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One-Sided Multiplier Approximation of Unbounded Functions

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Abstract

The main objective of this article is to study the degree of best one-sided multiplier approximation of unbounded functions $g \in L_{p,\psi_n}(Y), Y = [-1,1]$ by means of the average modulus of smoothness by using sequences of algebraic polynomials P_n of degree less than n, $n \geq r+1$, also in this search we shall prove a direct theorem by sequences P_n and some results.

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1. Introduction

Al-Saidy and Al-Saad [1] in 2014 obtained the degrees of pointwise summability in the "one-sided" approximation of functions.

Al-Saidy and Jawad [2] in 2015 studied best one-sided approximation by different operators in weighted spaces, also Al-Saidy and Abeer [3] in 2017 achieved the best one multiplier approximation of functions by Bernstein-Durrmeyer operators.

Al-Saidy and Ali [4] in 2020 obtained the degree of best multiplayer approximation of periodic unbounded functions using trigonometric operators.

2. Basic Concepts

In the beginning, we will put the most important definitions and basic lemmas which are used later in this paper.

Definition 1, [5]. A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergence if there is a sequence $\{\psi_n\}_{n=0}^{\infty}$, such that $\sum_{n=0}^{\infty} a_n \psi_n < \infty$ and we will say that $\{\psi_n\}_{n=0}^{\infty}$ is a multiplier for the convergence.

Definition 2, [5]. For any real valued function g, if there exists a sequence $\{\psi_n\}_{n=0}^{\infty}$, in which $\int_x g\,\psi_n(x)\,dx < \infty$, then we say that ψ_n is a multipliers for the integral.

Definition 3. Let $g \in L_{p,\psi_n}(Y), Y = [-1,1], p \in [1,\infty)$ be the space of all unbounded real valued functions f, such that $\int_y g\,\psi_n(y)\,dy < \infty$ with the norm:

$$\| g \|_{p,\psi_n} = \left(\int_x | g\,\psi_n(y) |^p \, dy \right)^{1/p}, \; y \in Y$$

Definition 4. For $g \in L_{p,\psi_n}(Y), Y = [-1,1]$, we will define the following concepts:

1. $\omega(g,\delta)_{p,\psi_n} = \sup_{|h|<\delta} \| g(y+h) - g(y) \|_{p,\psi_n}$ is the multiplier modulus of continuity of the function g for all $\delta > 0$.
2. $\tau_k(g,\delta)_{p,\psi_n} = \| \omega(g,.,\delta) \|_{p,\psi_n}$, $p \in [0,\infty)$, $k \in \mathbb{N}$, is the multiplier averaged modulus of smoothness of g of order k.

Definition 5. Let $g \in L_{p,\psi_n}(Y), Y = [-1,1]$ be the degree of best "one-sided" multiplier approximation of a function g with respect to algebraic polynomials is given by:

$$\tilde{E}_k(g)_{p,\psi_n} = \inf \left\{ \| p_n - q_n \|_{p,\psi_n} : p_n, q_n \in P_n, \; p_n(y) \leq g(y) \leq q_n(y) \right\}$$

where P_n be the set of all algebraic polynomials.

Lemma 1, [6]. Let $g \in W_\infty^r$ (W_∞^r be the class of all functions on [−1,1] with absolutely continuous (r − 1)-th derivative), r > 0, there is a sequence of algebraic polynomials P_n of degree less than n, $n \geq r + 1$, such that:

$$|g(y) - P_n(y)| \leq \frac{\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1} + C_r \frac{\ln n}{n^{r+1}}\left(\sqrt{1-y^2} + \frac{1}{n}\right)^r, \; y \in [-1,1]$$

where:

$$\tilde{K}_r = \frac{4\cos\left(\frac{r\pi}{2}\right)}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{r+1}}, \text{ if } 0 < r \leq \frac{1}{2}$$

Lemma 2, [5]. Let ω(t), $t \geq 0$ be the modulus of continuity and β > 0 is constant, then there is a sequence of algebraic polynomials $q_n(y)$ of degree less than or equal to n, such that for all $y \in [-1,1]$

$$0 \leq q_n(y) - \sqrt{1-y^2}\left(\frac{\sqrt{1-y^2}}{n}\right)^r \leq C\left(\sqrt{1-y^2} + \frac{1}{n}\right)\frac{1}{n^{2r}}, \; 0 < r < 1$$

where C is a constant which depends on β only.

