search sometimes, which makes it lose many solutions and this helps it in giving and searching high-efficiency solutions due to moving away from local search and approaching global search [7].

The method of Simulation Annealing (SA) is inspired by physical laws and is based on the principles of solid minerals. When metals melt at high temperatures and turn into a liquid state, metal particles become with high internal energy and move freely and quickly in various directions. When metals cool down, they gradually lose energy and form crystals [8].

- If the cooling process is slow enough to allow the formation of perfect crystals, the metal finds its state of minimum energy (optimal state) and crystals with a regular crystal structure are formed [7].
- If the cooling process is very fast, the metal will solidify more quickly and form amorphous, irregular, suboptimal shapes such as glass, and thus become brittle. Scale-down (SA) is used to improve the solution through a number of cooling stages until the optimum condition is achieved [7].

4.1 Steps of the simulated annealing [9]:

1. Initialize the SA control parameter (T, C, $N_{\max}$, $T_{\min}$).
2. Generate random initial solution (s_0).
3. The algorithm randomly searches for a new neighboring solution (S_n) through the current solution (S_c).
4. If the value of $f_{(S_n)}$ is better than the value of $f_{(S_c)}$, then the algorithm deletes the initial solution and replaces it with the new solution and returns to step (3).
5. If the new solution is worse than the current solution ($f_{(S_n)} > f_{(S_c)}$, the algorithm usually keeps the worst solution.
   This is done by calculating the probability value of Boltzmann as the following:
   $$p = e^{-\frac{\left(f_{(S_n)}-f_{(S_c)}\right)}{T}}$$
6. Generate a random value $u \in (1,0)$ if $u < p$, this means that the test solution (the worst solution) is accepted as the current solution, otherwise the current solution remains unchanged regardless of the action taken, go to step number (7).
7. If the current iteration number (N) is equal to the maximum number of iterations ($N_{\max}$) for the current level of the control parameter (T), then go to (8), otherwise increment the iteration number (N) by 1 and return to step (3).
8. Setting cooling temperature using the following relationship:
   $$T_n = T_c * C$$
   If the new T value is less than the stop parameter ($T_{\min}$), go to step (9), etc. Reset the current iteration number (N) to (1) and go back to step (3).
9. End of the program after obtaining the best solution.