Lemma 3. For any $r \in (0,1)$, there exists a sequence of algebraic polynomials $q_n^+(y)$, $y \in [-1,1]$, such that:

$$0 \leq q_n^+(y) - \left(\frac{1}{n} + \sqrt{1-y^2}\right) \leq \frac{C}{n^r}$$

Proof. Consider the function:

$$\phi(t) = \left(\frac{1}{n} + |t|\right)^r, \; t \in [-1,1]$$

for each $m \in \mathbb{N}$, let Z_m be the best approximation polynomial of ϕ. By Jackson's theorem [7], we have:

$$|Z_m(t) - \phi(t)| \leq \frac{C}{m^r}$$

Putting $t = \sqrt{1-y^2}$

$$\left|Z_m\left(\sqrt{1-y^2}\right) - \left(\frac{1}{n} + \sqrt{1-y^2}\right)^r\right| \leq \frac{C}{m^r}$$

If $m = n$

$$\left|Z_n\left(\sqrt{1-y^2}\right) - \left(\frac{1}{n} + \sqrt{1-y^2}\right)^r\right| \leq \frac{C}{n^r}$$

$$\left|Z_n(\Delta y) - \left(\frac{1}{n} + \sqrt{\Delta y}\right)^r\right| \leq \frac{C}{n^r}$$

where $\Delta y = 1 - y^2$.

$$-\frac{C}{n^r} \leq Z_n(\Delta y) - \left(\frac{1}{n} + \sqrt{\Delta y}\right)^r \leq \frac{C}{n^r}$$

$$0 \leq Z_n(\Delta y) + \frac{C}{n^r} - \left(\frac{1}{n} + \sqrt{\Delta y}\right)^r \leq \frac{2C}{n^r}$$

$$0 \leq q_n^+(y) - \left(\frac{1}{n} + \sqrt{\Delta y}\right)^r \leq \frac{2C}{n^r}$$

where $q_n^+(y) = Z_n(\Delta y) + \frac{C}{n^r}$. ■

Lemma 4. If $r \in (0,1)$, then there is a sequence of algebraic polynomials $q_n^-(y)$, such that:

$$-\frac{C}{n^r} \leq q_n^-(y) - \left(\frac{1}{n} + \sqrt{\Delta y}\right)^r \leq 0$$

Proof. From the proof of Lemma 3, we have:

$$-\frac{C}{n^r} \leq Z_n(\Delta y) - \left(\frac{1}{n} + \sqrt{\Delta y}\right)^r \leq \frac{C}{n^r}$$

$$-\frac{C}{n^r} - \frac{C}{n^2} \leq Z_n(\Delta y) - \frac{C}{n^r} - \left(\frac{1}{n} + \sqrt{\Delta y}\right)^r \leq 0$$

$$-\frac{2C}{n^r} \leq q_n^-(y) - \left(\frac{1}{n} + \sqrt{\Delta y}\right)^r \leq 0$$

where $q_n^-(y) = Z_n(\Delta y) - \frac{C}{n^r}$. Thus:

$$-\frac{C}{n^r} \leq q_n^-(y) - \left(\frac{1}{n} + \sqrt{\Delta y}\right)^r \leq 0 \quad ■$$

Lemma 5, [8]. For $g \in M$, then $\tau(g,\delta) = O(\delta)$, δ > 0 and $\delta \longrightarrow 0$.

The following lemma is easy to prove.

Lemma 6. For $g \in L_{p,\psi_n}(Y), Y = [-1,1]$, then $\tau(g,\delta)_{p,\psi_n} = O(\delta)$.

3. Main Results

In this section, we will be get the approximation for $g \in L_{p,\psi_n}(\Psi)$, $\Psi = [-1,1]$ by using two polynomials, $p_{n,r}^+(y)$ and $p_{n,r}^-(y)$.

Theorem 1. If $g \in L_{p,\psi_n}(\Psi)$, $\Psi = [-1,1]$, $0 < r < 1$, then there are two polynomials $p_{n,r}^+(y)$ and $p_{n,r}^-(y)$, for all $y = [-1,1]$, such that $p_{n,r}^-(y) \leq g(y) \leq p_{n,r}^+(y)$.

Proof. From Lemma 1, there is p_n, so that:

$$|p_n(y) - g(y)| \leq \frac{\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1} + \frac{C_r \ln}{n^{r+1}}\left(\frac{1}{n} + \sqrt{1-y^2}\right)^r$$

$$-\frac{\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}-\frac{C_r \ln}{n^{r+1}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r \leq$$

$$p_n(y)-g(y)\leq\frac{\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+$$

$$\frac{C_r \ln}{n^{r+1}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r$$

Adding $\frac{\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+\frac{C_r \ln}{n^{r+1}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r$ to the both sides of the last inequality, implies to:

$$0\leq p_n(y)-g(y)+\frac{\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+$$

$$\frac{C_r \ln}{n^{r+1}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r\leq\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+$$

$$\frac{2C_r \ln}{n^{r+1}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r \quad (1)$$

From Lemma 2, we have:

$$0\leq q_n(y)-\sqrt{1-y^2}\left(\frac{\sqrt{1-y^2}}{n}\right)^r$$

$$\leq C\left(\frac{1}{n}+\sqrt{1-y^2}\right)\frac{1}{n^{2r}} \quad (2)$$

The inequality (2) multiplied by $\tilde{K}_r$, we get:

$$0\leq \tilde{K}_r q_n(y)-\tilde{K}_r\sqrt{1-y^2}\left(\frac{\sqrt{1-y^2}}{n}\right)^r$$

$$\leq C\tilde{K}_r\left(\frac{1}{n}+\sqrt{1-y^2}\right)\frac{1}{n^{2r}} \quad (3)$$

From Lemma 3, we have:

$$0\leq q_n^+(y)-\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r\leq\frac{C}{n^r} \quad (4)$$

The inequality (3) multiplies by $\frac{C_r \ln}{n^{r+1}}$, we get:

$$0\leq q_n^+(y)\frac{C_r \ln}{n^{r+1}}-\frac{C_r \ln}{n^{r+1}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r\leq\frac{C}{n^r}\frac{C_r \ln}{n^{r+1}} \quad (5)$$

Adding (1), (3) and (5), we get:

$$0\leq p_{n,r}^+(y)-g(y)\leq\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+$$

$$\frac{2C_r \ln}{n^{r+1}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r+\frac{C\tilde{K}_r}{n^r}\left(\frac{1}{n}+\sqrt{1-y^2}\right)+\frac{C_r \ln}{n^{2r+1}}$$

where:

$$p_{n,r}^+(y)=p_n(y)+\tilde{K}_r q_n(y)+\frac{C_r \ln}{n^{2r+1}}q_n^+(y)$$

Since:

$$\frac{C\tilde{K}_r}{n^{2r}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)+\frac{CC_r \ln}{n^{2r+1}}\leq\frac{C\tilde{K}_r}{n^{2r}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)+$$

$$\frac{CC_r \ln}{n^{2r+1}}$$

$$\leq\frac{C\max\left\{\tilde{K}_r,C_r\right\}}{n^{2r}}\left(\frac{2}{n}+\sqrt{1-y^2}\right)$$

$$\leq\frac{2C\max\left\{\tilde{K}_r,C_r\right\}\ln}{n^{2r}}\left(\frac{2}{n}+\sqrt{1-y^2}\right)$$