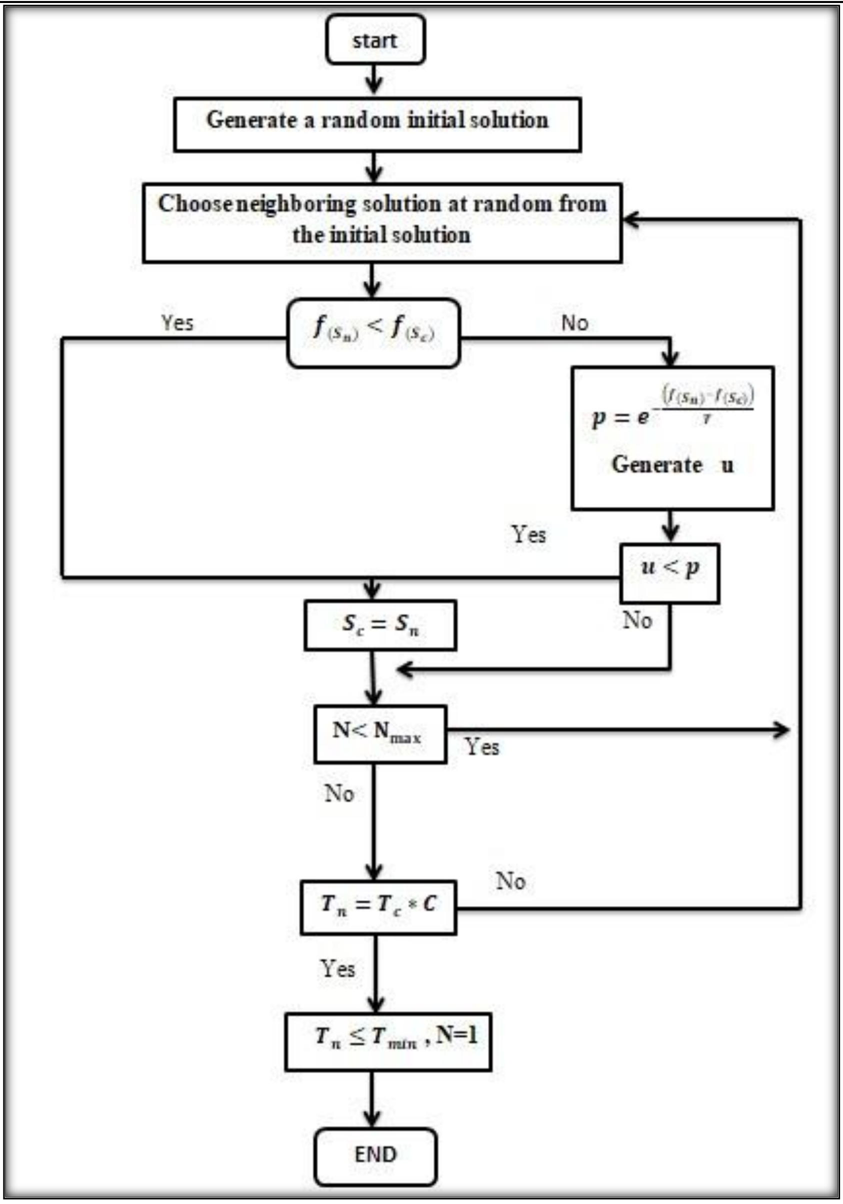

Figure 4. Steps of simulated annealing algorithm.

5. Simulation Study

The simulation program was written using the Matlab programming language.

The simulation experiment includes four stages to estimate the survival function of the POLO distribution and compare between the estimation methods:

a. Choosing different sample sizes as well as default values for parameters and forming models:

$\theta = 0.5, 1, 1.5$; $\beta = 2, 1.5, 2.5$; $\lambda = 1, 2, 3$; $n = (10, 25, 50, 75, 100$

The simulation experiment was repeated (1000 times).

b. Generating data for the POLO distribution according to the Monte Carlo method and using the inverse of the cumulative function As follows [10]:

$$F(t) = 1 - \beta^{\theta}(t^{\lambda} + \beta)^{-\theta} \quad (20)$$

Let $F(t) = u$

$$u = 1 - \beta^{\theta}(t^{\lambda} + \beta)^{-\theta} \quad (21)$$

$$t = \beta^{\frac{1}{\lambda}}\left[(1-u)^{-\frac{1}{\theta}} - 1\right]^{\frac{1}{\lambda}} \quad (22)$$

c. Estimate the survival function for the POLO distribution using the above estimation methods.

d. comparison between the methods according to the statistical comparison standard (IMSE) [11].

$$\text{IMSE}\left[\hat{S}(t)\right] = \frac{1}{r}\sum_{i=1}^{r}\left[\frac{1}{n_t}\sum_{j=1}^{n_t}\left[\hat{S}_i(t_j) - S(t_j)\right]^2\right]$$

6. Analysis of the Results

In this section, we review the simulation results. The table below shows the results of the simulation experiment to estimate the survival function of the POLO distribution according to the estimation methods (MLE,MOM) and using the algorithm SA.

Table 1. Value of IMSE for the estimated survival function by MLE and MOM.

N	Method		Best
	IMSE ($\hat{S}$) MLE_{SA}	IMSE ($\hat{S}$) MOM_{SA}	
$\theta = 0.5, \beta = 2, \lambda = 1$			
10	0.002086	0.121506	MLE
25	0.010491	0.03253	MLE
50	0.000143	0.190347	MLE
75	0.001042	0.199812	MLE
100	0.000112	0.142203	MLE
$\theta = 1, \beta = 1.5, \lambda = 2$			
10	0.022505	0.033525	MLE
25	0.005603	0.001748	MOM
50	0.000519	0.033046	MLE
75	0.007261	0.253094	MLE
100	0.000204	0.328626	MLE
$\theta = 1.5, \beta = 2.5, \lambda = 3$			
10	0.007383	0.031020	MLE
25	0.013936	0.036086	MLE
50	0.003104	0.002259	MOM
75	0.001508	0.033093	MLE
100	0.000642	0.027667	MLE

According to the simulation experience in the table above, the results for the three models showed and for default parameter values and different sample sizes, in the first model with parameter values ($\theta = 0.5, \beta = 2, \lambda = 1$); we found that the MLE method is better than the MOM method for all sample sizes, to possess less IMSE; while in the second model with parameter values ($\theta = 1, \beta = 1.5, \lambda = 2$); the MLE method outperformed the MOM method with most sample sizes, except for the sample size 25, in which the preference method was shown, As for the third model When increasing the value of the parameters ($\theta = 1, \beta = 1.5, \lambda = 2$); the preference was the MLE method in most sample sizes, with the exception of the sample size 50 in which the moment method showed their superiority.

7. Conclusion

The interest in estimating the survival function of statistical distributions continues to take a wide space by researchers because of their importance and connection to human life. in this paper, we reviewed one of the new distributions extending from the Lomax distribution, which is a Power Lomax distribution and some of its statistical properties such as the probability density function, the cumulative distribution function, and the survival function, we estimated the survival function for POLO distribution using MLE and MOM. According to the simulation experience, it was compared the two methods were compared according to the comparison standard IMSE it was found that the best method to estimate according to the parameter values and sample sizes that were assumed in this paper is MLE method that outperformed MOM method, as well as by employing the Simulated Annealing algorithm, the problem of nonlinear equations for the estimation methods was solved and to reach efficient estimators.

Conflicts of Interest

The authors declare that there is no conflict of interest.

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