Putting $2C\max\left\{\tilde{K}_r,C_r\right\}$ to be C, we have:

$$0\leq p_{n,r}^+(y)-g(x)\leq\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+$$

$$\frac{2C_r \ln}{n^{r+1}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r+\frac{C \ln}{n^r}\left(\frac{1}{n}+\sqrt{1-y^2}\right)$$

$$\leq\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+\frac{C_r \ln}{n^{2r}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r \quad (6)$$

Hence $0\leq p_{n,r}^+(y)-g(y)$, and therefore:

$$g(y)\leq p_{n,r}^+(y) \quad (7)$$

Similarly, we can prove that:

$$-\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}-\frac{C_r \ln}{n^{2r}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)^r\leq$$

$$p_{n,r}^-(y)-g(y)\leq 0$$

and so:

$$p_{n,r}^-(y)\leq g(y) \quad (8)$$

From (7) and (8), we get:

$$p_{n,r}^-(y)\leq g(y)\leq p_{n,r}^+(y) \quad \blacksquare$$

Theorem 2. Let $g\in L_{p,\psi_n}(Y), Y=[-1,1]$, then:

$$\left\|p_{n,r}^+(y)-g(y)\right\|_{p,\psi_n}\leq\tau\left(f,\frac{1}{n^r}\right)_{p,\psi_n}$$

Proof. From (6), we have:

$$p_{n,r}^+(y)-g(y)\leq\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+$$

$$\frac{C_r \ln n}{n^{2r}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)$$

$$\left\|p_{n,r}^+(\cdot)-g(\cdot)\right\|_{p,\psi_n}\leq\left\|\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}\right\|_{p,\psi_n}+$$

$$\left\|\frac{C_r \ln n}{n^{2r}}\left(\frac{1}{n}+\sqrt{1-y^2}\right)\right\|_{p,\psi_n}$$

$$\leq\frac{2\tilde{K}_r}{n^r}\left\|\left(\sqrt{1-y^2}\right)^{r+1}\right\|_{p,\psi_n}+\left\|\frac{C_r \ln}{n^{2r+1}}\right\|_{p,\psi_n}+$$

$$\left\|\frac{C_r \ln n}{n^{2r}}\left(\sqrt{1-y^2}\right)\right\|_{p,\psi_n}$$

$$\leq \frac{2\tilde{K}_r}{n^r}\int_{-1}^{1}\left(\sqrt{1-y^2}\right)^{r+1}\mathrm{d}y+\frac{C_r\ln}{n^{2r+1}}+$$
$$\frac{C_r\ln n}{n^{2r}}\int_{-1}^{1}\sqrt{1-y^2}\mathrm{d}y$$
$$=\frac{C_1 r}{n^r}+\frac{C_r\ln n}{n^{2r+1}}+\frac{\pi C_r\ln n}{n^{2r}}$$
$$=\frac{C_1 r\ln n+nC_r\ln n}{n^{2r+1}}$$
$$\leq\frac{C_1 r}{n^{2r}}+\frac{C_1}{n^{2r-1}}\leq\frac{C}{n^r}$$

By Lemma 6, $\tau\left(g,\frac{1}{n^r}\right)\simeq O\left(\frac{1}{n^r}\right)$. Then:

$$\tau\left(g,\frac{1}{n^r}\right)_{p,\psi_n}\simeq O\left(\frac{1}{n^r}\right)\quad\blacksquare$$

Theorem 3. Let $g\in L_{p,\psi_n}(Y), Y=[-1,1]$, then:

$$\left\|g(y)-p_{n,r}^{-}(y)\right\|_{p,\psi_n}\leq\tau\left(f,\frac{1}{n^r}\right)_{p,\psi_n}$$

Proof. Since:

$$-\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}-\frac{C_r\ln n}{n^r}\left(\frac{1}{n}+\sqrt{1-y^2}\right)\leq$$
$$g(x)-p_{n,r}^{-}(y)\leq 0$$

Then:

$$\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+\frac{C_r\ln n}{n^r}\left(\frac{1}{n}+\sqrt{1-y^2}\right)\geq$$
$$p_{n,r}^{-}(y)-g(y)\geq 0$$
$$0\leq p_{n,r}^{-}(y)-g(y)\leq\frac{2\tilde{K}_r}{n^r}\left(\sqrt{1-y^2}\right)^{r+1}+$$
$$\frac{C_r\ln n}{n^r}\left(\frac{1}{n}+\sqrt{1-y^2}\right)$$

Similarly, as in the proof of Theorem 2, it may be proved that:

$$\left\|g(y)-p_{n,r}^{-}(y)\right\|_{p,\psi_n}\leq\tau\left(g,\frac{1}{n^r}\right)_{p,\psi_n}\quad\blacksquare$$

Theorem 4. Let $g\in L_{p,\psi_n}(Y), Y=[-1,1]$, then:

$$\tilde{E}_n(g)_{p,\psi_n}\leq C\,\tau\left(g,\frac{1}{n^r}\right)_{p,\psi_n}$$

Proof.

$$\left\|p_{n,r}^{+}(\cdot)-p_{n,r}^{-}(\cdot)\right\|_{p,\psi_n}=\left\|p_{n,r}^{+}(y)-g(y)+g(y)-p_{n,r}^{-}(y)\right\|_{p,\psi_n}$$
$$\leq\left\|p_{n,r}^{+}(y)-g(y)\right\|_{p,\psi_n}+\left\|g(y)-p_{n,r}^{-}(y)\right\|_{p,\psi_n}$$

From Theorems 2 and 3, we get:

$$\left\|p_{n,r}^{+}(\cdot)-p_{n,r}^{-}(\cdot)\right\|_{p,\psi_n}\leq\tau\left(g,\frac{1}{n^r}\right)_{p,\psi_n}+\tau\left(g,\frac{1}{n^r}\right)_{p,\psi_n}$$
$$\leq 2\tau\left(g,\frac{1}{n^r}\right)_{p,\psi_n}$$

Hence:

$$\tilde{E}_n(g)_{p,\psi_n}\leq C\,\tau\left(g,\frac{1}{n^r}\right)_{p,\psi_n}\quad\blacksquare$$

Conflicts of Interest

The authors declare that there is no conflict of interest.

References

[1] Al-Saidy S. and Al-Saad H. A.; "On the degree of pointwise summability in the one-sided approximation of functions". M.Sc. Thesis, Mathematics Department, College of Science, Al-Mustansiryh University, 2014.
[2] Al-Saidy S. and Jawad K. J.; "Best one-sided approximation by different operators in weighted spaces ", Ph.D. Thesis, Department of Mathematics, College of Science, Mustansiryh University, 2015.
[3] Al-Saidy S. and Abeer M. S.; "Multiplier approximation of functions by Bernstein-Durrmeyer operators", Ph.D. Thesis, Department of Mathematics, College of Science, Mustansiryh University, 2017.
[4] Al-Saidy S. and Ali H. Z.; " Best multiplier approximation of periodic unbounded functions using trigonometric operators ", Ph.D. Thesis, Department of Mathematics, College of Science, Mustansiryh University, 2020.
[5] Hardy G.; "Divergent Series", ed: Oxford University Press, 1949.
[6] Pasko A.; "The pointwise estimation of the one-sided approximation of the class W_∞^r, $0<r<1$", Researches in Mathematics, 28 (1): 22-28, 2020.
[7] DeVore R. A. and Lorentz G. G., "Constructive approximation", Springer Science & Business Media, 1993.
[8] Sendov B. C. and Popov V. A.; "The averaged moduli of smoothness: application in numerical methods and approximation", Chichester, 1988.